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Stability of Contact Discontinuities Under Perturbations of Bounded Variation.

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Introduction.

In the middle of the sixties James Glimm proved in a celebrated paper, [Gl], that strictly hyperbolic systems of conservation laws in one space variable admit global solutions provided that the initial data are close to some constant background state and their variation is small. Our paper falls within that framework: we consider however a strong contact discontinuity as background state and give a stability condition which insures the global existence of perturbed solutions when initial data are close to the background contact discontinuity and have again small variation. Moreover for these solutions the structure of contact discontinuity persists.

In a previous paper, [CST1] (see also [CST2]), we considered the case of the BV perturbation of a strong shock wave and showed that Majda's stability condition for the background state yields global existence of perturbed shock solutions; we proved also that the perturbed shock curve is uniformly lipschitzean and its speed as well as the traces of the solutions at either side have bounded variations. The main tools were a suitable Glimm's scheme, which was previously carried out for mixed problems in [ST], and for traces the approximate characteristics of Glimm-Lax, [GL]. We refer the reader to that paper for some more motivations and references on the subject.

In the present paper we exploit to some extent the ideas of [CST1]. The stability condition we were speaking of just above is stated in Sec-

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tion 1: its main feature is that it does not require the existence of the contact flow from the left to the right part of the background state; when this happens it is equivalent to the stability condition of [Sc] and [Co]. We point out that under that condition Schochet [Sc] obtain as a byproduct a result of global existence of solutions when initial data are as ours; unfortunately his method is not useful to show that the solutions are still contact discontinuities. An analogous result is stated by Chern [Ch].

In Section 2 then we solve the Riemann problem for states close to the stable background contact discontinuity: the main point there is in showing how small waves colliding with the strong discontinuity are transmitted, reflected or absorbed. Solutions are constructed in Section 3 through a Glimm-type scheme; though it is similar to that we used in [CST1], nevertheless proofs of the estimates are somewhat complicated by the special attention that waves with the same number of the strong contact require. The last fourth section deals with traces: only at that point we manage to prove that the solutions previously constructed were contact discontinuities.

1. - The results.

Let f be a C^3 function defined in an open subset $\Omega \subset \boldsymbol{R}^N$, with values in $\boldsymbol{R}^N$, and consider the following system of conservation laws in one space dimension

$$\partial_t u + \partial_x f(u) = 0. \tag{1.1}$$

Strict hyperbolicity in Ω is assumed, i.e., the eigenvalues $\lambda_1(u), \ldots, \lambda_N(u)$ of the matrix $A(u) := Df(u)$ are real and distinct for $u \in \Omega$; we suppose that they are already ranged in the increasing order and denote by $r_i(u)$ a right eigenvector of the matrix $A(u)$ associated to the eigenvalue $\lambda_i(u)$. We assume that every mode is either genuinely nonlinear or linearly degenerate, that is, either $D\lambda_i(u)\cdot r_i(u) \neq 0$ or $D\lambda_i(u)\cdot r_i(u) = 0$ holds for every $u \in \Omega$ and $i = 1, \ldots, N$; for genuinely nonlinear modes let the normalization $D\lambda_i \cdot r_i = 1$ hold. Let κ (greek kappa) be an integer between 1 and N and assume that the eigenvalue λ_κ is linearly degenerate.

Let $\underline{u}^-$, $\underline{u}^+$ be two constant states in Ω, $\underline{q} \in \boldsymbol{R}$, and $\underline{u}$ the function defined by $\underline{u}^\pm$ for $\pm(x - \underline{q}t) > 0$; we call $\underline{u}$ the background state and assume that it is a κ-contact discontinuity, that is, $\underline{q} = \lambda_\kappa(\underline{u}^\pm)$ and the jump conditions $\underline{q}[\underline{u}] - [f(\underline{u})] = 0$ are satisfied. In order to handle contact discontinuities of arbitrary strength we make the following hypothesis:

($\mathcal{H}$) *in neighborhoods* $\underline{\Omega}^{\pm} \subset \Omega$ *of* $\underline{u}^{\pm}$ *there exist two hypersurfaces* $\Sigma^{\pm}$, *which are transverse to* $r_{\kappa}(\underline{u}^{\pm})$ *at points* $\underline{u}^{\pm}$, *and a diffeomorphism* $\psi\colon \Sigma^{-} \to \Sigma^{+}$ *whose graph defines the pairs of* κ*-contact discontinuities in* $\Sigma^{-} \times \Sigma^{+}$.

This condition is satisfied if there exists in the whole connected set Ω a family of $N-1$ κ-Riemann invariants $\{R_1, \ldots, R_{N-1}\}$ with linearly independent gradients (we recall that a κ-Riemann invariant is a scalar function I such that $DI(u) \cdot r_{\kappa}(u) = 0$). In fact in this case we can choose $\Sigma^{\pm} = \underline{u}^{\pm} + \operatorname{span}\{DR_1(\underline{u}^{\pm}), \ldots, DR_{N-1}(\underline{u}^{\pm})\} \cap \Omega$ and then ψ is defined through the integral curves of r_{κ}. In general however such a family exists only locally; the papers [Sc] and [Co] fall into this frame.

By shrinking in case the neighborhoods $\underline{\Omega}^{\pm}$ we extend ψ to a diffeomorphism $\Psi\colon \underline{\Omega}^{-} \to \underline{\Omega}^{+}$ by means of the integral curves of r_{κ}; in this way each pair $(v^{-}, v^{+}) \in \underline{\Omega}^{-} \times \underline{\Omega}^{+}$ lying on a product of integral curves of r_{κ} through $u^{-} \in \underline{\Omega}^{-}$ and $\Psi(u^{-}) \in \underline{\Omega}^{+}$, respectively, is a κ-contact discontinuity. As a consequence, assumption ($\mathcal{H}$) does not depend on the choice of the hypersurfaces $\Sigma^{\pm}$ as long as transversality holds. In particular we can take

$$\Sigma^{\pm} = \{\underline{u}^{\pm} + \operatorname{span}\{r_j(\underline{u}^{\pm});\ j = 1, \ldots, N,\ j \neq \kappa\}\} \cap \underline{\Omega}^{\pm} \tag{1.2}$$

so that $T_{\underline{u}^{\pm}} \Sigma^{\pm} = \operatorname{span}\{r_j(\underline{u}^{\pm});\ j = 1, \ldots, N, j \neq \kappa\}$, where $T_{\underline{u}^{\pm}} \Sigma^{\pm}$ denotes the tangent space of $\Sigma^{\pm}$ at $\underline{u}^{\pm}$. This assumption will simplify a bit the statements of our results and the proofs.

Assumption ($\mathcal{H}$) can be restated in the following geometric way. Let $F\colon \Omega \to \boldsymbol{R}^{N+1}$ be the C^2 function defined by $F(u) = (f(u) - \lambda_{\kappa}(u)u, \lambda_{\kappa}(u))$; remark that $u^{-} \in \underline{\Omega}^{-}$ and $u^{+} \in \underline{\Omega}^{+}$ are a κ-contact discontinuity if and only if $F(u^{+}) = F(u^{-})$. Since λ_{κ} is linearly degenerate, then $\operatorname{rank}(DF(u)) = N-1$ for every $u \in \Omega$ and therefore $M^{\pm} := F(\underline{\Omega}^{\pm})$ are submanifolds of $\boldsymbol{R}^{N+1}$ of dimension $N-1$. By the rank theorem (possibly by shrinking $\underline{\Omega}^{\pm}$) we obtain then two diffeomorphisms of manifolds $\phi^{\pm}\colon \Sigma^{\pm} \to M^{\pm}$, with $\phi^{\pm} = F|_{\Sigma^{\pm}}$. We claim that assumption ($\mathcal{H}$) holds if and only if $M^{+} = M^{-}$.

In fact, if $M^{+} \neq M^{-}$ then there exists $v^{+} \in \underline{\Omega}^{+}$ such that $F(v^{+}) \neq F(u^{-})$, for every $u^{-} \in \underline{\Omega}^{-}$. But $\Psi^{-1}(v^{+}) \in \underline{\Omega}^{-}$ and $F(v^{+}) = F(\Psi^{-1}(v^{+}))$, a contradiction. Conversely, if $M^{+} = M^{-}$ we can define $\psi = (\phi^{+})^{-1} \phi^{-}$; then $u^{+} \in \Sigma^{+}$ and $u^{-} \in \Sigma^{-}$ are a κ-contact discontinuity iff $\phi^{+}(u^{+}) = \phi^{-}(u^{-})$, that is, $u^{+} = \psi(u^{-})$.

For $u^{\pm} \in \Omega^{\pm}$ we shall call *causal* eigenvalues the eigenvalues

$$\lambda_{\kappa+1}(u^{-}), \ldots, \lambda_N(u^{-}), \lambda_1(u^{+}), \ldots, \lambda_{\kappa-1}(u^{+})$$

whereas the noncausal eigenvalues are

$$\lambda_1(u^-), \ldots, \lambda_{\kappa-1}(u^-), \lambda_{\kappa+1}(u^+), \ldots, \lambda_N(u^+).$$

We point out that the eigenvalue λ_κ does not appear neither in the first set nor in the latter. We shall make use of notations analogous to those in [CST1], i.e.,

$$r_{\mathrm{nc}}^-(u^-) = (r_1(u^-), \ldots, r_{\kappa-1}(u^-)), \quad r_{\mathrm{c}}^-(u^-) = (r_{\kappa+1}(u^-), \ldots, r_N(u^-)),$$

$$r_{\mathrm{c}}^+(u^+) = (r_1(u^+), \ldots, r_{\kappa-1}(u^+)), \quad r_{\mathrm{nc}}^+(u^+) = (r_{\kappa+1}(u^+), \ldots, r_N(u^+)).$$

The stability condition for κ-contact discontinuities is

$$\text{rank}\,(T_{\underline{u}^-}\psi(r_{\mathrm{nc}}^-(\underline{u}^-)), \ r_{\mathrm{nc}}^+(\underline{u}^+)) = N-1 \tag{1.3}$$

where $T_{\underline{u}^-}\psi$: $T_{\underline{u}^-}\Sigma^- \to T_{\psi(\underline{u}^-)}\Sigma^+$ is the tangent map of ψ. In the case of global existence of a family of κ-Riemann invariants as above it is equivalent to

$$\det(DR(\underline{u}^-)\,r_{\mathrm{nc}}^-(\underline{u}^-), \ DR(\underline{u}^+)r_{\mathrm{nc}}^+(\underline{u}^+)) \neq 0 \tag{1.4}$$

where we denoted by $R(u)$ the function valued in $\boldsymbol{R}^{N-1}$ whose components are the invariants $R_1(u), \ldots, R_{N-1}(u)$; it is the stability condition of [Sc] and [Co]. In terms of the map F introduced above condition (1.3) reads $\text{rank}\,(DF(\underline{u}^-)(r_{\mathrm{nc}}^-(\underline{u}^-)), DF(\underline{u}^+)(r_{\mathrm{nc}}^+(\underline{u}^+))) = N-1$, since $T_{\underline{u}^\pm}\phi^\pm(r_{\mathrm{nc}}^\pm(\underline{u}^\pm))$ is diffeomorphic to $T_{\underline{u}^\pm}F(r_{\mathrm{nc}}^\pm(\underline{u}^\pm))$.

In the general case, i.e., for choices of $\Sigma^\pm$ different from (1.2), condition (1.3) must be changed as follows. Let $\Sigma_1^\pm$ be two hypersurfaces as in ($\mathcal{H}$) and ψ_1: $\Sigma_1^- \to \Sigma_1^+$ the related diffeomorphism. We define then diffeomorphisms $\beta^\pm$: $\Sigma^\pm \to \Sigma_1^\pm$ by following the flow of r_κ in $\underline{\Omega}^\pm$; therefore $\psi = (\beta^+)^{-1} \circ \psi_1 \circ \beta^-$ and condition (1.3) becomes

$$\text{rank}\,(T_{\underline{u}^-}\psi_1 T_{\underline{u}^-}\beta^- r_{\mathrm{nc}}^-(\underline{u}^-), \ T_{\underline{u}^+}\beta^+ r_{\mathrm{nc}}^+(\underline{u}^+)) = N-1.$$

We give now the general definition of κ-contact discontinuity. Let $u \in BV_{\mathrm{loc}}(\boldsymbol{R}_t^+ \times \boldsymbol{R}_x)$, $\chi \in \mathrm{Lip}\,(\boldsymbol{R}_t^+)$, and denote by $u^\pm$ the restrictions of u to the regions $\pm(x-\chi(t)) > 0$. We say that u is a κ-contact discontinuity with κ-contact curve $x - \chi(t) = 0$ if u is a weak solution to (1.1) and

$$\lambda_\kappa(u^-(t, \chi(t))) = \chi'(t) = \lambda_\kappa(u^+(t, \chi(t)))$$

holds a.e.; the Rankine-Hugoniot conditions are then satisfied a.e. on the curve $x = \chi(t)$.

THEOREM 1.1. *Let assumption ($\mathcal{H}$) holds and let $\underline{u}$ be a stable, piecewise constant κ-contact discontinuity. If $h^{\pm} \in BV(\boldsymbol{R}^{\pm})$ satisfies*

$$\|h^{\pm} - \underline{u}^{\pm}\|_{L^\infty} \leqslant \delta, \qquad V(h^{\pm}) \leqslant \delta, \tag{1.5}$$

for some $\delta > 0$ sufficiently small, then the initial-value problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(0, x) = h^{\pm}(x), \qquad \pm\, x > 0, \end{cases} \tag{1.6}$$

has a weak solution u in $\boldsymbol{R}_t^+ \times \boldsymbol{R}_x$ with a large jump along a curve $x - \chi(t) = 0$. The function χ is in $\mathrm{Lip}(\boldsymbol{R}_t^+)$, $\chi(0) = 0$, *and $q := \chi' \in BV(\boldsymbol{R}_t^+)$. Moreover the following estimates hold:*

$$\|u^{\pm} - \underline{u}^{\pm}\|_{L^\infty} \leqslant C\{\|h^- - \underline{u}^-\|_{L^\infty} + \|h^+ - \underline{u}^+\|_{L^\infty}\}, \tag{1.7}$$

$$V(u^{\pm}(t, \cdot)) + V(q) \leqslant \tag{1.8}$$

$$\leqslant C\{V(h^-) + V(h^+) + |h(0-) - \underline{u}^-| + |h(0+) - \underline{u}^+|\},$$

for some positive constant C.

We point out that Theorem 1.1 does not say whether the solution u is a κ-contact discontinuity; this is the content of the following result.

THEOREM 1.2. *With the same assumptions of the previous theorem, let u be a solution of* (1.6) *obtained as a limit of Glimm's approximations with equidistributed sampling; then u is a κ-contact discontinuity. More precisely $\chi'(t) = \lambda_\kappa(u^{\pm}(t, \chi(t)))$ a.e., the traces on the contact curve $x = \chi(t)$ of the functions $f(u) - \lambda_\kappa(u)u$ are equal and have bounded variation, and we have the estimates*

$$\|q - \underline{q}\|_{L^\infty} \leqslant C\{\|h^- - \underline{u}^-\|_{L^\infty} + \|h^+ - \underline{u}^+\|_{L^\infty}\}, \tag{1.9}$$

$$V(\lambda_\kappa(u^{\pm})|_{(t, \chi(t))}) + V((f(u^{\pm}) - \lambda_\kappa(u^{\pm})u^{\pm})|_{(t, \chi(t))}) \leqslant \tag{1.10}$$

$$\leqslant C\{V(h^-) + V(h^+) + |h(0-) - \underline{u}^-| + |h(0+) - \underline{u}^+|\}.$$

We do not know whether the complete traces of the solution u on the contact curve are of bounded variations, but we believe that the answer is in the negative.

Our results apply to Euler system of gasdynamics: in this case Riemann invariants associated to the entropic mode exist globally, if the vacuum is avoided, and condition (1.4) holds. We can also consider the system of magnetohydrodynamics (see [JT]) in regions of strict hyperbolicity and in absence of the vacuum. The entropic and both the Alfvén modes are linearly degenerate, Riemann invariants exist globally, and it comes out after some cumbersome calculations that condition (1.4) is still satisfied in each case. Another example of stable contact discontinuities is given in [Co] for a system of conservation laws arising in elastodynamics.

2. – The Riemann problem near a strong contact discontinuity.

For $u \in \Omega$ we denote as usual Lax's curves through u by $\varepsilon_i \mapsto \Phi_i(\varepsilon_i, u)$, $i = 1, \ldots, N$. We say below that $(u^-, u^+) \in \underline{\Omega}^- \times \underline{\Omega}^+$ is a contact discontinuity of parameter p (modulo Ψ) if $u^+ = \Phi_\kappa(p, \Psi(u^-))$. The diffeomorphism Ψ was described in Section 1; from now on we choose Ψ defined explicitly for $u^- \in \underline{\Omega}^-$ by

$$\Psi(u^-) = \Phi_\kappa(\alpha_\kappa, \psi(\Phi_\kappa(-\alpha_\kappa, u^-))) \tag{2.1}$$

for some $\alpha_\kappa = \alpha_\kappa(u^-)$.

The following lemma concerns the Hugoniot curve of a strong contact discontinuity; its proof is a consequence of the definition of the diffeomorphism Ψ. We remark that while for shock waves the existence of the Hugoniot curve was merely a consequence of the fact that the shock is noncharacteristic (see [CST1]), for contact discontinuities assumption $(\mathcal{H})$ is needed.

LEMMA 2.1. *Assume* $(\mathcal{H})$ *and let* $\underline{u}$ *be a piecewise constant* κ*-contact discontinuity. Then there exist two disjoint neighborhoods* $\Omega^\pm \subset \underline{\Omega}^\pm$ *of* $\underline{u}^\pm$ *and* $I \subset \mathbf{R}$ *of* 0 *such that for* $u^- \in \Omega^-$ *the curve*

$$H(\cdot, u^-)\colon I \to \Omega^+, \qquad p \mapsto \Phi_\kappa(p, \Psi(u^-)),$$

defines the κ*-contact discontinuity of parameter* p *(modulo* Ψ*) with left state* u^-*. Moreover if* $(u^-, u^+) \in \Omega^- \times \Omega^+$ *is a* κ*-contact discontinuity of parameter* p*, then* $u^+ = H(p, u^-)$*.*

From the definition of H it follows that

$$\begin{cases} D_p H(p, u^-) = r_\kappa(H(p, u^-)), \\ D_{u^-} H(0, u^-) = D\Psi(u^-). \end{cases} \tag{2.2}$$

In the following we call briefly contact discontinuities of strength p the contact discontinuities of the previous lemma. Notations will be analogous to those in [CST1]; however we draw attention on the fact that the term «causal» has here a different meaning. Therefore for instance we denote

$$\Phi^-_{\mathrm{nc}}(\varepsilon^-_{\mathrm{nc}}, u) = \Phi_{\kappa-1}(\varepsilon_{\kappa-1}, \Phi_{\kappa-2}(\varepsilon_{\kappa-2}, \ldots, \Phi_1(\varepsilon_1, u))\ldots),$$

$$\Phi^-_{\mathrm{c}}(\varepsilon^-_{\mathrm{c}}, u) = \Phi_N(\varepsilon_N, \Phi_{N-1}(\varepsilon_{N-1}, \ldots, \Phi_{\kappa+1}(\varepsilon_{\kappa+1}, u))\ldots),$$

and analogously Φ^+_{c} and Φ^+_{nc}. Let $\omega^\pm$, $\omega^\pm_{\mathrm{c}}$, $\omega^\pm_\kappa$, $\omega^\pm_{\mathrm{nc}}$, be neighborhoods of 0 and $\Omega^\pm_1$ a subset of $\Omega^\pm \subset \boldsymbol{R}^N$ such that their images by the Lax's functions, $\Phi(\omega^\pm, \Omega^\pm_1)$, $\Phi^\pm_{\mathrm{c}}(\omega^\pm_{\mathrm{c}}, \Omega^\pm_1)$, $\Phi^\pm_\kappa(\omega_\kappa, \Omega^\pm_1)$, $\Phi^\pm_{\mathrm{nc}}(\omega^\pm_{\mathrm{nc}}, \Omega^\pm_1)$, are contained in $\Omega^\pm$. Then we can find neighborhoods $\omega^\pm_1 \subset \omega^\pm$ of 0 and $\Omega^\pm_2 \subset \Omega^\pm_1$ of $\underline{u}^\pm$ in such a way that every Riemann problem with initial data $(u^\pm_l, u^\pm_r) \in \Omega^\pm_2 \times \Omega^\pm_2$ can be solved by some strength $\varepsilon^\pm \in \omega^\pm_1$, i.e., $u^\pm_r = \Phi(\varepsilon^\pm, u^\pm_l)$; moreover $\varepsilon^\pm = O(1)|u^\pm_l - u^\pm_r|$ and intermediate states $v^\pm \in \Omega^\pm_1$ satisfy $|v^\pm - \underline{u}^\pm| = O(1)\max(|u^\pm_l - \underline{u}^\pm|, |u^\pm_r - \underline{u}^\pm|)$.

We show now how the stability condition (1.3) allows to solve Riemann problems for states close to a strong κ-contact discontinuity; we point out that no wave but the κ-th one is strong.

PROPOSITION 2.2. *Assume* ($\mathcal{H}$) *holds and let* $\underline{u}$ *be a stable, piecewise constant* κ*-contact discontinuity. Unless of shrinking the neighborhoods* $\Omega^\pm_2$ *and* $\omega^\pm_1$*, there exists a neighborhood* $I_1 \subset I$ *of* 0 *such that for every pair of states* $(u^-, u^+) \in \Omega^-_2 \times \Omega^+_2$ *we can find in a unique way strengths* $\gamma^\pm_{\mathrm{nc}} \in \omega^\pm_{1,\mathrm{nc}}$ *and a parameter* $p \in I_1$ *satisfying*

$$u^+ = \Phi^+_{\mathrm{nc}}(\gamma^+_{\mathrm{nc}}, H(p, \Phi^-_{\mathrm{nc}}(\gamma^-_{\mathrm{nc}}, u^-))).$$

Moreover $(\gamma^-_{\mathrm{nc}}, p, \gamma^+_{\mathrm{nc}}) = O(1)\max(|u^- - \underline{u}^-|, |u^+ - \underline{u}^+|)$ *and intermediate states* $v^\pm \in \Omega^\pm_1$ *satisfy* $|v^\pm - \underline{u}^\pm| = O(1)\max(|u^- - \underline{u}^-|, |u^+ - \underline{u}^+|)$.

PROOF. We consider the function F: $\omega^-_{\mathrm{nc}} \times I \times \omega^+_{\mathrm{nc}} \times \Omega^-_1 \to \Omega^-_1 \times \Omega^+_1$ defined by

$$F(\gamma^-_{\mathrm{nc}}, p, \gamma^+_{\mathrm{nc}}; u^-) = (u^-, \Phi^+_{\mathrm{nc}}(\gamma^+_{\mathrm{nc}}, H(p, \Phi^-_{\mathrm{nc}}(\gamma^-_{\mathrm{nc}}, u^-))));$$

then $F(0, 0, 0; \underline{u}^-) = (\underline{u}^-, \underline{u}^+)$ and

$$D_{\gamma^-_{\mathrm{nc}}, p, \gamma^+_{\mathrm{nc}}}F(0, p, 0; u^-) = \begin{pmatrix} 0 & 0 & 0 \\ D_u H(p, u^-) r^-_{\mathrm{nc}}(u^-) & r_\kappa(u^+) & r^+_{\mathrm{nc}}(u^+) \end{pmatrix}$$

for $u^+ = H(p, u^-)$. Then we introduce the $N \times N$ matrix

$$(2.3)\quad \mathcal{B}(u^-, p, u^+) = D_{\gamma^-_{\mathrm{nc}}, p, \gamma^+_{\mathrm{nc}}} \Phi^+_{\mathrm{nc}}(\gamma^+_{\mathrm{nc}}, H(p, \Phi^-_{\mathrm{nc}}(\gamma^-_{\mathrm{nc}}, u^-)))|_{(0, p, 0; u^-)} =$$

$$= (D_u H(p, u^-) r^-_{\mathrm{nc}}(u^-), \qquad D_\varepsilon H(p, u^-), \;\; r^+_{\mathrm{nc}}(u^+))$$

for $u^+ = H(p, u^-)$. This matrix will play a very important role in what follows; remark that the stability condition implies that the matrix

$$\mathcal{B}(\underline{u}^-, 0, \underline{u}^+) = (D\Psi(\underline{u}^-) r^-_{\mathrm{nc}}(\underline{u}^-), \;\; r^+_\kappa(\underline{u}^+), \;\; r^+_{\mathrm{nc}}(\underline{u}^+))$$

is nonsingular. From (2.2) we deduce

$$D_{\gamma^-_{\mathrm{nc}}, p, \gamma^+_{\mathrm{nc}}, u^-} F(0, 0, 0; \underline{u}^-) = \begin{pmatrix} 0 & Id \\ \mathcal{B}(\underline{u}^-, 0, \underline{u}^+) & D\Psi(\underline{u}^-) \end{pmatrix};$$

we used above here the diffeomorphism Ψ for simplicity, but formulas depend as a matter of fact only on ψ. Then the inverse function theorem applies and gives neighborhoods $\Omega_2^\pm$, I_1, $\omega^\pm_{1,\mathrm{nc}}$. This proves the proposition.

In order to consider the interaction of weak waves with the strong κ-contact discontinuity we introduce now the transmission, absorption, reflection matrices, $T_{(\pm)}$, $A_{(\pm)}$, $R_{(\pm)}$, respectively, as follows:

$$\mathcal{B}(u^-, p, u^+)^{-1} r^+_{\mathrm{c}}(u^+)|_{u^+ = H(p, u^-)} = \begin{pmatrix} T_{(+)}(p, u^-) \\ A_{(+)}(p, u^-) \\ R_{(+)}(p, u^-) \end{pmatrix},$$

$$\mathcal{B}(u^-, p, u^+)^{-1} D_{u^-} H(p, u^-) r^-_{\mathrm{c}}(u^-)|_{u^+ = H(p, u^-)} = \begin{pmatrix} R_{(-)}(p, u^-) \\ A_{(-)}(p, u^-) \\ T_{(-)}(p, u^-) \end{pmatrix}.$$

The matrix $\mathcal{B}$ was defined in (2.3); the sizes of the matrices $T_{(+)}$, $A_{(+)}$, $R_{(+)}$ are respectively $(\kappa - 1) \times (\kappa - 1)$, $1 \times (\kappa - 1)$, $(N - \kappa) \times (\kappa - 1)$, while the matrices $R_{(-)}$, $A_{(-)}$, $T_{(-)}$ have sizes $(\kappa - 1) \times (N - \kappa)$, $1 \times$ $\times (N - \kappa)$, $(N - \kappa) \times (N - \kappa)$. No confusion is possible between the row vectors $A_{(\pm)}(p, u^-)$ and the matrix $A(u)$.

LEMMA 2.3 (Weak-strong interaction). *There exist neighborhoods* $\omega_2^\pm \subset \omega_1^\pm$ *of* 0, $I_2 \subset I_1$ *of* 0 *and* $\Omega_3^- \subset \Omega_2^-$ *of* $\underline{u}^-$ *such that the following developments hold.*

((+) CASE) *Let us take strengths and parameters* $(\delta^-_{\rm nc}, p, \gamma^+_{\rm nc}) \in \in \omega^-_{2,\rm nc} \times I_2 \times \omega^+_{2,\rm nc}$, $\varepsilon^+_{\rm c} \in \omega^+_{2,\rm c}$, $\varepsilon^+_{\kappa} \in \omega^+_{2,\kappa}$, $(\Gamma^-_{\rm nc}, P, \Gamma^+_{\rm nc}) \in \omega^-_{1,\rm nc} \times I_1 \times \times \omega^+_{1,\rm nc}$, *and suppose that they are linked by the relation*

$$(2.4)\qquad \Phi^+_{\rm nc}(\Gamma^+_{\rm nc}, H(P, \Phi^-_{\rm nc}(\Gamma^-_{\rm nc}, u^-))) = \\ = \Phi^+_{\kappa}\left(\varepsilon^+_{\kappa}, \Phi^+_{\rm c}\left(\varepsilon^+_{\rm c}, \Phi^+_{\rm nc}(\gamma^+_{\rm nc}, H(p, \Phi^-_{\rm nc}(\delta^-_{\rm nc}, u^-)))\right)\right).$$

Then in $\omega^-_{2,\rm nc} \times \omega^+_{2,\rm nc} \times \omega^+_{2,\rm c} \times \omega^+_{2,\kappa}$ we have

$$(2.5)\qquad \begin{pmatrix} \Gamma^-_{\rm nc} \\ P \\ \Gamma^+_{\rm nc}\end{pmatrix} = \begin{pmatrix} \delta^-_{\rm nc} \\ p + \varepsilon^+_{\kappa} \\ \gamma^+_{\rm nc}\end{pmatrix} + \begin{pmatrix} T_{(+)}(p, u^-)\varepsilon^+_{\rm c} \\ A_{(+)}(p, u^-)\varepsilon^+_{\rm c} \\ R_{(+)}(p, u^-)\varepsilon^+_{\rm c}\end{pmatrix} + \\ + O(1)(|\varepsilon^+_{\rm c}|(|\delta^-_{\rm nc}| + |\gamma^+_{\rm nc}| + |\varepsilon^+_{\rm c}|) + |\varepsilon^+_{\kappa}|(|\gamma^+_{\rm nc}| + |\varepsilon^+_{\rm c}|)).$$

((−) CASE) *Let* $\delta^-_{\kappa} \in \omega^-_{2,\kappa}$, $\delta^-_{\rm c} \in \omega^-_{2,\rm c}$, $(\gamma^-_{\rm nc}, p, \varepsilon^+_{\rm nc}) \in \omega^-_{2,\rm nc} \times I_2 \times \times \omega^+_{2,\rm nc}$, $(\Gamma^-_{\rm nc}, P, \Gamma^+_{\rm nc}) \in \omega^-_{1,\rm nc} \times I_1 \times \omega^+_{1,\rm nc}$ *be another set of strengths and parameters such that*

$$\Phi^+_{\rm nc}\Big(\Gamma^+_{\rm nc}, H\big(P, \Phi^-_{\rm nc}(\Gamma^-_{\rm nc}, u^-)\big)\Big) = \\ = \Phi^+_{\rm nc}\Big(\varepsilon^+_{\rm nc}, H\Big(p, \Phi^-_{\rm nc}\big(\gamma^-_{\rm nc}, \Phi^-_{\rm c}(\delta^-_{\rm c}, \Phi^-_{\kappa}(\delta^-_{\kappa}, u^-))\big)\Big)\Big).$$

Then in $\omega^-_{2,\kappa} \times \omega^-_{2,\rm c} \times \omega^-_{2,\rm nc} \times \omega^+_{2,\rm nc}$ *we have*

$$(2.6)\qquad \begin{pmatrix} \Gamma^-_{\rm nc} \\ P \\ \Gamma^+_{\rm nc}\end{pmatrix} = \begin{pmatrix} \gamma^-_{\rm nc} \\ p + \delta^-_{\kappa} \\ \varepsilon^+_{\rm nc}\end{pmatrix} + \begin{pmatrix} R_{(-)}(p, u^-)\delta^-_{\rm c} \\ A_{(-)}(p, u^-)\delta^-_{\rm c} \\ T_{(-)}(p, u^-)\delta^-_{\rm c}\end{pmatrix} + \\ + O(1)(|\delta^-_{\rm c}|(|\delta^-_{\rm c}| + |\gamma^-_{\rm nc}| + |\varepsilon^+_{\rm nc}|) + |\delta^-_{\kappa}|(|\gamma^-_{\rm nc}| + |\delta^-_{\rm c}|)).$$

The term $O(1)$ *is uniform in* (p, u^-) *for* $(p, u^-) \in I_2 \times \Omega^-_3$, *in both cases.*

PROOF ((+) *case*). We introduce the notations $W = (\Gamma^-_{\rm nc}, P, \Gamma^+_{\rm nc})$, $w = (\delta^-_{\rm nc}, \gamma^+_{\rm nc}, \varepsilon^+_{\rm c}, \varepsilon^+_{\kappa})$, and for $W \in \omega^-_{1,\rm nc} \times I_1 \times \omega^+_{1,\rm nc}$, $w \in \omega^-_{1,\rm nc} \times \times \omega^+_{1,\rm nc} \times \omega^+_{1,\rm c} \times \omega^+_{1,\kappa}$, $p \in I_1$, $u^- \in \Omega^-_2$, we define the function

$$F(W; w; p, u^-) = \Phi^+_{\rm nc}(\Gamma^+_{\rm nc}, H(P, \Phi^-_{\rm nc}(\Gamma^-_{\rm nc}, u^-))) - \\ - \Phi^+_{\kappa}\left(\varepsilon^+_{\kappa}, \Phi^+_{\rm c}\left(\varepsilon^+_{\rm c}, \Phi^+_{\rm nc}(\gamma^+_{\rm nc}, H(p, \Phi^-_{\rm nc}(\delta^-_{\rm nc}, u^-)))\right)\right).$$

Since $F(0;0;0,\underline{u}^-)=0$ and $D_W F(0;0;0,\underline{u}^-)=\mathcal{B}(\underline{u}^-,0,\underline{u}^+)$, then the implicit function theorem gives some neighborhoods

$$\omega_{2,\,\mathrm{nc}}^- \times \omega_{2,\,\mathrm{nc}}^+ \times \omega_{2,\,\mathrm{c}}^+ \times \omega_{2,\,\kappa}^+ \subset \omega_{1,\,\mathrm{nc}}^- \times \omega_{1,\,\mathrm{nc}}^+ \times \omega_{1,\,\mathrm{c}}^+ \times \omega_{1,\,\kappa}^+$$

of $w=0$, $I_2\subset I_1$ of 0, $\Omega_3^-\subset\Omega_2^-$ of $\underline{u}^-$, and a function $W=W(w,p,u^-)$, such that (2.4) holds if and only if $W=W(w,p,u^-)$ therein.

Remark now that $F(\delta_{\mathrm{nc}}^-,p,\gamma_{\mathrm{nc}}^+;\delta_{\mathrm{nc}}^-,\gamma_{\mathrm{nc}}^+,0,0;p,u^-)=0$ and also, as a consequence of (2.1), $F(\delta_{\mathrm{nc}}^-,p+\varepsilon_\kappa^+,0;\delta_{\mathrm{nc}}^-,0,0,\varepsilon_\kappa^+;p,u^-)=0$; therefore by uniqueness

$$\text{(2.7)}\qquad \begin{pmatrix}\Gamma_{\mathrm{nc}}^-\\ P\\ \Gamma_{\mathrm{nc}}^+\end{pmatrix}\Bigg|_{\varepsilon_c^+=0,\,\varepsilon_\kappa^+=0}=\begin{pmatrix}\delta_{\mathrm{nc}}^-\\ p\\ \gamma_{\mathrm{nc}}^+\end{pmatrix},\quad \begin{pmatrix}\Gamma_{\mathrm{nc}}^-\\ P\\ \Gamma_{\mathrm{nc}}^+\end{pmatrix}\Bigg|_{\varepsilon_c^+=0,\,\gamma_{\mathrm{nc}}^+=0}=\begin{pmatrix}\delta_{\mathrm{nc}}^-\\ p+\varepsilon_\kappa^+\\ 0\end{pmatrix}$$

for every $(\delta_{\mathrm{nc}}^-,\gamma_{\mathrm{nc}}^+;p,u^-)$ and $(\delta_{\mathrm{nc}}^-,\varepsilon_\kappa^+;p,u^-)$, respectively. For $u^+=H(p,u^-)$ we have $D_{\varepsilon_c^+}F(W(0,p,u^-);0;p,u^-)=-r_c^+(u^+)$, and then the coefficient of ε_c^+ in the Taylor development of $(\Gamma_{\mathrm{nc}}^-,P,\Gamma_{\mathrm{nc}}^+)$ at the point $(0,p,u^-)$ is

$$\text{(2.8)}\qquad -((D_W F)^{-1}D_{\varepsilon_c^+}F)(W(0,p,u^-);0;p,u^-)=$$

$$=\mathcal{B}(u^-,p,u^+)^{-1}r_c^+(u^+).$$

Formulas (2.7), (2.8) imply (2.5).

(($-$) *case*). In order to apply again the implicit function theorem we define

$$F(\Gamma_{\mathrm{nc}}^-,P,\Gamma_{\mathrm{nc}}^+;\ \delta_\kappa^-,\delta_c^-,\gamma_{\mathrm{nc}}^-,\varepsilon_{\mathrm{nc}}^+;p,u^-)=F(W;w;p,u^-)=$$

$$=\Phi_{\mathrm{nc}}^+(\Gamma_{\mathrm{nc}}^+,H(P,\Phi_{\mathrm{nc}}^-(\Gamma_{\mathrm{nc}}^-,u)))-$$

$$-\Phi_{\mathrm{nc}}^+\big(\varepsilon_{\mathrm{nc}}^+,H\big(p,\Phi_{\mathrm{nc}}^-\big(\gamma_{\mathrm{nc}}^-,\Phi_c^-(\delta_c^-,\Phi_\kappa^-(\delta_\kappa^-,u^-))\big)\big)\big).$$

Locally we obtain that $F=0$ is equivalent to $W=W(w,p,u^-)$ for some regular function W which satisfies

$$W((0,0,\gamma_{\mathrm{nc}}^-,\varepsilon_{\mathrm{nc}}^+),p,u^-)=(\gamma_{\mathrm{nc}}^-,p,\varepsilon_{\mathrm{nc}}^+),$$

$$W((\delta_\kappa^-,0,0,\varepsilon_{\mathrm{nc}}^+),p,u^-)=(0,p+\delta_\kappa^-,\varepsilon_{\mathrm{nc}}^+),$$

according to (2.1). Then

$$D_{\delta_c^-} F(W(0, p, u^-); 0; p, u^-) = -D_{u^-} H(p, u^-) r_c^-(u^-)$$

for $u^+ = H(p, u^-)$ implies (2.6). The lemma is therefore proved.

Away from the strong κ-contact discontinuity the interactions among weak waves are described by the following well known Glimm's lemma. For strengths δ, ε we denote $\Delta(\delta, \varepsilon) = \sum_{i>j} |\delta_i \varepsilon_j| + \sum_i' |\delta_i \varepsilon_i|$; the summation in the second sum is made for the indices i such that the eigenvalue λ_i is genuinely nonlinear and either $\delta_i < 0$ or $\varepsilon_i < 0$. Let $\Omega_3^+ = \Omega_2^+$ so that notations are symmetric with respect to the contact curve.

LEMMA 2.4 (Weak-weak interaction). *If γ satisfies $\Phi(\varepsilon, \Phi(\delta, u)) = \Phi(\gamma, u)$ for $u \in \Omega_3^- \cup \Omega_3^+$ then, unless of shrinking ω_2 and $\Omega_3^\pm$,*

$$\gamma = \varepsilon + \delta + O(1)\Delta(\delta, \varepsilon).$$

The term $O(1)$ is uniform in u for $u \in \Omega_3^- \cup \Omega_3^+$.

We denote by c_0 an upper bound of all $O(1)$ terms in Lemmas 2.1, 2.4, and by c_1 an upper bound for the norms of the matrices $T_{(\pm)}$, $A_{(\pm)}$, $R_{(\pm)}$, in $I_2 \times \Omega_3^-$.

3. – Glimm's scheme for a strong contact discontinuity.

This section concerns the proof of Theorem 1.1. At first we introduce the version of the Glimm's scheme that fits here: though it is similar to that already exploited in [CST1] for the shock case, nevertheless it is resumed here for the reader's convenience. Then we define the linear and quadratic functionals and the related estimates: no weight is assigned to characteristic modes, since at first order they do not participate neither in transmission nor in reflection. The key point in the L^∞ estimates is Lemma 3.4 which translates the jump condition for a suitable set of dependent variables; clearly also in the new variables the characteristic components do not enter at first order in the determination of the noncausal components. The section ends with the convergence of the scheme.

3.1. *The scheme.*

Let us start by fixing the time and space mesh sizes, Δt, Δx, satisfying the Courant-Friedrichs-Lewy condition $\Delta x / \Delta t > 2 \sup \{ |\lambda_j(u)|;$

$u \in \Omega$, $j = 1, \ldots, N\}$, and a sequence $\{\theta_k\}$ in $]-1, 1[$. From an initial data $h^{\pm} \in BV(\boldsymbol{R}^{\pm}, \Omega_2^{\pm})$ we construct the step function $\overline{u}_0$ for $x \in](2n - 2)\Delta x, 2n\Delta x]$ as $\overline{u}_0(x) = U_{0,n} = h\big((2n-1)\Delta x - 0\big)$ and then let n run in $\boldsymbol{Z}$. At the jump points $2n\Delta x$ we solve the Riemann problems of Cauchy data $U_{0,n}$, $U_{0,n+1}$, within the class of admissible simple waves; for $n = 0$ we use Proposition 2.2 which gives

$$U_{0,1} = \Phi^+_{\mathrm{nc}}\big(\gamma^+_{\mathrm{nc}}(0), H(p_0, \Phi^-_{\mathrm{nc}}(\gamma^-_{\mathrm{nc}}(0), U_{0,0}))\big) \equiv \Phi^+_{\mathrm{nc}}(\gamma^+_{\mathrm{nc}}(0), H(p_0, t_0^-))$$

and we denote $q_0 = \lambda_\kappa(t_0^-)$ the speed of the strong κ-contact discontinuity. We paste then together the solutions of all such problems and obtain in the strip $\boldsymbol{R} \times [0, \Delta t]$ a function u_0; the contact curve is $x = \chi_0(t) = q_0 t$ and t_0^- is the trace of u_0 at the left of this curve. By taking traces at $t = \Delta t$ of u_0 we define another step function $\overline{u}_1$ as $\overline{u}_1(x) = U_{1,n} = u_0(\Delta t, (2n - 1 + \theta_1)\Delta x + q_0 \Delta t - 0)$ for $x \in q_0 \Delta t +](2n-2)\Delta x, 2n\Delta x]$ and $n \in \boldsymbol{Z}$, and so on. In general the step function $\overline{u}_k$ is defined for $x \in \pi_{k-1}\Delta t +](2n-2)\Delta x, 2n\Delta x]$ by

$$\overline{u}_k(x) = U_{k,n} = u_{k-1}(k\Delta t, (2n - 1 + \theta_k)\Delta x + \pi_{k-1}\Delta t - 0) \tag{3.1}$$

where $\pi_{k-1} = \sum_{j=0}^{k-1} q_j$; here $k = 1, 2, \ldots$, and $n \in \boldsymbol{Z}$. From $\overline{u}_k$ we define then u_k; the contact curve in the $(k+1)$-th strip $[k\Delta t, (k+1)\Delta t]$ is therefore $x = \chi_k(t) = q_k \cdot (t - k\Delta t) + \pi_{k-1}\Delta t$. At last we paste together the functions u_k, χ_k thus obtained in the intervals $[k\Delta t, (k+1)\Delta t[$ and denote the outcome by $u_{\Delta t, \Delta x}$, $\chi_{\Delta t, \Delta x}$, respectively.

For the iterative scheme we introduce now the following notations; for the k-th row:

$$U_{k,n+1} = \Phi(\gamma(n), U_{k,n}), \qquad n \in \boldsymbol{Z} \setminus \{0\},$$

$$U_{k,1} = \Phi^+_{\mathrm{nc}}\big(\gamma^+_{\mathrm{nc}}(0), H(p, \Phi^-_{\mathrm{nc}}(\gamma^-_{\mathrm{nc}}(0), U_{k,0}))\big),$$

$$t_k^- = \Phi^-_{\mathrm{nc}}(\gamma^-_{\mathrm{nc}}(0), U_{k,0}), \qquad t_k^+ = H(p, t_k^-),$$

and for the $(k+1)$-th row:

$$U_{k+1,n+1} = \Phi(\Gamma(n), U_{k,n}), \qquad n \in \boldsymbol{Z} \setminus \{0\},$$

$$U_{k+1,1} = \Phi^+_{\mathrm{nc}}\big(\Gamma^+_{\mathrm{nc}}(0), H(P, \Phi^-_{\mathrm{nc}}(\Gamma^-_{\mathrm{nc}}(0), U_{k+1,0}))\big),$$

$$t_{k+1}^- = \Phi^-_{\mathrm{nc}}(\Gamma^-_{\mathrm{nc}}(0), U_{k+1,0}), \qquad t_{k+1}^+ = H(P, t_{k+1}^-).$$

Sometimes we shall use the notation $\gamma(k, n)$ in order to stress dependence on the row.

According to the slope of the line joining the points $(k\Delta t, 2n\Delta x +$

$+ \pi_k \Delta t)$ and $((k+1)\Delta t, (2n \pm 1 + \theta_{k+1})\Delta x + \pi_k \Delta t)$, where $\mp \theta_{k+1} > 0$, for $n \in \boldsymbol{Z}\setminus\{0\}$ we split $\gamma(n) = \varepsilon(n) + \delta(n)$. If $n = 0$ we write

$$\gamma_{\mathrm{nc}}^{-}(0) = (\gamma_1(0), \ldots, \gamma_{\kappa-1}(0)), \qquad \gamma_{\mathrm{nc}}^{+}(0) = (\gamma_{\kappa+1}(0), \ldots, \gamma_N(0))$$

and decompose consequently $\gamma_{\mathrm{nc}}^{\pm}(0) = \varepsilon_{\mathrm{nc}}^{\pm}(0) + \delta_{\mathrm{nc}}^{\pm}(0)$.

The variation of the traces at the left and right-hand side of the piecewise linear contact curve from the k-th strip to the $(k+1)$-th is

$$\dot{t}_{k+1}^{\pm} = |t_{k+1}^{\pm} - t_k^{\pm}|;$$

the variation of the κ-contact parameter is

$$\dot{p}_{k+1} = |P - p|.$$

From Lax's theory and the results of Section 2 we can choose neighborhoods $\Omega_4^{\pm} \subset \Omega_3^{\pm}$ of $\underline{u}^{\pm}$, $I_3 \subset I_2$ of $0 \in \boldsymbol{R}$ and $\omega_3 \subset \omega_2$ of $0 \in \boldsymbol{R}^N$ such that

(i) if $u^- \in \Omega_4^-$, $\Phi(\gamma, u^-) \in \Omega_4^-$, then $\gamma \in \omega_3$; conversely, if $u^- \in \Omega_4^-$, $\gamma \in \omega_3$, then $\Phi(\gamma, u^-) \in \Omega_3^-$;

(ii) if $u^+ \in \Omega_4^+$, $\Phi(\gamma, u^+) \in \Omega_4^+$, then $\gamma \in \omega_3$; conversely, if $u^+ \in \Omega_4^+$, $\gamma \in \omega_3$, then $\Phi(\gamma, u^+) \in \Omega_3^+$;

(iii) if $u^- \in \Omega_4^-$ and $\Phi_{\mathrm{nc}}^{+}(\gamma_{\mathrm{nc}}^{+}, H(p, \Phi_{\mathrm{nc}}^{-}(\gamma_{\mathrm{nc}}^{-}, u^-))) \in \Omega_4^+$, then $(\gamma_{\mathrm{nc}}^{-}, p, \gamma_{\mathrm{nc}}^{+}) \in \omega_{3,\mathrm{nc}}^{-} \times I_3 \times \omega_{3,\mathrm{nc}}^{+}$; and conversely, again, if $u^- \in \Omega_4^-$ and $(\gamma_{\mathrm{nc}}^{-}, p, \gamma_{\mathrm{nc}}^{+}) \in \omega_{3,\mathrm{nc}}^{-} \times I_3 \times \omega_{3,\mathrm{nc}}^{+}$, then $\Phi_{\mathrm{nc}}^{+}(\gamma_{\mathrm{nc}}^{+}, H(p, \Phi_{\mathrm{nc}}^{-}(\gamma_{\mathrm{nc}}^{-}, u^-))) \in \Omega_3^+$.

3.2. *BV estimates.*

Let $h^{\pm} \in BV(\boldsymbol{R}^{\pm}, \Omega^{\pm})$ and assume temporarily that h is constant for $|x|$ large in order to relieve us from problems of convergence of series. Strengths of weak and strong waves will be controlled by the linear functionals

$$\mathcal{L}(k) = \mathcal{L}_{\mathrm{nc}}(k) + |\gamma_{\mathrm{nc}}(0)| + \mathcal{L}_{\kappa}(k) + K\mathcal{L}_{\mathrm{c}}(k) + \mathcal{C}^{-}(k) + \mathcal{P}(k), \qquad k \geq 0,$$

where $\mathcal{L}_{\mathrm{nc}}(k) = \mathcal{L}_{\mathrm{nc}}^{-}(k) + \mathcal{L}_{\mathrm{nc}}^{+}(k)$, $\mathcal{L}_{\kappa}(k) = \mathcal{L}_{\kappa}^{-}(k) + \mathcal{L}_{\kappa}^{+}(k)$, $\mathcal{L}_{\mathrm{c}}(k) = \mathcal{L}_{\mathrm{c}}^{-}(k) + \mathcal{L}_{\mathrm{c}}^{+}(k)$ with

$$\mathcal{L}_{\mathrm{nc}}^{\pm}(k) = \sum_{\pm n \geq 1} |\gamma_{\mathrm{nc}}^{\pm}(n)|, \quad \mathcal{L}_{\kappa}^{\pm}(k) = \sum_{\pm n \geq 1} |\gamma_{\kappa}^{\pm}(n)|, \quad \mathcal{L}_{\mathrm{c}}^{\pm}(k) = \sum_{\pm n \geq 1} |\gamma_{\mathrm{c}}^{\pm}(n)|.$$

Moreover $|\gamma_{\mathrm{nc}}(0)| = |\gamma_{\mathrm{nc}}^{-}(0)| + |\gamma_{\mathrm{nc}}^{+}(0)|$; the weight K is a positive constant to be chosen later on and the functionals

$$\mathcal{T}^-(k) = \sum_{h=0}^{k} \dot{t}_h^-, \qquad \dot{t}_0^- = |t_0^- - \underline{u}^-|,$$

$$\mathcal{P}(k) = \sum_{h=0}^{k} \dot{p}_h, \qquad \dot{p}_0 = |p_0|,$$

bound the variation of the left traces and the parameters. These functionals control also the related functionals

$$\mathcal{S}(k) = \sum_{h=0}^{k} |\lambda_\kappa(t_h^\pm) - \lambda_\kappa(t_{h-1}^\pm)|,$$

$$\mathcal{T}^+(k) = \sum_{h=0}^{k} |t_h^+ - t_{h-1}^+|, \qquad t_{-1}^+ = \underline{u}^+,$$

respectively variation of the speed of the strong κ-contact curve and variation of the traces at the right side of this curve.

We shall need also the «inner» functional

$$\mathcal{L}^\circ(k) = \mathcal{L}_{\mathrm{nc}}(k) + |\gamma_{\mathrm{nc}}(0)| + \mathcal{L}_\kappa(k) + K\mathcal{L}_{\mathrm{c}}(k).$$

We shall prove that if K sufficiently large and δ sufficiently small then for any $k \geqslant 0$ the following statements hold:

($\mathfrak{I}_k$) *for* $h = 1, \ldots, k$:
the states $U_{h,n}$ *belong to* Ω_4^- *(respectively,* Ω_4^+ *) for* $n \leqslant 0$ $(n > 0)$,
the strengths $\gamma(h, n)$ *are in* ω_3 *for* $n \neq 0$ *and* $\gamma_{\mathrm{nc}}^\pm(h, 0) \in \omega_{3,\mathrm{nc}}^\pm$,
the parameter p_h *is in* I_2;
$\mathcal{L}(k) \leqslant 2\mathcal{L}(0)$.

The proof is by induction and will take up the whole section; the final results are collected in Proposition 3.4.

The interaction potential is

$$\Lambda_{(\pm)}(k) = \Lambda_{(\pm)}^-(k) + \lambda_{(\pm)}(k) + \Lambda_{(\pm)}^+(k)$$

where

$$\Lambda_{(+)}^\pm(k) = \sum_{\pm n \geqslant 1} \Delta(\delta(n), \varepsilon(n+1)), \qquad \Lambda_{(-)}^\pm(k) = \sum_{\pm n \geqslant 1} \Delta(\delta(n-1), \varepsilon(n)),$$

take into account interactions among weak waves and

$$\lambda_{(+)}(k) = |\varepsilon_{\mathrm{c}}^+(1)|(|\delta_{\mathrm{nc}}^-(0)| + |\gamma_{\mathrm{nc}}^+(0)| + |\varepsilon_{\mathrm{c}}^+(1)|) +$$
$$+ |\varepsilon_\kappa^+(1)|(|\gamma_{\mathrm{nc}}^+(0)| + |\varepsilon_{\mathrm{c}}^+(1)|)$$

$$\lambda_{(-)}(k) = |\delta_c^-(-1)|(|\delta_c^-(-1)| + |\gamma_{nc}^-(0)| + |\varepsilon_{nc}^+(0)|) +$$
$$+ |\delta_\kappa^-(-1)|(|\delta_c^-(-1)| + |\gamma_{nc}^-(0)|),$$

the interactions between the weak waves and the strong contact discontinuity.

PROPOSITION 3.1 (Linear estimates). *If assumption* $(\mathfrak{J}_k)$ *holds and the weight* K *is sufficiently large then we have the estimates (according to* $\pm\theta_{k+1} > 0$)

$$\mathcal{L}(k+1) \leqslant \mathcal{L}(k) + 2c_0 K \Lambda_{(\pm)}(k).$$

PROOF. We consider only the case $\theta_{k+1} \geqslant 0$ since the case $\theta_{k+1} < 0$ is analogous. In view of Lemma 2.4 and hypothesis $(\mathfrak{J}_k)$ we have the estimates:

$$(3.2) \qquad \mathcal{L}_{nc}^-(k+1) \leqslant \mathcal{L}_{nc}^-(k) + |\varepsilon_{nc}^-(0)| + c_0 \Lambda_{(+)}^-(k),$$

$$\mathcal{L}_\kappa^-(k+1) \leqslant \mathcal{L}_\kappa^-(k) + c_0 \Lambda_{(+)}^-(k), \qquad \mathcal{L}_c^-(k+1) \leqslant \mathcal{L}_c^-(k) + c_0 \Lambda_{(+)}^-(k),$$

$$(3.3) \qquad \mathcal{L}_c^+(k+1) \leqslant \mathcal{L}_c^+(k) - |\varepsilon_c^+(1)| + c_0 \Lambda_{(+)}^+(k),$$

$$\mathcal{L}_\kappa^+(k+1) \leqslant \mathcal{L}_\kappa^+(k) - |\varepsilon_\kappa^+(1)| + c_0 \Lambda_{(+)}^+(k), \qquad \mathcal{L}_{nc}^+(k+1) \leqslant \mathcal{L}_{nc}^+(k) + c_0 \Lambda_{(+)}^+(k).$$

On the other hand Lemma 2.3 (together with hypothesis $(\mathfrak{J}_k)$) yields

$$|\Gamma_{nc}^-(0)| \leqslant |\delta_{nc}^-(0)| + c_1 |\varepsilon_c^+(1)| + c_0 \lambda_{(+)}(k),$$

$$|\Gamma_{nc}^+(0)| \leqslant |\gamma_{nc}^+(0)| + c_1 |\varepsilon_c^+(1)| + c_0 \lambda_{(+)}(k)$$

and also

$$\dot{t}_{k+1}^- \leqslant c_2 (c_1 |\varepsilon_c^+(1)| + c_0 \lambda_{(+)}(k)),$$

$$\dot{p}_{k+1} \leqslant |\varepsilon_\kappa^+(1)| + c_1 |\varepsilon_c^+(1)| + c_0 \lambda_{(+)}(k),$$

for a positive constant c_2 depending only on Φ and H; then

$$\mathcal{T}^-(k+1) \leqslant \mathcal{T}^-(k) + c_2 (c_1 |\varepsilon_c^+(1)| + c_0 \lambda_{(+)}(k)),$$

$$\mathcal{P}(k+1) \leqslant \mathcal{P}(k) + |\varepsilon_\kappa^+(1)| + c_1 |\varepsilon_c^+(1)| + c_0 \lambda_{(+)}(k).$$

Therefore

$$\mathcal{L}(k+1) \leqslant \mathcal{L}(k) + 2c_0 K \Lambda_{(+)}(k)$$

if K is sufficiently large. The proposition is proved.

The quadratic functionals which match well with the interactions potentials defined above are

$$Q^-(k) = \sum_{n<m\leqslant -1} \Delta(\gamma(n), \gamma(m)), \qquad Q^+(k) = \sum_{1\leqslant m<n} \Delta(\gamma(m), \gamma(n)),$$

$$q^-(k) = \sum_{n\leqslant -1} \Delta(\gamma(n), \gamma_{\mathrm{nc}}^-(0)), \qquad q^+(k) = \sum_{n\geqslant 1} \Delta(\gamma_{\mathrm{nc}}^+(0), \gamma(n)),$$

$$Q(k) = Q^-(k) + Q^+(k), \qquad q(k) = q^-(k) + q^+(k).$$

LEMMA 3.2. *Under assumption* $(\mathfrak{H}_k)$ *we have*

$$\begin{aligned}(3.4)\qquad & Q(k+1) + q(k+1) \leqslant Q(k) + q(k) - \\ & -(\Lambda_{(+)}^-(k) + |\gamma_{\mathrm{nc}}^+(0)|(|\varepsilon_{\mathrm{c}}^+(1)| + |\varepsilon_{\kappa}^+(1)|) + \Lambda_{(+)}^+(k)) + \\ & +3c_0\Lambda_{(+)}(k)(\mathcal{L}^\circ(k) + \mathcal{L}^\circ(k+1)) + \\ & +c_1|\varepsilon_{\mathrm{c}}^+(1)|\{\mathcal{L}^-(k) + \mathcal{L}^+(k) + |\varepsilon_{\mathrm{nc}}^-(0)| - |\varepsilon_{\mathrm{c}}^+(1)| - |\varepsilon_{\kappa}^+(1)|\},\end{aligned}$$

$$\begin{aligned} & Q(k+1) + q(k+1) \leqslant Q(k) + q(k) - \\ & -(\Lambda_{(-)}^-(k) + |\gamma_{\mathrm{nc}}^-(0)|(|\delta_{\mathrm{c}}^-(-1)| + |\delta_{\kappa}^-(-1)|) + \Lambda_{(-)}^+(k)) + \\ & +3c_0\Lambda_{(-)}(k)(\mathcal{L}^\circ(k) + \mathcal{L}^\circ(k+1)) + \\ & +c_1|\delta_{\mathrm{c}}^-(-1)|\{\mathcal{L}^-(k) + \mathcal{L}^+(k) + |\delta_{\mathrm{nc}}^+(0)| - |\delta_{\mathrm{c}}^-(-1)| - |\delta_{\kappa}^-(-1)|\},\end{aligned}$$

according to $\theta_{k+1} \geqslant 0$ *or* $\theta_{k+1} < 0$, *respectively.*

PROOF. We consider only the case $\theta_{k+1} \geqslant 0$. Proceeding as in [CST1] one finds that

$$\begin{aligned} Q^-(k+1) \leqslant Q^-(k) - \Lambda_{(+)}^-(k) + c_0\Lambda_{(+)}^-(k)(\mathcal{L}^-(k) + \mathcal{L}^-(k+1)) + \\ + \sum_{n\leqslant -1} \Delta(\gamma(n), \varepsilon_{\mathrm{nc}}^-(0)),\end{aligned}$$

$$\begin{aligned} q^-(k+1) \leqslant q^-(k) - \sum_{n\leqslant -1} \Delta(\gamma(n), \varepsilon_{\mathrm{nc}}^-(0)) + \\ +c_1|\varepsilon_{\mathrm{c}}^+(1)|(\mathcal{L}^-(k) + |\varepsilon_{\mathrm{nc}}^-(0)|) + c_0\lambda_{(+)}(k)\big(\mathcal{L}^-(k) + |\gamma_{\mathrm{nc}}^-(0)|\big),\end{aligned}$$

$$Q^+(k+1) \leqslant Q^+(k) - \varLambda^+_{(+)}(k) + c_0\varLambda^+_{(+)}(\mathcal{L}^+(k) + \mathcal{L}^+(k+1)) - \\ - \sum_{n\geqslant 2} \varDelta(\varepsilon^+(1), \gamma(n)),$$

$$q^+(k+1) \leqslant q^+(k) - |\gamma^+_{\mathrm{nc}}(0)|(|\varepsilon^+_{\mathrm{c}}(1)| + |\varepsilon^+_{\kappa}(1)|) + \\ + c_1|\varepsilon^+_{\mathrm{c}}(1)|(\mathcal{L}^+(k) - |\varepsilon^+_{\mathrm{c}}(1)| - |\varepsilon^+_{\kappa}(1)|) + c_0\lambda_{(+)}(k)\mathcal{L}^-(k),$$

and then it suffices to add these estimates to obtain (3.4).

The definitive quadratic functional needed to balance the possible increase of the linear functional $\mathcal{L}(k)$ is

$$\mathcal{Q}(k) = Q(k) + q(k) + q^\circ(k)$$

where $q^\circ(k) = \mathcal{L}^\circ(k)K\mathcal{L}_{\mathrm{c}}(k)$.

PROPOSITION 3.3 (Quadratic estimates). *Under hypothesis* ($\mathfrak{J}_k$) *the estimates*

$$\mathcal{Q}(k+1) \leqslant \mathcal{Q}(k) - \varLambda_{(\pm)}(k) + CK\varLambda_{(\pm)}(k)\Big\{\mathcal{L}^\circ(k) + CK\varLambda_{(\pm)}(k)\Big\} \tag{3.5}$$

hold according to $\pm\theta_{k+1} \geqslant 0$, *where* C *is a positive constant depending only on* c_0.

PROOF. We consider the case $\theta_{k+1} \geqslant 0$. From (3.2)-(3.3) we deduce

$$\mathcal{L}^\circ(k+1) \leqslant \mathcal{L}^\circ(k) - |\varepsilon^+_{\kappa}(1)| + 2c_0K\varLambda_{(+)}(k), \tag{3.6}$$

$$\mathcal{L}_{\mathrm{c}}(k+1) \leqslant \mathcal{L}_{\mathrm{c}}(k) - |\varepsilon^+_{\mathrm{c}}(1)| + c_0\varLambda_{(+)}(k),$$

and then

$$q^\circ(k+1) \leqslant \\ \leqslant q^\circ(k) - K|\varepsilon^+_{\mathrm{c}}(1)|\mathcal{L}^\circ(k) + c_0K\varLambda_{(+)}(k)\{3\mathcal{L}^\circ(k) + 2c_0K\varLambda_{(+)}(k)\}. \tag{3.7}$$

The term $-K|\varepsilon^+_{\mathrm{c}}(1)|\mathcal{L}^\circ(k)$ in (3.7) allows to compensate the term $c_1|\varepsilon^+_{\mathrm{c}}(1)|\{\mathcal{L}^-(k) + \mathcal{L}^+(k) + |\varepsilon^-_{\mathrm{nc}}(0)| - |\varepsilon^+_{\mathrm{c}}(1)| - |\varepsilon^+_{\kappa}(1)|\}$ in the right-hand side of (3.4) and to obtain a part of the missing terms to $\lambda_{(+)}$:

$$c_1|\varepsilon^+_{\mathrm{c}}(1)|\{\mathcal{L}^-(k)+\mathcal{L}^+(k)+|\varepsilon^-_{\mathrm{nc}}(0)|-|\varepsilon^+_{\mathrm{c}}(1)|-|\varepsilon^+_{\kappa}(1)|\} - K|\varepsilon^+_{\mathrm{c}}(1)|\mathcal{L}^\circ(k) \leqslant \\ -K|\varepsilon^+_{\mathrm{c}}(1)|(|\delta^-_{\mathrm{nc}}(0)| + |\varepsilon^+_{\mathrm{c}}(1)| + |\varepsilon^+_{\kappa}(1)|).$$

Then we easily obtain

$$\mathcal{Q}(k+1) \leqslant \mathcal{Q}(k) - \Lambda_{(+)}(k) +$$

$$+3c_0\Lambda_{(+)}(k)\{\mathcal{L}^\circ(k) + \mathcal{L}^\circ(k+1)\} + c_0K\Lambda_{(+)}(k)\{3\mathcal{L}^\circ(k) + 2c_0K\Lambda_{(+)}(k)\}$$

whence (3.5) with $C = 9c_0$. As above the case $\theta_{k+1} < 0$ is treated analogously.

We can then prove the statement $(\mathfrak{J}_{k+1})$ in the same way as in [CST1], for K large enough and δ small enough; by finite propagation speed we get rid of the assumption that h is constant outside a compact set. We resume all what we proved in the following proposition.

PROPOSITION 3.4. *Let $\Omega^\pm$ be sufficiently small neighborhoods of $\underline{u}^\pm$ and assume that* (1.5) *holds with some sufficient small δ. Then the iterative scheme defined by* (3.1) *provides two functions $u_{\Delta t, \Delta x}(t, x)$ and $\chi_{\Delta t, \Delta x}(t)$ for $t \geqslant 0$, such that*

(i) $u_{\Delta t, \Delta x} \in BV_{\mathrm{loc}}$, $u^\pm_{\Delta t, \Delta x}$ *is valued in* $\Omega^\pm$, $u_{\Delta t, \Delta x}(t, \cdot) \in BV(\boldsymbol{R})$;

(ii) $\chi_{\Delta t, \Delta x}(t) \in \mathrm{Lip}(\boldsymbol{R}^+)$, $\chi_{\Delta t, \Delta x}$ *is piecewise linear*, $q_{\Delta t, \Delta x} = \chi'_{\Delta t, \Delta x} \in BV(\boldsymbol{R}^+)$;

(iii) $u_{\Delta t, \Delta x}$ *and* $q_{\Delta t, \Delta x}$ *satisfy the estimates*

$$\|u^\pm_{\Delta t, \Delta x}(t, \cdot) - \underline{u}^\pm\|_{L^\infty} + \|q_{\Delta t, \Delta x} - \underline{q}\|_{L^\infty} \leqslant \tag{3.8}$$

$$\leqslant C\{\|h^- - \underline{u}^-\|_{L^\infty} + \|h^+ - \underline{u}^+\|_{L^\infty} + V(h^-) + V(h^+)\},$$

$$V(u^\pm_{\Delta t, \Delta x}(t, \cdot)) + V(q_{\Delta t, \Delta x}) \leqslant C\{V(h^-) + V(h^+)\},$$

$$V(u^\pm_{\Delta t, \Delta x}(t, \chi_{\Delta t, \Delta x}(t))) \leqslant C\{V(h^-) + V(h^+)\},$$

for some positive constant C; $\pm$ refer to $\pm(x - \chi_{\Delta t, \Delta x}(t)) > 0$.

3.3. *L^∞ estimates.*

In this section we improve estimate (3.8): we shall obtain that deviations in the sup-norm of the functions $u^\pm_{\Delta t, \Delta x}(t, \cdot)$ from the constant states $\underline{u}^\pm$ are bounded by the deviations of the initial data. As in [Gl] we pass to coordinates $w^\pm$ centered at the background state $\underline{u}^\pm$, i.e., $w^\pm(\underline{u}^\pm) = 0$, and such that $D_{u^\pm} w_i^\pm(\underline{u}^\pm) \cdot r_j(\underline{u}^\pm) = \delta_{ij}$ for $i, j = 1, \ldots, N$; we may assume that these new dependent variables $w^\pm$ are defined in the whole $\Omega^\pm$. Under the usual notations $w^\pm_{\mathrm{nc}}$, $w^\pm_\kappa$, $w^\pm_{\mathrm{c}}$, we define

$$|w^-_{\mathrm{nc}}(u^-)| = \max\{|w_j^-(u^-)|;\ j = 1, \ldots, \kappa - 1\}$$

and so on. We define moreover

$$\|u_{\Delta t,\,\Delta x}(t,\,\cdot)\|_{K,\,\infty} = \max \|u^{\pm}_{\Delta t,\,\Delta x}(t,\,\cdot)\|_{K,\,\infty}$$

for

$$\|u^{\pm}_{\Delta t,\,\Delta x}(t,\,\cdot)\|_{K,\,\infty} = \sup\left\{ |u^{\pm}_{\Delta t,\,\Delta x}(t,\,x)|_K;\ \pm(x - \chi_{\Delta t,\,\Delta x}(t)) > 0\right\},$$

$$|u^{\pm}|_K = \max\left\{ |w^{\pm}_{\mathrm{nc}}(u^{\pm})|,\ |w^{\pm}_{\kappa}(u^{\pm})|,\ K|w^{\pm}_{\mathrm{c}}(u^{\pm})|\right\}.$$

As for BV also the proof of the L^{∞} estimates is by induction: our claim is that the estimate

$$(\mathfrak{J}_k) \qquad \|u_k((k+1)\Delta t,\,\cdot)\|_{K,\,\infty} \leqslant 2\|\overline{u}_0(\cdot)\|_{K,\,\infty}$$

holds for every k, unless of increasing K and decreasing δ; this would imply the desidered estimate

$$\|u^{\pm}_{\Delta t,\,\Delta x}(t,\,\cdot) - \underline{u}^{\pm}\|_{L^{\infty}} \leqslant C\{\|h^- - \underline{u}^-\|_{L^{\infty}} + \|h^+ - \underline{u}^+\|_{L^{\infty}}\}.$$

Let $k = 0$. We consider a Riemann problem with initial data u_l, u_r in Ω_2^- and let $v^- = \Phi_i(\delta_i,\, \Phi_{i-1}(\varepsilon_{i-1}\ldots,\, \Phi_1(\varepsilon_1,\, u_l))\ldots)$ be an intermediate state: if λ_i is linearly degenerate then $\delta_i = \varepsilon_i$, otherwise either $\delta_i = \varepsilon_i$ if $\varepsilon_i \leqslant 0$ or $0 \leqslant \delta_i \leqslant \varepsilon_i$ if $\varepsilon_i > 0$. In the same way as in [CST1], for $\delta \leqslant \delta(K)$ small enough we obtain

$$|v^-|_K \leqslant \|\overline{u}_0\|_{K,\,\infty} + c_0 K\|\overline{u}_0\|^2_{1,\,\infty}$$

for some constant c_0, and then $|v^-|_K \leqslant 2\|\overline{u}_0\|_{K,\,\infty}$. Analogous estimates hold for Riemann problems with data in $\Omega_2^{\pm}$.

The case of Riemann problems with data u_l in Ω_2^- and u_r in Ω_2^+ is more interesting. If $v = \Phi_i(\varepsilon_i,\, \Phi_{i-1}(\varepsilon_{i-1},\, \ldots,\, \Phi_1(\varepsilon_1,\, u_l))\ldots) = v^-$, for some $i \leqslant \kappa - 1$, then

$$(3.9) \qquad |w_j^-(v^-) - w_j^-(u^-)| \leqslant$$
$$\leqslant O(1)(|\varepsilon_{i+1}| + \ldots + |\varepsilon_{\kappa-1}|)\max\{|u_l - \underline{u}^-|,\ |u_r - \underline{u}^+|\}, \qquad j \leqslant i,$$

$$(3.10) \qquad |w_j^-(v^-) - w_j^-(u_l)| \leqslant$$
$$\leqslant O(1)(|\varepsilon_1| + \ldots + |\varepsilon_i|)\max\{|u_l - \underline{u}^-|,\ |u_r - \underline{u}^+|\}, \qquad j > i,$$

where $u^{\pm}$ denotes the traces of the solution at the sides of $x = q_0 t$.

LEMMA 3.5. *If the neighborhoods $\Omega^{\pm}$ and I of Lemma 2.1 are sufficiently small, then the Rankine-Hugoniot condition $u^+ = H(p,\, u^-)$*

holds in $\Omega^- \times I \times \Omega^+$ *if and only if*

$$w_{\mathrm{nc}}^{\pm}(u^{\pm}) = W_{\mathrm{nc}}^{\pm}(w_{\kappa}^{-}(u^{-}), w_{\mathrm{c}}^{-}(u^{-}), w_{\mathrm{c}}^{+}(u^{+}), w_{\kappa}^{+}(u^{+})),$$

$$p = P(w_{\kappa}^{-}(u^{-}), w_{\mathrm{c}}^{-}(u^{-}), w_{\mathrm{c}}^{+}(u^{+}), w_{\kappa}^{+}(u^{+})),$$

for some smooth functions $W_{\mathrm{nc}}^{\pm}$ *and* P. *Moreover these functions satisfy*

$$D_{w_{\kappa}^{-}} W_{\mathrm{nc}}^{\pm}(0) = 0, \qquad D_{w_{\kappa}^{+}} W_{\mathrm{nc}}^{\pm}(0) = 0.$$

The proof of this lemma is a bare application of the implicit function theorem, since the stability condition holds. Therefore we have the estimate

$$(3.11) \qquad |w_j^-(u^-)| \leqslant$$
$$\leqslant \alpha_{\kappa}^- |w_{\kappa}^-(u^-)| + \alpha_{\mathrm{c}}^- |w_{\mathrm{c}}^-(u^-)| + \beta_{\mathrm{c}}^- |w_{\mathrm{c}}^+(u^+)| + \beta_{\kappa}^- |w_{\kappa}^+(u^+)|$$

for $j = 1, \ldots, \kappa - 1$, where α_{κ}^-, α_{c}^-, β_{c}^-, β_{κ}^- are positive constants, the first and last one beeing as small as necessary. The same calculations leading to (3.9)-(3.10) give now

$$(3.12) \qquad |w_{\kappa}^-(u^-) - w_{\kappa}^-(u_l)| + |w_{\mathrm{c}}^-(u^-) - w_{\mathrm{c}}^-(u_l)| \leqslant$$
$$\leqslant O(1)(|\varepsilon_1| + \ldots + |\varepsilon_{\kappa-1}|) \max\{|u_l - \underline{u}^-|, |u_r - \underline{u}^+|\},$$

$$(3.13) \qquad |w_{\mathrm{c}}^+(u^+) - w_{\mathrm{c}}^+(u_r)| + |w_{\kappa}^+(u^+) - w_{\kappa}^+(u_r)| \leqslant$$
$$\leqslant O(1)(|\varepsilon_{\kappa+1}| + \ldots + |\varepsilon_N|) \max\{|u_l - \underline{u}^-|, |u_r - \underline{u}^+|\}.$$

Therefore from one hand (3.9) implies

$$K|w_{\mathrm{c}}^-(v^-)| \leqslant K|w_{\mathrm{c}}^-(u_l)| + O(1)K\|\overline{u}_0\|_{1,\infty}^2,$$

$$|w_j^-(v^-)| \leqslant |w_j^-(u_l)| + O(1)\|\overline{u}_0\|_{1,\infty}^2, \qquad i < j \leqslant \kappa$$

while from the other hand (3.12)-(3.13) yield

$$|w_{\mathrm{nc}}^-(u^-)| \leqslant$$
$$\leqslant (K^{-1}(\alpha_{\mathrm{c}}^- + \beta_{\mathrm{c}}^-) + \alpha_{\kappa}^- + \beta_{\kappa}^-) \max\{|u_l|_K, |u_r|_K\} + O(1)\|\overline{u}_0\|_{1,\infty}^2.$$

So unless of shrinking once more $\Omega^{\pm}$, increasing K and decreasing δ, we obtain again $|v^-|_K \leqslant 2\|\overline{u}_0\|_{K,\infty}$ and since the case $v = v^+$ is analogous we have proved $(\mathfrak{J}_0)$.

We sketch now the proof of how $(\mathfrak{J}_k)$ implies $(\mathfrak{J}_{k+1})$. Let $u_{k+1,n}$

denote the restriction to $]\chi_{k+1}((k+2)\Delta t)+(2n-1)\Delta x, \chi_{k+1}((k+2)\Delta t)+(2n+1)\Delta x[$ of the function $u_{k+1}((k+2)\Delta t, \cdot)$, for $n \in \boldsymbol{Z}$.

For causal modes and the κ-th mode the proof of [CST1] works and for $n \leqslant 0$ yields, if for instance $\theta_{k+1} \geqslant 0$,

$$\max\{|w_\kappa^-(u_{k+1,n}(x))|, K|w_c^-(u_{k+1,n}(x))|\} \leqslant 2\|\overline{u}_0\|_{K,\infty},$$

for δ small enough. The same process can be used for noncausal modes if we reach $t=0$ following a Glimm's polygonal curve [Gl], without approaching the large contact discontinuity.

Then let us consider noncausal modes, for instance when $x^- = \chi_{k+1}((k+2)\Delta t)-0$, so $j \leqslant \kappa - 1$, and $\theta_{k+1} > 0$. We use again estimate (3.10) to obtain

$$\begin{aligned} |w_j^-(u_{k+1,0}(x^-))| &\leqslant \alpha_\kappa^- |w_\kappa^-(u_{k+1,0}(x^-))| + \alpha_c^- |w_c^-(u_{k+1,0}(x^-))| + \\ &\quad + \beta_c^- |w_c^+(u_{k+1,0}(x^+))| + \beta_\kappa^- |w_\kappa^+(u_{k+1,0}(x^+))| \leqslant \\ &\leqslant 2(K^{-1}(\alpha_c^- + \beta_c^-) + \alpha_\kappa^- + \beta_\kappa^-)\cdot\|\overline{u}_0\|_{K,\infty} \leqslant \|\overline{u}_0\|_{K,\infty} \end{aligned}$$

for $\Omega^{\pm}$ small and K large enough. At last, when we approach the curve $x=\chi_{\Delta t,\Delta x}(t)$ for $x<\chi_{k+1}((k+2)\Delta t)$ and $j \leqslant \kappa-1$, the proof goes on as in [CST1]. This proves the claim $(\mathfrak{J}_k)$ for every k.

3.4. *Convergence of the scheme.*

The convergence of the scheme is made after a change of variables which straightens the contact curve $x=\chi_{\Delta t,\Delta x}(t)$ to $\widetilde{x}=0$; then in the following proposition $\pm$ will refer to $\pm\widetilde{x}>0$. For a given space mesh Δx let $\Delta t = a\Delta x$ be a time mesh such that condition (CFL) holds, for a positive parameter a; we write consequently $u_\Delta = u_{a\Delta x,\Delta x}$, $q_\Delta = q_{a\Delta x,\Delta x}$, $\chi_\Delta = \chi_{a\Delta x,\Delta x}$, and denote by $\widetilde{u}_\Delta$, $\widetilde{q}_\Delta$, $\widetilde{\chi}_\Delta$ these functions after the change of variables $\widetilde{x} = x - \chi_\Delta(t)$.

PROPOSITION 3.6 (Convergence). *For each given sequence θ in $]-1,1[$, every sequence $\{\Delta_\nu\}_\nu$ of positive numbers with limit 0 has a subsequence, still denoted for simplicity by $\{\Delta_\nu\}_\nu$, such that*

(i) *$\{\widetilde{u}^{\pm}_{\Delta_\nu}(\widetilde{t},\cdot)\}$ is convergent in $L^1_{\mathrm{loc}}(\boldsymbol{R}^{\pm})$ to $\widetilde{u}^{\pm}_\theta(\widetilde{t},\cdot) \in BV(\boldsymbol{R}^{\pm})$, for every $\widetilde{t} \in \boldsymbol{R}^+$;*

(ii) *$\{\widetilde{q}_{\Delta_\nu}\}$ is convergent in $L^1_{\mathrm{loc}}(\boldsymbol{R}^+)$ and almost everywhere to $\widetilde{q}_\theta \in BV(\boldsymbol{R}^+)$;*

(iii) *$\{\widetilde{\chi}_{\Delta_\nu}\}$ is pointwise convergent to $\widetilde{\chi}_\theta \in \mathrm{Lip}(\boldsymbol{R}^+)$ and $\widetilde{\chi}_\theta(\widetilde{t}) = \int_0^{\widetilde{t}} \widetilde{q}_\theta(s)\,ds$.*

Moreover for $\tilde{t}$, $\tilde{s} \geq 0$ we have the following estimates:

$$\|\tilde{u}_\theta^\pm(\tilde{t}, \cdot) - \underline{u}^\pm\|_{L^\infty} \leq C\{\|h^- - \underline{u}^-\|_{L^\infty} + \|h^+ - \underline{u}^+\|_{L^\infty}\},$$

$$V(\tilde{u}_\theta^\pm(\tilde{t}, \cdot)) + V(\tilde{q}_\theta) \leq C\{V(h^-) + V(h^+)\},$$

$$\int_{\pm\tilde{x} > 0} |\tilde{u}_\theta^\pm(\tilde{t}, \tilde{x}) - \tilde{u}_\theta^\pm(\tilde{s}, \tilde{x})| d\tilde{x} \leq C|\tilde{t} - \tilde{s}|\{V(h^-) + V(h^+)\}.$$

The proof does not require any essential change with respect to that of [Gl]. The last step in the proof of Theorem 1.1 consists in making an inverse change of variables which gives functions $u_\theta^\pm$, q_θ, χ_θ in $\pm(x - \chi_\theta(t)) > 0$; as in [CST1] the smoothness of χ_θ assures then that for almost every θ the function $u_\theta^\pm$ is a weak solution to (1.6).

4. - Traces.

In this last section we prove that each solution constructed as above and issued from an equidistributed scheme is a κ-contact discontinuity and establish as well the estimates of Theorem 1.2. The tool to be used are the approximate characteristics of Glimm-Lax [GL] and the extension of that theory to cover the linearly degenerate case given by Liu [Li].

The differences with the case of a strong shock wave treated in [CST1] are worth mentioning: in that case Lax's entropy conditions (and the equidistribution of the sequence θ) made κ-approximate characteristics starting from points close to the strong shock curve coalesce into it; this allowed to apply the conservation laws for strengths in domains bounded by two approximate characteristics of the same number κ (one of them was the strong shock curve) and by a space-like curve. On the contrary, in the present case κ-approximate characteristics are parallel, roughly speaking, and we are impelled to deal with domains bounded by the strong κ-contact discontinuity (a priori an approximate characteristic) and by a $\kappa - 1$ or $\kappa + 1$ characteristic according to we are at its right or left side.

As we already pointed out, we prove below that Riemann invariants of our solutions have traces of bounded variation at the contact curve; this is achieved by a local equicontinuity result at either side (whence the existence of traces as in [GL]). This is sufficient to prove that u is a contact discontinuity.

Let θ be an equidistributed sequence and $u = u_\theta$ the related solution provided by Theorem 1.1; for simplicity we drop dependence on θ. Let $\{\Delta_\nu\}_\nu$ be a sequence of positive numbers (or, when it is needed, a subsequence of it) converging to 0, $q_\nu = q_{a\Delta_\nu}$, $\chi_\nu = \chi_{a\Delta_\nu}$, and $u_\nu^\pm = u^\pm_{a\Delta_\nu, \Delta_\nu}$ for $\pm(x - \chi_\nu(t)) > 0$.

LEMMA 4.1 *Let $t_0 \in \boldsymbol{R}^+$; for every neighborhood V of $(t_0, \chi(t_0))$ there exists $\eta > 0$, $\nu_0 \in \boldsymbol{Z}^+$ such that*

(i) *for $\nu \geqslant \nu_0$ every $(\kappa - 1)$-approximate characteristic ψ_ν issuing from some point in the interval $\{t_\nu\} \times]\chi_\nu(t_\nu), \chi_\nu(t_\nu) + \eta[$, where t_ν is defined by $t_\nu = k_\nu a\Delta_\nu$, $k_\nu a\Delta_\nu \leqslant t_0 < (k_\nu + 1)a\Delta_\nu$, runs into χ_ν in V for positive times;*

(ii) *for $\nu \geqslant \nu_0$ every $(\kappa + 1)$-approximate characteristic φ_ν issuing from some point $(t_\nu^-, \chi_\nu(t_\nu^-))$ of V such that $t_\nu^- \in \boldsymbol{Z}^+ a\Delta_\nu$, $0 < t_\nu - t_\nu^- < \eta$, runs into $\{t = t_\nu\}$ in V.*

The proof of this lemma is simple: $(\kappa - 1)$-characteristics on the right of χ_ν point toward χ_ν, $(\kappa + 1)$-characteristics point away from χ_ν, owing to the equidistribution of the sequence θ.

We define now the characteristics, the amounts of waves, and the domains which are going to be employed in the following. Let φ_ν be a $(\kappa + 1)$-approximate characteristic issuing from $(t_\nu^-, \chi_\nu(t_\nu^-))$, $t_\nu^- = k_\nu^- a\Delta_\nu$, with zero $(\kappa + 1)$-strength on its left; it runs into $\{t = t_\nu\}$ in V at a point (t_ν, x_ν^-), $x_\nu^- > \chi_\nu(t_\nu)$. Let ψ_ν be a $(\kappa - 1)$-approximate characteristic issuing from (t_ν, x_ν), $x_\nu < x_\nu^-$, $0 < x_\nu - \chi_\nu(t_\nu) < \eta$; it runs into χ_ν in V at a point $(t_\nu^+, \chi_\nu(t_\nu^+))$, $t_\nu^+ = k_\nu^+ a\Delta_\nu$. Let Σ_ν be the curve made of edges of diamonds just above $\{t = t_\nu\}$ and on the right of the approximate contact curve; we denote with $\mathcal{D}_\nu^-$ the closed domain contained in V and bounded by χ_ν, φ_ν and Σ_ν, $\mathcal{D}_\nu^+$ the domain contained in V and bounded by χ_ν, ψ_ν and Σ_ν; at last $\mathcal{D}_\nu = \mathcal{D}_\nu^- \cup \mathcal{D}_\nu^+$, $\sigma_\nu \equiv \Sigma_\nu \cap \{\chi_\nu(t_\nu) < x < x_\nu\}$.

Let l_ν be anyone of the following curves made of edges of diamonds: just on the left of φ_ν in $\mathcal{D}_\nu^-$, just on the left of ψ_ν in $\mathcal{D}_\nu^+$, σ_ν or Σ_ν. We denote $X_<(l_\nu) = \sum_{i=1}^{\kappa-1} \{X_i^+(l_\nu) + |X_i^-(l_\nu)|\}$ the amount of waves with numbers less than κ crossing l_ν, where the sign is that of strengths, and analogously $X_>(l_\nu)$. We shall write briefly $X_>(\varphi_\nu^-)$ for $X_>(l_\nu)$ when l_ν is just on the left of φ_ν in $\mathcal{D}_\nu^-$ and analogously for the other curves.

For χ_ν, as in [GL], we can only define $X_<(l_\nu)$, l_ν being the curve made of edges of diamonds just on the right of $\chi_\nu \cap \mathcal{D}_\nu^\pm$, or $\chi_\nu \cap \mathcal{D}_\nu$; as before we shall write $X(\chi_\nu)$, $X(\chi_\nu^\pm)$. We shall need also the following amounts of waves related to reflections and transmissions of waves through the strong contact discontinuity:

$$B_>(\chi_\nu) = \sum_{\substack{k_\nu^- \leqslant k \leqslant k_\nu^+ - 1 \\ \theta_{k+1} > 0}} |R_{(+)}(p_k, U_{k,0})\,\varepsilon_c^+(k, 1)| + \\ + \sum_{\substack{k_\nu^- \leqslant k \leqslant k_\nu^+ - 1 \\ \theta_{k+1} < 0}} |T_{(-)}(p_k, U_{k,0})\,\delta_c^-(k, -1)|\,.$$

At last we denote with $\dot{\mathcal{D}}_\nu^\pm$, $(\overline{\mathcal{D}}_\nu^\pm)$ the largest (smallest) domain of whole diamonds contained (which contains) $\mathcal{D}_\nu^\pm$ and analogously $\dot{\mathcal{D}}_\nu, (\overline{\mathcal{D}}_\nu)$; adding to $\dot{\mathcal{D}}_\nu^+$ half diamonds on the left of ψ_ν^+ we obtain $\dot{\mathcal{D}}_\nu^+ \cup \psi_\nu/2$ and analogously for $\dot{\mathcal{D}}_\nu^- \cup \varphi_\nu^-/2$; adding to $\dot{\mathcal{D}}_\nu^+$ whole diamonds which contain ψ_ν^+ we obtain $\dot{\mathcal{D}}_\nu^+ \cup \psi_\nu^+$ and analogously $\dot{\mathcal{D}}_\nu^- \cup \varphi_\nu^-$. Let $\mathcal{D}$ be anyone of the domains $\dot{\mathcal{D}}_\nu$, $\dot{\mathcal{D}}_\nu^\pm$, $\dot{\mathcal{D}}_\nu^+ \cup \psi_\nu^+$, $\dot{\mathcal{D}}_\nu^+ \cup \psi_\nu^+/2$, $\dot{\mathcal{D}}_\nu^- \cup \varphi_\nu^-$. We denote with $C_<(\mathcal{D})$ the amount of waves with numbers less than κ cancelled in $\mathcal{D}$ and analogously $C_>(\mathcal{D})$. If $\mathcal{D}$ is either one of the previous domains or $\overline{\mathcal{D}}_\nu$, $\overline{\mathcal{D}}_\nu^\pm$, we denote with $\Lambda(\mathcal{D})$ the amount of interaction in $\mathcal{D}$; let us point out that $\Lambda(\overline{\mathcal{D}}_\nu)$ contains also terms like $\lambda_{(\pm)}(k)$.

PROPOSITION 4.2. (i) *Under the assumptions of Theorem* 1.2 *we have*

$$(4.1) \qquad X_<(\sigma_\nu) \leqslant \\ \leqslant X_<(\chi_\nu^+) + 2C_<(\dot{\mathcal{D}}_\nu^+ \cup \psi_\nu^+) + |\varepsilon_{\kappa-1}(k_\nu^+ - 1, 1)| + O(1)\Lambda(\overline{\mathcal{D}}_\nu^+)\,,$$

$$(4.2) \qquad X_>(\sigma_\nu) \leqslant B_>(\chi_\nu^-) + O(1)\Lambda(\overline{\mathcal{D}}_\nu^-)\,.$$

(ii) *All the terms in the right-hand sides of* (4.1) *and* (4.2) *are bounded by a constant times* $\mathcal{L}_\nu(0)$.

PROOF. To prove (4.1) we apply the conservation law (2.12) of [GL] to the domain $\dot{\mathcal{D}}_\nu^+ \cup \psi_\nu^+/2$: waves with numbers less than κ entering through σ_ν are leaving through χ_ν or (the $\kappa-1$-th ones) are absorbed from the left by ψ_ν^+; these last waves cannot exceed

$$|\varepsilon_{\kappa-1}(k_\nu^+ - 1, 1)| + 2C_{\kappa-1}(\psi_\nu^+) + O(1)\Lambda(\psi_\nu^+)\,,$$

$C_{\kappa-1}(\psi_\nu^+)$, $\Lambda(\psi_\nu^+)$ denoting cancellations and interactions along ψ_ν^+, whence (4.1).

At last the proofs of (4.2) and of (ii) are those of [CST1].

The proof of Theorem 1.2 is now obtained in the following way: with the notations of Lemma 4.1, proceeding as in [CST1] along the lines of [GL], we deduce from Proposition 4.2 that for every κ-Riemann invariant g^+ in Ω^+, the functions $g^+(u_\nu^+)$ are right equicontinuous at points $(t_\nu, \chi_\nu(t_\nu))$ when the limit t_0 of t_ν does not belong to an at most countable set. Since the sequence $\{\mathcal{C}_\nu^+\}_\nu$ is bounded, we can assume that the sequence $\{g^+(u_\nu^+(t, \chi_\nu(t)+0))\}_\nu$ converge towards $v^+ \in BV(\boldsymbol{R}_t^+)$ in L^1_{loc}. For the solution u of Theorem 1.1 issued from an equidistributed scheme, we obtain then $g^+(u^+(t, \chi(t)+0)) = v^+(t)$ a.e. and in particular $g^+(u^+(t, \chi(t)+0)) \in BV(\boldsymbol{R}_t^+)$.

We proceed in the same way with a κ-Riemann invariant g^- in Ω^-; thus the traces of $\lambda_\kappa(u)$ and of every component of $f(u) - \lambda_\kappa(u)u$ at each side of the curve $x = \chi(t)$ are in $BV(\boldsymbol{R}_t^+)$. Moreover, passing to the limit, the equality

$$\lambda_\kappa(u_\nu^-(t, \chi_\nu(t))) = \chi_\nu'(t) = \lambda_\kappa(u_\nu^+(t, \chi_\nu(t)))$$

yields a.e.

$$\lambda_\kappa(u^-(t, \chi(t))) = \chi'(t) = \lambda_\kappa(u^+(t, \chi(t))).$$

At last the Rankine-Hugoniot relation

$$[f(u)] = \chi'(t)[u] \equiv \lambda_\kappa(u^+(t, \chi(t)))[u],$$

which holds a.e. on $x = \chi(t)$, insures a null jump for $f(u) - \lambda_\kappa(u)u$ on this curve. All this proves (1.10). Moreover the L^∞ estimate of χ' in (1.9) follows from that of u since, for example, $\chi'(t) = \lambda_\kappa(u(t, \chi(t)-0)))$.

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