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On the Rationality of the Moduli Schemes of Vector Bundles on Curves.

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It is often true that certain moduli problems of varieties or of vector bundles over a fixed variety have as coarse moduli scheme an integral variety. In this case it is very interesting to know if the moduli scheme is unirational or rational. Often the unirationality follows easily from the proof that the moduli scheme is irreducible. Here we will consider the case of stable vector bundles over a smooth curve.

Let X be a smooth complete algebraic curve of genus $g \geqslant 2$ defined over an algebraically closed base field $\boldsymbol{K}$ with $\mathrm{char}(\boldsymbol{K}) = 0$. Fix integers r, d with $r \geqslant 1$. Fix $L \in \mathrm{Pic}^d(X)$. Let $S_L(r, d)$ be the moduli scheme of stable rank r vector bundles on X with determinant L. It is well known ([S]) that $S_L(r, d)$ is smooth, irreducible of dimension $(r^2 - 1)(g - 1)$ and unirational. P. E. Newstead ([N]) proved in many cases that $S_L(r, d)$ is rational, for example when $d \equiv \pm 1 \bmod (r)$. The variety $S_L(r, d)$ is a fine moduli scheme if and only if $(r, d) = 1$ ([S], [MBN], [Ra]). Thus the case when $(r, d) = 1$ is particularly interesting. An integral variety W is called stably rational of level $\leqslant w$ if $W \times \boldsymbol{P}^w$ is rational. A stably rational variety is unirational but there are examples (see [BCSS]) of stably rational varieties which are not rational. It was proved in [B] that $S_L(r, d)$ is stably rational if $(r, d) = 1$. Our main result is the following theorem which gives a huge number of cases in which $S_L(r, d)$ is rational.

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THEOREM 0.1. *Let X be a smooth complete algebraic curve of genus $g \geqslant 2$ defined over an algebraically closed field $\boldsymbol{K}$ with* $\operatorname{char}(\boldsymbol{K}) = 0$. *Set $q' \equiv d \bmod r$ and $q'' \equiv -d \bmod (r)$ with $r(g-1) \leqslant q' < rg$ and $r(g-1) \leqslant q'' < rg$. Set $r' := rg - q'$ and $r'' := rg - q''$. Assume $(r, d) = 1$ and that one of the following conditions* (i) *or* (ii) *is satisfied:*

(i) $(r', q') = 1$ *and* $r'^2 \leqslant (r^2 - r'^2)(g-1)$;

(ii) $(r'', q'') = 1$ *and* $r''^2 \leqslant (r^2 - r''^2)(g-1)$.

Then for every $L \in \operatorname{Pic}^d(X)$ the variety $S_L(r, d)$ of stable rank r vector bundles with determinant L is rational. In particular, if $(r, d) = (r', q') = (r'', q'') = 1$, then $S_L(r, d)$ is rational.

Theorem 0.1 will be proved in section one. A key step for the proof of Theorem 0.1 is to give a good upper bound (i.e. r^2) on the level of stable rationality for every variety $S_L(r, d)$ with $(r, d) = 1$ (see 1.2). This bound is better than the explicit bound u^2 with $r < u < 2r$, $u \equiv d \bmod (r)$ which, although not explicitely stated in [He], follows in a trivial way from the main result in [He]. Our improved bound allows us to give more examples of rational $S_L(r, d)$. We think that Theorem 1.2 and its proof have an independent interest. A key point for the proof of Theorem 1.2 (hence of Theorem 0.1) is contained in the proof of Lemma 1.1 and will be discussed there. An important tool for all the results of this paper is the use of elementary transformations of vector bundles. The referee pointed out that in the recent preprint [BY] H. Boden and K. Yokogawa proved that for every $g \geqslant 2$, every $r \geqslant 2$ and every $L \in \operatorname{Pic}(X)$ the moduli space $M(r, L)$ of rank r semistable bundles with determinant L is stably rational of level $\leqslant r-1$. This implies that if $(r, d) = 1$, $0 < d < r$ and either $(g, d) = 1$ or $(g, r-d) = 1$, then $S_L(r, d)$ is rational. We are very much indebt with the referee for this important refence as clear by the following remark.

IMPORTANT REMARK 0.2. Assume $g \geqslant 2$ and $(r, d) = 1$. Note that for every integer s with $0 < s < r$ we have $s - 1 \leqslant (r^2 - s^2)(g-1)$. Hence if we use the result just quoted from [BY] instead of Theorem 1.2 in the first sentence of the proof of Theorem 0.1, we obtain the rationality of $S_L(r, d)$ if $(r, d) = 1$ and either $(r', q') = 1$ or $(r'', q'') = 1$.

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1. – Proof of Theorem 0.1.

To prove Theorem 0.1 we need the following lemma.

LEMMA 1.1. *For every r, d, L with $d > rg$ there is a rational variety $V := V(L, r, d)$ with a family of stable vector bundles on V such that the associated morphism $V \to S_L(r, d)$ is dominant and with fibers equidimensional and non empty, each of them a Zariski open subset of an affine space of dimension r^2.*

PROOF. Fix $P \in X$. Set $x := gr - 1$. It was proved in [N] that for every $M \in \mathrm{Pic}^x(X)$, the variety $S_M(r, x)$ is rational. It was essentially implicitely proved in [N] (but see [Gr] for a detailed and different proof) that the general $W \in S_M(r, x)$ has $h^0(X, W) = r - 1$. Fix $y \equiv x \bmod (r)$ with $y - r < d \leq y$ and set $A := \big(M \otimes \boldsymbol{O}(((y-x))P)\big) \in \mathrm{Pic}^y(X)$. We have $S_A(r, y) \cong S_M(r, x)$ because both varieties are coarse moduli schemes of isomorphic functors, the isomorphism at the functorial level being induced by the twist with $\mathbf{O}(((y-x)/r)P)$. Hence $S_A(r, y)$ is rational. Consider the varieties $S' := \{(E, j)$: $E \in S_L(r, d)$, j is an isomorphism of vector spaces of the fiber $E|\{P\}$ with $\boldsymbol{K}^r\}$ and $S'' := \{(E, j)$: $E \in S_A(r, y)$, j is an isomorphism of vector spaces of the fiber $E|\{P\}$ with $\boldsymbol{K}^r\}$. We claim that S' is birational to $S_L(r, d) \times \boldsymbol{K}^{r^2}$, and that S'' is birational to $S_A(r, y) \times \boldsymbol{K}^{r^2}$. To check the claim, first note that any two bases of $\boldsymbol{K}^r$ differ by an element of $GL(r)$ which is an open subset of $\boldsymbol{K}^{r^2}$. As proved in [G] (or see [Do], Proposition at page 9) the group $GL(r)$ is special, i.e. every principal fibration for $GL(r)$ which is locally trivial in the étale topology is locally trivial in the Zariski topology. This is the key point which is false for the projective linear group $PGL(r)$ (and this is the error in the proof of rationality of all varieties $S_L(a, b)$ published by Tyurin in 1965). It is trivial to check that in our situation we have a bundle in the étale topology since every smooth morphism has local sections in the étale topology. If we fix a rank r degree y vector bundle E on X and we make an elementary transformation along a $(y-d)$-dimensional subspace of $E|\{P\}$ we obtain a vector bundle F of rank r and degree d. Viceversa, if we start from a rank r degree d vector bundle F on X, we make an elementary transformation along a subspace of dimension $r + d - y$ of $F|\{P\}$ and then we twist it by $\mathbf{O}(P)$, we obtain a vector bundle E with rank r and degree y. It is easy to see that for general stable E and a suitable choice of the $(y - d)$-dimensional subspace, the corresponding F is stable (and viceversa, going from F to E). This shows that S' is birational to S'' and that the family of degree y bundles on S'' induced by the universal family of

$S_A(r, y)$ induces a family, T, of stable degree d bundles on a Zariski dense subset $V := V(L, r, d)$ of S'. the result follows easily. ■

THEOREM 1.2. *For all integers r, d with $(r, d) = 1$ the variety $S_L(r, d)$ is stably rational of level $\leq r^2$.*

PROOF. Since $S_L(r, d) \cong S_{L(kP)}(r, d + kr)$ for every $P \in X$ and every integer k, we may assume $d > gr$. We use 1.1. Since $(r, d) = 1$ the existence of a universal family of bundles on $S_L(r, d)$ gives that the fibration $V(L, r, d) \to S_L(r, d)$ has rational sections. Hence it is birationally a product ([Se]). ■

PROOF OF THEOREM 0.1. Itwas proved in [N] (see [N], Lemmata 5 and 6, and [B], Proof of Lemma 4) that if $(r', q') = 1$, then for every $M \in \in \mathrm{Pic}^{q'}(X)$ the variety $S_L(r, d)$ is birational to $S_M(r', q') \times \boldsymbol{P}^{(r^2 - r'^2)(g-1)}$. Hence if (i) is satisfied, then $S_L(r, d)$ is rational by Theorem 1.2. Since every bundle has a dual, we have $S_L(r, d) \cong S_{L^*}(r, -d)$ because both varieties are coarse moduli schemes of isomorphic functors. Hence if (ii) is satisfied, then $S_L(r, d)$ is rational by the statement just proved. Since at least one of the two conditions: $r'^2 \leq (r^2 - r'^2)(g-1)$ or $r''^2 \leq \leq (r^2 - r''^2)(g-1)$ is satisfied, the last part of the statement of 0.1 follows. ■

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