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A Remark on Semiglobal Existence for $\bar{\partial}$.

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ABSTRACT - It is proved in [6] that a domain Ω of $X = \mathbb{C}^n$ is pseudoconvex if and only if $\bar{\partial}$-closed forms (of positive degree) are $\bar{\partial}$-exact in Ω. We prove here by elementary techniques that pseudoconvexity of Ω is still characterized by solvability of $\bar{\partial}$-forms over relatively compact subsets of Ω.

1. - Semiglobal solvability in $\mathbb{C}^2$.

Let $X = \mathbb{C}^n$ and let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. Let Ω be a domain of X with C^2-boundary $M = \partial\Omega$, and denote by $\delta(z)$, $z \in \Omega$ the Euclidian distance to M. Let z be a point of M; one defines the Levi form of M at z (from the exterior side of Ω) by

$$L_M(z) = -\partial\bar{\partial}(\delta)(z)|_{T_z^{\mathbb{C}}M},$$

where $T_z^{\mathbb{C}}M = T_zM \cap \sqrt{-1}T_zM$. One denotes by $s_M^+(z)$, $s_M^-(z)$, and $s_M^0(z)$ the numbers of respectively positive, negative, and null eigenvalues of $L_M(z)$.

One denotes by $H^j(\Omega, \mathcal{O}_X)$, $0 \leq j \leq n$, the space of $\bar{\partial}$-closed $(0, j)$-forms on Ω (with C^∞-coefficients) modulo $\bar{\partial}$-exact forms.

We begin our discussion in $X = \mathbb{C}^2$. The proof of the following statement was suggested word by word by professor Alexander Tumanov.

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PROPOSITION 1.1. *Let Ω' be a domain of $\mathbb{C}^2\backslash\{z_0\}$ such that z_0 belongs to a compact component of $\mathbb{C}^2\backslash\Omega'$. Then*

$$H^1(\mathbb{C}^2\backslash\{z_0\}, \mathcal{O}_X)|_{\Omega'} \neq 0 . \tag{1.1}$$

PROOF. Assume $z_0 = 0$. Define:

$$f = \begin{cases} \bar\partial\left(\dfrac{\bar z_2}{z_1(|z_1|^2 + |z_2|^2)}\right), & z_1 \neq 0, \\ -\bar\partial\left(\dfrac{\bar z_1}{z_2(|z_1|^2 + |z_2|^2)}\right), & z_2 \neq 0 . \end{cases}$$

It is easy to prove that f is a (closed) 1-form in $\mathbb{C}^2\backslash\{z_0\}$. Assume by absurd that $H^1(\mathbb{C}^2\backslash\{z_0\}, \mathcal{O}_X)|_{\Omega'} = 0$. Then there exists g in Ω': $\bar\partial g = f$ in Ω' whence

$$h := z_1 g - \frac{\bar z_2}{|z_1|^2 + |z_2|^2}$$

is holomorphic in Ω' since it is holomorphic in $\Omega'\backslash\{z_1 = 0\}$ and bounded in Ω' far from 0. By Hartogs' theorem h extends to the compact components of $\mathbb{C}^2\backslash\Omega'$ and in particular to 0. But $h|_{z_1=0} = -1/z_2$ does not. Q.E.D.

PROPOSITION 1.2. *Let Ω be a domain of $\mathbb{C}^2$ with C^2 boundary $M = \partial\Omega$. For $z_0 \in \partial\Omega$, assume $L_M(z_0) < 0$. Then*

$$H^1(\mathbb{C}^2\backslash\{z_0\}, \mathcal{O}_X)|_{\Omega'} \neq 0 , \tag{1.2}$$

for suitable $\Omega' \subset\subset \Omega$.

PROOF. Let $z_0 = 0$. (*a*) We prove that for convenient $\Omega' \subset\subset \Omega$, we can find $\tilde\Omega' \supset \Omega'$, $z_0 \in \tilde\Omega'$, connected along a complex curve L through z_0 such that any $h \in \mathcal{O}_X(\Omega')$ extends to $\tilde\Omega'$. In fact in complex coordinates $z = (z_1, z_2)$, $z = x + iy$, we can assume:

$$\Omega = \{z;\, x_2 > -x_1^2 + o(x_1, y_1, y_2)^2\} . \tag{1.3}$$

Define

$$I_t = \left\{(x_1, x_2);\ x_2 = \frac{t^2}{4},\ -t \leqslant x_1 \leqslant t\right\} \cup \left\{(x_1, x_2);\ x_1 = t,\ \frac{-t^2}{2} \leqslant x_2 \leqslant \frac{t^2}{4}\right\},$$

$$J_t = \left\{(x_1, x_2);\ -t < x_1 < t,\ \frac{t^2}{4} - \frac{t}{3}(x_1 + t) < x_2 < \frac{t^2}{4}\right\}.$$

Let $K_{ct} = \{(y_1, y_2);\ |y_1, y_2| < ct\}$. We have $\Omega \supset K_{ct} \times I_t$, $\forall c \gg 1$ and for suitable t. Let B be a fixed neighborhood of z_0 and $\Omega' = \Omega'_t \subset\subset \Omega$ s.t. $\Omega' \cap B \supset K_{ct} \times I_t$. By [8,Theorem 5] any holomorphic function on $K_{ct} \times I_t$ extends to $K_{ct/2} \times J_t$. In particular (a) follows with L being the z_2 axis.

(b) Choose complex coordinates in $\mathbb{C}^2$ s.t. the curve L of (a) coincides with the z_2 axis. Define f as in the preceding Proposition, solve $\bar\partial g = f$ in Ω' and set $h = z_1 g - \bar z_2/(|z_1|^2 + |z_2|^2)$ (holomorphic in Ω'). Take $\Omega' \subset\subset \Omega$ such that (a) holds. Then h (and therefore also $h|_L = -1/z_2$) extends at $z_0 = 0$ which is a contradiction.

2. – Semiglobal solvability in $\mathbb{C}^n$.

Let $X = \mathbb{C}^n$, $M = \partial\Omega$ a C^2 real submanifold of $\mathbb{C}^n$, $\Omega = \{z \in X: \delta(z) > 0\}$.

DEFINITION 2.1. *Ω is pseudoconvex if and only if $s^-_M(z) \equiv 0$, $\forall z \in M$.*

It is classical that Ω is pseudoconvex if and only if $H^j(\Omega, \mathcal{O}_X) = 0$, $\forall j > 0$. With the notation $\Omega_\varepsilon = \{z \in \Omega;\ \delta > \varepsilon\}$ we can generalize the above characterization as follows:

THEOREM 2.2. *Ω is pseudoconvex if and only if*

$$H^j(\Omega_\varepsilon, \mathcal{O}_X)|_{\Omega_{\varepsilon'}} = 0, \qquad \forall\varepsilon,\ \forall\varepsilon',\ \textit{with}\ \varepsilon' > \varepsilon \geqslant 0,\ \forall j > 0. \tag{2.1}$$

PROOF. The «Only if» follows from the fact that Ω_ε is pseudoconvex [6, Theorem 2.6.12].

«If»: (a) Let L be a complex submanifold of X, and set $\omega = L \cap \Omega$, $\omega_\varepsilon = \{z \in \omega;\ \delta_L(z) > \varepsilon\}$ where δ_L is the distance along L. Then (2.1) implies that $\forall\varepsilon$ $\exists\varepsilon'$ such that

$$H^j(\omega_\varepsilon, \mathcal{O}_L)|_{\omega_{\varepsilon'}} = 0 \qquad \text{with } \varepsilon' > \varepsilon \geqslant 0, \tag{2.2}$$

with $\varepsilon' \to 0$ as $\varepsilon \to 0$. This follows from adapting the proof of [6, Theorem 4.2.9] as we show here. Let f be a $\bar\partial$-closed form in $\{z \in \omega;\ \delta_L > \varepsilon\}$. Let $\phi \in C^\infty$, $\phi = 1$ in a neighborhood of ω and $\phi = 0$ in a neighborhood of $\{z \in \Omega;\ \pi z \notin \omega\}$ (π: $X \mapsto L$ the orthogonal projection). By a recoursive argument it is not restrictive to assume $\mathrm{cod}_X L = 1$. Choose coordinates so that $L = \{z\colon z_n = 0\}$. Solve

$$\bar\partial v = z_n^{-1} \bar\partial \phi \wedge \pi^* f \qquad \text{in } \Omega_{\varepsilon'}$$

thus

$$F := \phi\pi^* f - z_n v$$

is $\bar\partial$-closed in $\Omega_{\varepsilon'}$. Solve $\bar\partial G = F$ in $\Omega_{\varepsilon''}$, $\varepsilon'' > \varepsilon'$. Then $g := j^* G$ (j: $L \hookrightarrow X$) solves $\bar\partial g = f$ in $\omega_{\varepsilon'''}$, $\varepsilon''' > \varepsilon''$ with $\varepsilon''' \to 0$ as $\varepsilon \to 0$. This proves (2.2).

(*b*) If $n = 1$ there is nothing to prove, and if $n = 2$ the theorem follows from § 1. Set $n \geqslant 3$, take $z_0 \in \partial\Omega$, $v \in T^{\mathbf{C}}_{z_0} M$ and consider $L = \mathbf{C}v \oplus \oplus \mathbf{C}u$ (u transversal to M). Then (2.2) holds and this implies in particular $H^1(\mathbf{C}^2 \setminus \{z_0\}, \mathcal{O}_L)|_{\omega'} = 0$, $\forall \omega' \subset\subset \omega$. Thus with $N = \partial\omega$,

$$L_M(z_0)\, v^t \bar v = L_N(z_0) v^t \bar v \geqslant 0\,. \qquad \text{Q.E.D.}$$

We can localize our statement:

DEFINITION 2.3. *Ω is pseudoconvex at z_0 if and only if $s_{\bar M}^-(z) = 0$, $\forall z \in M$ close to z_0.*

Let B (or B') range through the family of neighborhoods of the points z close to z_0 on M.

THEOREM 2.4. *For any B there exists $B' \subset B$ such that:*

(2.3) $$H^1(\Omega \cap B, \mathcal{O}_X)|_{\Omega' \cap B'} = 0\,, \qquad \forall \Omega' \subset\subset \Omega$$

if and only if Ω is pseudoconvex at z_0.

PROOF. «If»: by [1] we may find $\tilde\Omega$: $\tilde\Omega$ is pseudoconvex, $\tilde\Omega \subset \Omega \cap B$, $\tilde\Omega \cap B' = \Omega \cap B'$. Then (2.3) follows.

«Only if»: (*a*) $X = \mathbf{C}^2$. If Ω is not pseudoconvex at z_0 one may find $z_\nu \to z_0$ with $s_{\bar M}^-(z_\nu) = 1$. Then the proof of Proposition (1.2) shows that

$$H^1(\mathbf{C}^2 \setminus \{z_\nu\}, \mathcal{O}_X)|_{\Omega' \cap B'} \neq 0$$

for any B' neighborhood of z_ν and for suitable $\Omega' = \Omega'_{B'} \subset\subset \Omega$. In particular (2.3) is violated.

(*b*) Let $X = \mathbb{C}^n$, $n \geq 3$, take L with $\operatorname{cod}_X L \geq 1$, define $b = B \cap L$, $\omega = \Omega \cap L$ and consider f $\bar{\partial}$-closed in $\omega \cap b$. By the same argument as in Theorem 2.2 f can be «lifted» to F $\bar{\partial}$-closed in $\Omega' \cap B'$ for any Ω' relatively compact in Ω and for suitable B'. For any ε we take Ω' (dependent on ε) and B'' (independent of ε) s.t.

$$-n\varepsilon + (\Omega \cap B'') \subset \Omega' \cap B'$$

(where n is the outward normal) and define

$$F_1(x) = F(x - n\varepsilon) \qquad \text{in } \Omega \cap B''.$$

Solve $\bar{\partial} G_1 = F_1$ in $\Omega'' \cap B'''$ (independent of ε) then g_1 solves $\bar{\partial} g_1 = f_1$ in $\omega'' \cap b'''$ (independent of ε) and thus $\bar{\partial} g = f$ in $-n\varepsilon + (\omega'' \cap b''') \supset \omega''' \cap \cap b''''$ where ω''' is any relatively compact open subset of ω and b'''' a suitable neighborhood.

(*c*) The end of the proof follows from combining (*a*) and (*b*).

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