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# The Spectrum of the Transport Operator with a Potential Term under the Spatial Periodicity Condition.

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ABSTRACT - Under the spatial periodicity condition we deal with the transport operator with a potential term in the space of square-integrable complex-valued functions. The purpose of this paper is to prove that there exists a positive constant $C$ such that, for each $\theta \in [0, 1)$, the intersection of $\{\mu \in \mathbb{C};\ \mathrm{Re}\,\mu > -\theta C\}$ and the spectrum of the transport operator is a finite set of points which consists only of eigenvalues of the transport operator with finite (algebraic) multiplicity.

## 1. - Introduction.

The transport equation describes the evolution of the density of particles under certain conditions of rarefaction and interaction. In particular, if an external force $\mathbb{F} = -\nabla_x \phi(x)$ acts on the particles ($\phi = \phi(x)$ is an external-force potential), then the equation has the form,

$$\partial f / \partial t = Bf, \tag{1.1}$$

where

$$B \equiv -\Lambda - \nu + K, \qquad \Lambda \equiv \xi \cdot \nabla_x - \nabla_x \phi \cdot \nabla_\xi . \tag{1.2}$$

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$f = f(t, x, \xi)$ is the unknown function of $(t, x, \xi) \in [0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3$, which represents the density of the particles at time $t$, at a point $x$, and with a velocity $\xi$. $\nu = \nu(x, \xi)$ is a multiplication operator. $K$ is an integral operator of the form,

$$(1.3) \qquad (Kf(t, \cdot, \cdot))(x, \xi) = \int_{\eta \in \mathbb{R}^3} K(x, \xi, \eta) f(t, x, \eta)\, d\eta\,.$$

The operator $B$ is called the transport operator with a potential term. We consider $B$ in the space of square-integrable complex-valued functions of $(x, \xi)$. We are concerned with the structure of the spectrum of $B$ in the case where $\phi = \phi(x)$ is not identically equal to a constant (see, e.g., [16] for the details of (1.1) when $\phi(x) \equiv$ constant).

In [1], existence, uniqueness, dissipativity, and positivity results for solutions of (1.1) have been proved under very general conditions. However, there have been few studies on the structure of the spectrum of $B$ in the case where $\phi = \phi(x)$ is not identically equal to a constant. Noting that (1.1) describes the evolution of the density of particles acted upon by $\mathbb{F} = -\nabla_x \phi(x)$, we see that the structure of the spectrum of $B$ is closely connected to the behavior of solutions of the following system of ordinary differential equations:

$$(\text{SODE}) \qquad dx/dt = \xi, \qquad d\xi/dt = -\nabla\phi(x)\,.$$

Hence, if we try to investigate the structure of the spectrum of $B$, we need not only to study the operator $B$ by methods in the theory of functional analysis but also to investigate the behavior of the solutions of (SODE) by methods in the theory of dynamical systems. However, in general, the behavior of the solutions of (SODE) is very complicated. Hence it is very difficult to investigate these subjects at the same time. This is the reason for the difficulty encountered in trying to study the structure of the spectrum of $B$ (this difficulty will be discussed again in §5).

Taking the difficulty into account, we reasonably conclude that it is advisable to simplify the behavior of the solutions of (SODE) by imposing some restrictive assumptions. For this reason, we will impose the spatial periodicity condition in this paper, i.e., we assume that all the functions considered are periodic with respect to the space variable $x$ and have the same period. The transport equation with the spatial periodicity condition is essentially the same as that in a 3-dimensional torus. Hence, by virtue of the periodicity condition, we can regard that the solution $x = x(t)$ of (SODE) is always contained in the torus; the behavior of the solutions of (SODE) is simplified very much. This fact will play an essential role in the present paper (see Remark 8.2).

We do not impose any restrictive assumptions on $\phi = \phi(x)$. We only assume that $\phi = \phi(x)$ is sufficiently smooth.

The main result of this paper is as follows: there exists a positive constant $C$ such that, for each $\theta \in [0, 1)$, the intersection of $\{\mu \in \mathbb{C};\ \mathrm{Re}\,\mu > -\theta C\}$ and the spectrum of $B$ is a finite set of points which consists only of eigenvalues of $B$ with finite (algebraic) multiplicity (see Main Theorem in §3).

The present paper has 8 sections. §2 presents preliminaries. In §3 we will prove Main Theorem. In this proof, by making use of Lemma 3.2 and Theorem 3.3, we overcome the difficulty which is connected mainly to the theory of functional analysis. Theorem 3.3 and Lemma 3.2 play an essential role in this proof. In particular, Lemma 3.2 is a key lemma. In §8 we prove Lemma 3.2. In §4-7 we prepare for the proof. The purpose of §4 is to obtain estimates for the operator $A \equiv -\Lambda - \nu$. In §5 we discuss Lemma 3.2 and the difficulty encountered in trying to prove Lemma 3.2. In order to prove Lemma 3.2, we need to investigate the behavior of the solutions to (SODE). The necessity for this investigation is explained in §5. In §6-7 we obtain estimates for the rank of a Jacobian matrix with respect to the solutions of (SODE). By making use of these estimates, in §8 we prove Lemma 3.2.

REMARK 1.1. (i) The transport equation with the spatial periodicity condition is essentially the same as that in a 3-dimensional torus, as already mentioned above. Hereafter we will consider our problem in a 3-dimensional torus for simplicity. We denote the 3-dimensional torus in which our problem is studied by $\Omega$.

(ii) It must be noted that the subject of the present paper is closely related not only to the theory of functional analysis but also to the theory of dynamical systems. See [15, pp. 742-746 and pp. 754-756].

(iii) We can simplify the behavior of the solutions of (SODE) also by imposing restrictive assumptions on $\phi = \phi(x)$. In [15] we simplify the behavior of the solutions of (SODE) by assuming that $\phi = \phi(x)$ is spherically symmetric.

(iv) For further details of (1.1), see, e.g., [2], [4], [9-11], and [17] in addition to those presented in [1]. For the nonlinear Boltzmann equation with a potential term, see, e.g., [3] and [6-7].

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## 2. – Preliminaries.

FUNCTION SPACES. By $\mathbb{B}(X)$ ($\mathbb{C}(X)$ respectively) we denote the set of all bounded (compact respectively) linear operators in a Banach space $X$. We consider the transport operator $B$ in the space of square-integrable functions of $(x, \xi) \in \Omega \times \mathbb{R}^3$, i.e., in $L^2(\Omega \times \mathbb{R}^3)$, as already mentioned in §1 (see Remark 1.1, (i)). Write $\|\cdot\|$ as the norm of $L^2(\Omega \times \times \mathbb{R}^3)$. Write $|||\cdot|||$ as the norm of operators of $\mathbb{B}(L^2(\Omega \times \mathbb{R}^3))$.

ASSUMPTIONS. We will impose assumptions on $\nu$, $K$ and $\phi$ (see (1.2)-(1.3)).

ASSUMPTION $\nu$. *$\nu = \nu(x, \xi)$ is a real-valued continuous function of $(x, \xi) \in \Omega \times \mathbb{R}^3$ such that $\nu_0 \leqq \nu(x, \xi)$ for any $(x, \xi)$, where $\nu_0$ is a positive constant.*

We define an operator-valued function $\mathbb{K} = \mathbb{K}(x)$ as follows:

$$(\mathbb{K}(x)u(\cdot))(\xi) \equiv \int_{\eta \in \mathbb{R}^3} K(x, \xi, \eta)\, u(\eta)\, d\eta, \qquad x \in \Omega,$$

where the kernel $K(x, \xi, \eta)$ is that in (1.3).

ASSUMPTION $K$. (i) *$\mathbb{K} = \mathbb{K}(x)$ is a continuous operator-valued function from $\Omega_x$ to $\mathbb{B}(L^2(\mathbb{R}^3_\xi))$.*

(ii) *$\mathbb{K}(x) \in \mathbb{C}(L^2(\mathbb{R}^3_\xi))$ for each $x \in \Omega$.*

ASSUMPTION $\phi$. *$\phi = \phi(x)$ is a real-valued function defined in $\Omega$ which have continuous partial derivatives of order up to and including* 2.

REMARK 2.1. If $\inf_{x, \xi} \nu(x, \xi) = 0$, then we need to take an approach similar to that in [15, pp. 742-746 and pp. 754-756]. However, by virtue of Assumption $\nu$, we do not need to take such an approach in this paper.

OPERATORS. We consider the following operators (see (1.1)-(1.3)):

$$\Lambda \equiv \xi \cdot \nabla_x - \nabla_x \phi \cdot \nabla_\xi , \qquad A \equiv -\Lambda - \nu , \qquad B \equiv A + K .$$

We denote the domain of an operator $L$ by $D(L)$. We define $D(\Lambda) \equiv \equiv \{v = v(x, \xi) \in L^2(\Omega \times \mathbb{R}^3);\ \Lambda v \in L^2(\Omega \times \mathbb{R}^3)\}$. We similarly define $D(A) \equiv \{v = v(x, \xi) \in L^2(\Omega \times \mathbb{R}^3);\ Av \in L^2(\Omega \times \mathbb{R}^3)\}$. Making use of Assumption $K$, we deduce that $K \in \mathbb{B}(L^2(\Omega \times \mathbb{R}^3))$. Hence we can define $D(B) \equiv D(A)$.

SYMBOLS. We denote the Lebesgue measure of a set $Y$ by meas $Y$. In §7-8 the letter $c$ denotes some positive constant. We will use $c$ as a generic constant replacing any other constants (such as $c^3$ or $c^{1/2}$) by $c$.

From Assumption $\phi$ we see that the Cauchy problem for (SODE) with initial data $(x, \xi)(0) = (X, \Xi) \in \Omega \times \mathbb{R}^3$ has a unique solution globally in time. We denote the solution by $(x, \xi) = (x(t, X, \Xi), \xi(t, X, \Xi))$. We denote by $m_{ij} = m_{ij}(t, X, \Xi)$ the $(i, j)$ component of the Jacobian matrix,

$$J = J(t, X, \Xi) \equiv \partial(x(-t, X, \Xi), \xi(-t, X, \Xi))/\partial(X, \Xi),$$

$i, j = 1, \dots, 6$, i.e., if $1 \leqq i, j \leqq 3$, then $m_{ij}(t, X, \Xi) \equiv \partial x_i(-t, X, \Xi)/ /\partial X_j$. If $1 \leqq i \leqq 3$ and $4 \leqq j \leqq 6$, then $m_{ij}(t, X, \Xi) \equiv \partial x_i(-t, X, \Xi)/\partial \Xi_{j-3}$. If $4 \leqq i \leqq 6$ and $1 \leqq j \leqq 3$, then $m_{ij}(t, X, \Xi) \equiv \partial \xi_{i-3}(-t, X, \Xi)/\partial X_j$. If $4 \leqq i, j \leqq 6$, then $m_{ij}(t, X, \Xi) \equiv \partial \xi_{i-3}(-t, X, \Xi)/\partial \Xi_{j-3}$. We easily obtain the following equality:

$$\det(J(t, X, \Xi)) = 1, \qquad \text{for each } (t, X, \Xi) \in \mathbb{R} \times \Omega \times \mathbb{R}^3 , \tag{2.1}$$

where we denote the determinant of a square matrix $M$ by $\det(M)$.

By $Q$ we denote the set of all the one-rank operators of the form,

$$(Mu(\cdot, \cdot))(x, \xi) = (u(x, \cdot), f(x, \cdot))\, g(x, \xi),$$

where the brackets $(\cdot, \cdot)$ denote the inner product in $L^2(\mathbb{R}^3_\xi)$. $f = f(x, \xi)$ and $g = g(x, \xi)$ are infinitely differentiable functions on $\Omega_x \times \mathbb{R}^3_\xi$ which satisfy the following conditions (cf. [14, p. 1836]): there exists $r > 0$ such that, for each $x \in \Omega$

$$\operatorname{supp} f(x, \cdot),\ \operatorname{supp} g(x, \cdot) \subseteq \{\xi \in \mathbb{R}^3_\xi ;\ |\xi| \leqq r\} . \tag{2.2}$$

Noting that $\Omega_x \times \{\xi \in \mathbb{R}^3_\xi\,;\ |\xi| \leqq r\}$ is compact, we see that

$$(2.3) \qquad \max_{x,\xi} |f(x, \xi)|\,, \qquad \max_{x,\xi} |g(x, \xi)| < +\infty\,.$$

LEMMA 2.2. *The operator $K$ can be approximated in $\mathbb{B}(L^2(\Omega \times \times \mathbb{R}^3))$ with a finite sum of operators of $Q$.*

PROOF. Noting that $\Omega$ is compact, and applying Assumption $K$, (i), we see that $\mathbb{K} = \mathbb{K}(x)$ is uniformly continuous in $\Omega$. Hence, making use of Assumption K, (ii), and performing calculations similar to those in [12, p. 200, Theorem VI.13], we can deduce that $\mathbb{K} = \mathbb{K}(x)$ can be approximated in $\mathbb{B}(L^2(\mathbb{R}^3_\xi))$ uniformly for $x \in \Omega$ with operators of the form, $\mathbb{K}_n = \mathbb{K}_n(x) \equiv \sum_{j=1}^{n} (\psi_j, \cdot)\mathbb{K}(x)\psi_j$, $n \in \mathbb{N}$, where $\{\psi_j = \psi_j(\xi)\}_{j \in \mathbb{N}}$ is a complete orthonormal system of $L^2(\mathbb{R}^3_\xi)$. The brackets $(\cdot, \cdot)$ denote the inner product in $L^2(\mathbb{R}^3_\xi)$. Applying the Friedrichs' mollifier to $\mathbb{K}_n = = \mathbb{K}_n(x)$, we obtain the present lemma.

## 3. – Main Theorem.

Define $\delta(\theta) \equiv \{\mu \in \mathbb{C};\ \mathrm{Re}\,\mu > -\theta\nu_0\}$, $0 \leqq \theta \leqq 1$, (see Assumption $\nu$ as for $\nu_0$). The following theorem is the main result of the present paper:

MAIN THEOREM. *The intersection of $\delta(1)$ and the spectrum of $B$ is a discrete set of points $\{\mu_k\}_{k \in \mathbb{N}}$ which satisfies the following* (i)-(iii):

(i) *$\{\mu_k\}_{k \in \mathbb{N}} \cap \delta(\theta)$ is a finite set for each $\theta \in [0, 1)$.*

(ii) *The operator $(\mu - B)$ is boundedly invertible for each $\mu \in \in \delta(1) \backslash \{\mu_k\}_{k \in \mathbb{N}}$.*

(iii) *Each $\mu_k$ is a pole of $(\mu - B)^{-1}$ and an eigenvalue of $B$ with finite algebraic multiplicity.*

PROOF. The following lemma will be proved in §4.

LEMMA 3.1. *$(\mu - A)^{-1}$ is an analytic operator-valued function of $\mu \in \delta(1)$.*

The following lemma is the key lemma, which will be proved in §5-8:

LEMMA 3.2. *$L = L(\mu) \equiv K(\mu - A)^{-1}$ is an analytic operator-*

*valued function of* $\mu \in \delta(1)$ *which satisfies the following* (i)-(ii) *for each* $\theta \in [0, 1)$:

(i) *If* $\mu \in \delta(\theta)$, *then* $L^4(\mu) \in \mathbb{C}(L^2(\Omega \times \mathbb{R}^3))$,

(ii) $|||L^4(\mu)||| \to 0$ *as* $|\mu| \to +\infty$, $\mu \in \delta(\theta)$.

The following theorem can be proved in the same way as [13, p. 107, Theorem XIII.13] and [12, p. 201, Theorem VI.14]:

THEOREM 3.3. *Let* $D$ *be a domain of* $\mathbb{C}$ *(i.e., a connected open subset of* $\mathbb{C}$*). Let* $f = f(\mu)$ *be an analytic operator-valued function of* $\mu \in D$ *such that* $f(\mu) \in \mathbb{C}(L^2(\Omega \times \mathbb{R}^3))$ *for each* $\mu \in D$. *Then, either*

(i) $(1 - f(\mu))^{-1}$ *exists for no* $\mu \in D$,

*or*

(ii) $(1 - f(\mu))^{-1}$ *exists for all* $\mu \in D \backslash S$, *where* $S$ *is a discrete subset of* $D$ *(i.e., a set which has no accumulation point in* $D$*). In this case,* $(1 - f(\mu))^{-1}$ *is meromorphic in* $D$ *and analytic in* $D \backslash S$. *For each* $\mu \in S$, *the coefficients of the negative terms of the Laurent series at* $\mu \in S$ *are finite rank operators.*

Let $\mu \in \delta(\theta)$, $0 \leqq \theta < 1$, in what follows. Making use of Lemma 3.1, we can set up

$$(\mu - B)^{-1} = (\mu - A)^{-1}(1 - L)^{-1} .$$

Applying the following equalities:

$$(I - L)(I + L + L^2 + L^3)(I - L^4)^{-1} = I ,$$

$$(I - L^4)^{-1}(I + L + L^2 + L^3)(I - L) = I ,$$

we see that

$$(\mu - B)^{-1} = (\mu - A)^{-1}(I + L + L^2 + L^3)(I - L^4)^{-1} .$$

Applying Assumption K, Lemmas 3.1-3.2 and Theorem 3.3 with $(f(\mu), D) = (L^4(\mu), \delta(\theta))$, $0 \leqq \theta < 1$, to this equality, and making use of the results in [8, pp. 180-181], we can obtain the Main Theorem.

REMARK 3.4. In §5 we will explain the reason for considering the 4-th power $L^4(\mu)$ in Lemma 3.2 in place of $L(\mu)$. In order to inspect $L^4(\mu)$, we need to investigate the behavior of the solutions of (SODE). The necessity for this investigation will also be discussed in §5.

## 4. - The Operator $A$.

PROOF OF LEMMA 3.1. We easily see that $A$ generates a strongly continuous semigroup. The semigroup $e^{tA}$ has the form,

$$(e^{tA}f(\cdot,\cdot))(X,\Xi)=f(x(-t,X,\Xi),\ \xi(-t,X,\Xi))\,e(t,X,\Xi),\tag{4.1}$$

where $e(t,X,\Xi)\equiv\exp\left(-\int_0^t \nu(x(-s,X,\Xi),\ \xi(-s,X,\Xi))\,ds\right)$.

Making use of Assumption $\nu$, we see that

$$e(t,X,\Xi)\leqq\exp(-\nu_0 t),\qquad t\geqq 0.\tag{4.2}$$

Applying (2.1) and (4.2) to the Laplace transformation of (4.1),

$$((\mu-A)^{-1}f(\cdot,\cdot))(X,\Xi)=\tag{4.3}$$

$$=\int_0^{+\infty}R(\mu,t,X,\Xi)f(x(-t,X,\Xi),\xi(-t,X,\Xi))\,dt,$$

we can obtain the present lemma, where

$$R(\mu,t,X,\Xi)\equiv e(t,X,\xi)\exp(-\mu t).$$

In view of (4.3), we define the following operator:

$$(\mathbb{R}(\mu,M)f(\cdot,\cdot))(X,\Xi)\equiv\tag{4.4}$$

$$\equiv\int_{t\in M}R(\mu,t,X,\Xi)f(x(-t,X,\Xi),\xi(-t,X,\Xi))\,dt,$$

where $M\subseteq[0,+\infty)$ is a Lebesgue measurable set. We can obtain the following lemma in the same way as [14, Lemma 3.2, (i)]:

LEMMA 4.1. *If $\beta\equiv\mathrm{Re}\,\mu+\nu_0>0$ and $M\subseteq[0,+\infty)$, then*

$$|||\mathbb{R}(\mu,M)|||\leqq\int_{t\in M}\exp(-\beta t)\,dt.$$

REMARK 4.2. Note that $\mathbb{R}(\mu,[0,+\infty))=(\mu-A)^{-1}$, and that $\mathbb{R}(\mu,\cdot)$ is an additive set function. Making use of Lemma 4.1, we can make an approximation of $(\mu-A)^{-1}$ with operators of the form $\mathbb{R}(\mu,[0,T])$, $1\leqq T<+\infty$. We will make use of this result in the next section.

## 5. – Discussion on Lemma 3.2.

If Lemma 3.2 is proved, then we can complete the proof of the Main Theorem. In this section we will discuss Lemma 3.2. The purpose of §6-8 is to prove Lemma 3.2.

Let us explain the reason for considering the 4-th power $L^4(\mu)$ in place of $L(\mu)$. If we were able to prove the following (5.1)-(5.2), then we could more easily obtain the Main Theorem (see §3 for $\delta(\theta)$):

$$L(\mu) \in \mathbb{C}(L^2(\Omega \times \mathbb{R}^3)) \quad \text{if} \quad \mu \in \delta(\theta) \text{ and } 0 \leqq \theta < 1, \tag{5.1}$$

$$|||L(\mu)||| \to 0 \quad \text{as} \quad |\mu| \to +\infty, \ \mu \in \delta(\theta), \quad \text{for each } 0 \leqq \theta < 1. \tag{5.2}$$

However, inspecting the integrand of (4.3), we see that $L(\mu)$ has the form,

$$(L(\mu)u(\cdot, \cdot))(X, \Xi) = \tag{5.3}$$

$$= \int_{\eta \in \mathbb{R}^3,\, t \geqq 0} K(X, \Xi, \eta) R(\mu, t, X, \eta) u(x(-t, X, \eta), \xi(-t, X, \eta))\, dt\, d\eta .$$

Inspecting the form of the integrand in (5.3), we can reasonably conclude that it is nearly impossible to prove that $L(\mu)$ satisfies estimates such as (5.1)-(5.2). However, reviewing the calculations performed in [5, p. 46], we recognize the possibility that a power $L^n(\mu)$, $n \in \mathbb{N}$, may satisfy estimates such as (5.1)-(5.2). If a power $L^n(\mu)$, $n \in \mathbb{N}$, satisfies such estimates, then we can prove the Main Theorem, as already performed in §3, under the difficult circumstances stated above. This is the reason for considering the 4-th power $L^4(\mu)$ in place of $L(\mu)$.

Next we will look for estimates which imply Lemma 3.2. Let us construct operators with which we can make an approximation of $L^4(\mu)$. By $\prod_{j=1}^{m} A_j$ we denote the product $A_m A_{m-1} \cdots A_2 A_1$ of the operators $A_j$, $j = 1, \dots, m$. Consider operators of the form,

$$\mathbb{G}(\mu, T) \equiv \prod_{j=1}^{4} \{k_j \mathbb{R}(\mu, [0, T))\}, \tag{5.4}$$

where $\mu \in \delta(\theta)$, $0 \leqq \theta < 1$, $1 \leqq T \leqq +\infty$, and $k_j \in Q$, $j = 1, \dots, 4$. Let $\theta \in [0, 1)$ be fixed. Making use of Lemma 4.1 and Remark 4.2, we see that $\mathbb{G}(\mu, T) \to \mathbb{G}(\mu, +\infty)$ in $\mathbb{B}(L^2(\Omega \times \mathbb{R}^3))$ as $T \to +\infty$ uniformly for $\mu \in \delta(\theta)$. Furthermore, applying Lemma 4.1 and Lemma 2.2, we can make an approximation of $L^4(\mu)$ with a finite sum of operators of the form (5.4) with $T = +\infty$ uniformly for $\mu \in \delta(\theta)$. Therefore Lemma 3.2

can be derived from the following conditions:

(5.5) $$\mathbb{G}(\mu, T) \in \mathbb{C}(L^2(\Omega \times \mathbb{R}^3))$$

$$\text{if } \mu \in \delta(\theta), \quad 0 \leqq \theta < 1, \quad \text{and } 1 \leqq T < +\infty\,;$$

(5.6) $$|||\mathbb{G}(\mu, T)||| \to 0 \quad \text{as } |\mu| \to +\infty, \quad \mu \in \delta(\theta),$$

$$\text{for each } 0 \leqq \theta < 1 \quad \text{and for each } 1 \leqq T < +\infty\,.$$

Consequently we have only to show (5.5)-(5.6), which will be proved in §8.

Let us inspect the integration kernel of $\mathbb{G}(\mu, T)$. Write $(x_4, \xi_4) \in \Omega \times \mathbb{R}^3$ as the variables of $\mathbb{G}(\mu, T)u$, i.e., $\mathbb{G}(\mu, T)u = (\mathbb{G}(\mu, T) \cdot u(\cdot, \cdot))(x_4, \xi_4)$. Iterating calculations in obtaining (5.3), we obtain

(5.7) $$(\mathbb{G}(\mu, T)u(\cdot, \cdot))(x_4, \xi_4) = \int_{0 \leqq t_j \leqq T,\ |\eta_j| \leqq \boldsymbol{R},\ j = 1, \ldots, 4} Gu(x_0, \xi_0)\,dt\,d\eta\,,$$

where $dt \equiv dt_1 \cdots dt_4$, $d\eta \equiv d\eta_1 \cdots d\eta_4$, and

(5.8) $$G = G(\mu, x_4, \xi_4; \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1) \equiv$$

$$\equiv \prod_{j=1}^{4} \{k_j(x_j, \xi_j, \eta_j) R(\mu, t_j, x_j, \eta_j)\}\,.$$

$(x_j, \xi_j)$, $j = 0, \ldots, 3$, are defined inductively by the following equalities:

(5.9) $$x_j = x(-t_{j+1}, x_{j+1}, \eta_{j+1}), \qquad \xi_j = \xi(-t_{j+1}, x_{j+1}, \eta_{j+1}),$$

$$j = 0, \ldots, 3\,.$$

$k_j(x_j, \xi_j, \eta_j)$, $j = 1, \ldots, 4$, denote the integration kernels of $k_j \in Q$, $j = 1, \ldots, 4$, respectively, i.e.,

$$(k_j f(\cdot, \cdot))(x_j, \xi_j) = \int k_j(x_j, \xi_j, \eta_j) f(x_j, \eta_j)\,d\eta_j, \quad j = 1, \ldots, 4\,.$$

$\boldsymbol{R} > 0$ is a constant so large that for any $x \in \Omega$ supp $k_j(x, \cdot, \cdot)$, $j = 1, \ldots, 4$, are contained in $\{\xi;\ |\xi| \leqq \boldsymbol{R}\} \times \{\eta;\ |\eta| \leqq \boldsymbol{R}\}$. In (5.7), for convenience, we extend the interval of integration with respect to $t$ from $[0, T)$ to $[0, T]$.

From (2.3), we see that $\max\limits_{x_j, \xi_j, \eta_j} |k_j(x_j, \xi_j, \eta_j)| \leqq c_j$, $j = 1, \ldots, 4$, where $c_j$, $j = 1, \ldots, 4$, are some positive constants. Applying these inequalities and (4.2), we deduce that if $t_j \geqq 0$, $j = 1, \ldots, 4$, and $\mu \in \delta(\theta)$,

$0 \leqq \theta < 1$, then

$$(5.10)\qquad |G(\mu, x_4, \xi_4; \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1)| \leqq$$
$$\leqq \prod_{j=1}^{4} \{c_j \exp(-(1-\theta)\nu_0 t_j)\}.$$

Making use of the conservation law of energy, $\phi(x_0) + |\xi_0|^2/2 = \phi(x_1) + |\eta_1|^2/2$, we see that if $|\eta_1| \leqq \boldsymbol{R}$, then $|\xi_0| \leqq (4\max_{x \in \Omega} |\phi(x)| + \boldsymbol{R}^2)^{1/2}$, where $\boldsymbol{R}$ is the constant defined above. Noting that $\xi_0$ is a function of $\omega_1 \equiv (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1)$, recalling the definition of $\boldsymbol{R}$, and noting that $G$ contains $k_1(x_1, \xi_1, \eta_1)$, we see that if $\omega_1$ satisfies the inequality, $|\xi_0| > (4\max_{x \in \Omega} |\phi(x)| + \boldsymbol{R}^2)^{1/2}$, then

$$(5.11)\qquad G(\mu, x_4, \xi_4; \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1) = 0.$$

We will make use of (5.10)-(5.11) in §8.

REMARK 5.1. (i) We have obtained the integration kernel of $\mathbb{G}(\mu, T)$ in (5.7). However the true character of the integration kernel is still obscure. We need to extract the integration kernel explicitly. Note that $u = u(\cdot, \cdot)$ contains $(x_0, \xi_0)$ as variable in (5.7), and that $(x_0, \xi_0)$ is a function of $\omega_1 \equiv (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1)$. By inspecting (5.7)-(5.9), we recognize the need to study the rank of the Jacobian matrix,

$$\mathbb{J} = \mathbb{J}(\omega_1) \equiv \partial(x_0, \xi_0)/\partial(\eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1),$$

in order to extract the kernel. For this purpose we need to first calculate the rank of the Jacobian matrix $J = J(t, X, \Xi)$, (see §2 for $J = J(t, X, \Xi)$). This is the reason why we need to investigate the behavior of the solutions of (SODE).

(ii) We can prove Lemma 3.2 also by the method in [14]. However, the method in [14] is very complicated. The method in the present paper is much simpler than that in [14]. In [14, Lemma 8.2] we consider $j(\omega_1) \equiv \partial(x_0, \xi_0)/\partial(\eta_4, \ldots, \eta_1)$. In this paper we consider $\mathbb{J} = \mathbb{J}(\omega_1)$ in place of $j(\omega_1)$. This approach represents the difference between the method in the present paper and that in [14].

## 6. - Estimates for the rank of a Jacobian matrix.

The main result of this section is Lemma 6.1, which deals with the rank of $J = J(t, X, \Xi)$. Lemma 6.3 will be employed in §7 in order to calculate the rank of $\mathbb{J} = \mathbb{J}(\omega_1)$ (see Remark 5.1, (i)).

We denote the i-th row vectors of $J = J(t, X, \Xi)$ by $J_i = J_i(t, X, \Xi)$, i.e., we define $J_i = J_i(t, X, \Xi) \equiv (m_{i1}, \dots, m_{i6})$, $i = 1, \dots, 6$, (see §2 for $m_{ij}$).

Let $b_j$, $j = 1, \dots, N$, be linearly independent vectors in $\mathbb{R}^n$. We orthogonalize these vectors, i.e., we define $b_{j,\perp}$, $j = 1, \dots, N$, as follows (we do not normalize them):

$$b_{1,\perp} \equiv b_1,$$

$$b_{m+1,\perp} \equiv b_{m+1} - \sum_{k=1}^{m} (b_{m+1} \cdot b_{k,\perp}) b_{k,\perp} / |b_{k,\perp}|^2, \qquad m = 1, \dots, N-1.$$

LEMMA 6.1. *Let $1 \leqq r, T < +\infty$. There exists a positive constant $c_{6.1}$ such that if $(X, \Xi) \in D(r) \equiv \{(X, \Xi) \in \Omega \times \mathbb{R}^3;\ |\Xi| \leqq r\}$ and if $t \in [0, T]$, then*

$$6^{-5/2} \exp(-5 c_{6.1} t) \leqq |J_{i,\perp}(t, X, \Xi)| \leqq |J_i(t, X, \Xi)| \leqq 6^{1/2} \exp(c_{6.1} t),$$
$$i = 1, \dots, 6.$$

REMARK 6.2. The constant $c_{6.1}$ is independent of $(t, X, \Xi)$. We will make use of this fact in the next section.

PROOF OF LEMMA 6.1. Let $(t, X, \Xi) \in [0, T] \times D(r)$ in what follows. Differentiating both sides of (SODE) with respect to $X$ and $\Xi$, and applying Assumption $\phi$, we have

$$\|J(t, X, \Xi)\| \leqq \|J(0, X, \Xi)\| \exp(c_{6.1} t), \qquad t \geqq 0, \tag{6.1}$$

where $\|\cdot\|$ denotes a norm of matrices defined as follows: $\|(a_{ij})\| \equiv \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}$. $c_{6.1}$ is some positive constant dependent on $\sup_{x \in \Omega,\, i,j = 1,2,3} |\partial^2 \phi(x)/\partial x_i \partial x_j|$ but independent of $(t, X, \Xi)$. We easily obtain the following inequalities and equalities:

$$|J_{i,\perp}(t, x, \Xi)| \leqq |J_i(t, X, \Xi)| \leqq \|J(t, X, \Xi)\|, \qquad i = 1, \dots, 6,$$

$$|J_{1,\perp}(t, X, \Xi)| \cdots |J_{6,\perp}(t, X, \Xi)| = |\det(J(t, X, \Xi))|,$$

$$\|J(0, X, \Xi)\| = 6^{1/2}.$$

Combining these, (2.1) and (6.1), we obtain the lemma.

If $a_j$, $j = 1, \dots, N$, are linearly independent vectors of $\mathbb{R}^M$, $N, M \in$

$\in \mathbb{N}$, then we define $W(a_1, \dots, a_N) \equiv \min_{j=1,\dots,N} |a_{j,\perp}|$. If those vectors are linearly dependent, then we define $W(a_1, \dots, a_N) \equiv 0$. Let $b_j = b_j(z)$, $j = 1, \dots, N$, be $\mathbb{R}^M$-vector-valued functions of $z \in D$, where $D$ is a domain of $\mathbb{R}^m$, $m \in \mathbb{N}$. If $\operatorname{ess\,inf}_{z \in D} W(b_1(z), \dots, b_N(z)) > 0$, then we say that $b_j = b_j(z)$, $j = 1, \dots, N$, are *uniformly linearly independent* for almost all $z \in D$.

We will make use of the following lemma in the next section.

LEMMA 6.3. (i) *Let* $1 \leqq r, T < +\infty$ *be constants. The* 6-*dimensional vectors* $J_i = J_i(t, X, \Xi)$, $i = 1, \dots, 6$, *are uniformly linearly independent for almost all* $(t, X, \Xi) \in [0, T] \times D(r)$ *and are uniformly bounded in* $[0, T] \times D(r)$. *See Lemma* 6.1 *for* $D(r)$.

(ii) *Let* $D$ *be a domain of* $\mathbb{R}^N$, $N \in \mathbb{N}$. *Let* $M = M(z)$ *be a* $3 \times 6$-*matrix-valued, Lebesgue measurable function of* $z \in D$ *whose row vectors are uniformly linearly independent for almost all* $z \in D$ *and are essentially bounded in* $D$. *Let* $n$ *be an integer such that* $4 \leqq n \leqq 6$. *Let* $v(j) = v(j; z)$, $j = 1, \dots, n$, *be 6-dimensional-column-vector-valued, Lebesgue measurable functions of* $z \in D$ *which are uniformly linearly independent for almost all* $z \in D$ *and are essentially bounded in* $D$. *Then we can choose* $(n-3)$ *vectors from* $M(z)v(j; z)$, $j = 1, \dots, n$, *for* $z \in D$ *almost everywhere so that those* $(n-3)$ *vectors are uniformly linearly independent for almost all* $z \in D$, *i.e., so that there exist Lebesgue measurable, integer-valued functions* $L_k = L_k(z)$, $k = 1, \dots, n-3$, *such that*

(1) $L_k(z) \in \{1, \dots, n\}$ *for each* $k = 1, \dots, n-3$,

(2) $L_i(z) \neq L_j(z)$ *if* $i \neq j$, $i, j = 1, \dots, n-3$,

(3) $M(z)v(L_k(z); z)$, $k = 1, \dots, n-3$, *are uniformly linearly independent for almost all* $z \in D$.

REMARK 6.4. Let $n$ be an integer such that $4 \leqq n \leqq 6$. If $M$ is a $3 \times 6$ matrix whose row vectors are linearly independent, and if $v_j$, $j = 1, \dots, n$, are linearly independent 6-dimensional column vectors, then we can choose $(n-3)$ vectors from $Mv_j$, $j = 1, \dots, n$, so that those $(n-3)$ vectors are linearly independent. Lemma 6.3,(ii) is an extension of this result.

PROOF OF LEMMA 6.3. (i) follows from Lemma 6.1 immediately. Let us prove (ii) when $n = 6$. We can prove (ii) when $n = 4, 5$ in the same way. Define

$$\boldsymbol{H} \equiv \{(n_1, n_2, n_3);\ n_k \in \{1, \dots, 6\},\ k = 1, 2, 3,\ n_1 < n_2 < n_3\}.$$

The number of elements of $\boldsymbol{H}$ is equal to ${}_6C_3 = 20$. We number the elements of $\boldsymbol{H}$ from 1 to 20, and denote them by $(n_{j1}, n_{j2}, n_{j3}), j = 1, \dots, 20$, in numerical order.

Let $\delta > 0$ be a sufficiently small constant, and define $\Phi_j = \Phi_j(\delta)$, $j = 1, \dots, 20$, as follows:

$$\Phi_j(\delta) \equiv \{z \in D;\ W(M(z)v(n_{j1}; z), M(z)v(n_{j2}; z), M(z)v(n_{j3}; z)) \geqq \delta\},$$
$$j = 1, \dots, 20.$$

We define $\Psi_j = \Psi_j(\delta)$, $j = 1, \dots, 20$, as follows: $\Psi_1 \equiv \Phi_1$, $\Psi_j \equiv \Phi_j \setminus \setminus(\Phi_1 \cup \cdots \cup \Phi_{j-1})$, $j = 2, \dots, 20$. Note that $\Psi_i$ and $\Psi_j$ are disjoint if $i \neq j$, and that $\bigcup_{j=1}^{20} \Psi_j = \bigcup_{j=1}^{20} \Phi_j$. We define $L_k = L_k(z)$, $k = 1, 2, 3$, in $\bigcup_{j=1}^{20} \Psi_j$ as follows: if $z \in \Psi_j$, then $L_k(z) \equiv n_{jk}$, $k = 1, 2, 3$, $j = 1, \dots, 20$.

If $D \setminus \bigcup_{j=1}^{20} \Phi_j(\delta)$ is a null set for some $\delta > 0$, then we can complete the proof. We will prove this condition by contradiction. Suppose that $D \setminus \bigcup_{j=1}^{20} \Phi_j(\delta)$ is not a null set for any $\delta > 0$. Making use of this assumption and the conditions of the present lemma, we see that there exists a sequence $\{z_n\}_{n \in \mathrm{N}} \subseteq D$ which satisfies the following conditions:

$$\text{(6.2)} \quad \sup_n \|M(z_n)\| < +\infty, \qquad \sup_n |v(l; z_n)| < +\infty, \qquad l = 1, \dots, 6,$$

$$\text{(6.3)} \quad \inf_n W(m_1(z_n), m_2(z_n), m_3(z_n)) > 0, \ \inf_n W(v(1; z_n), \dots, v(6; z_n)) > 0,$$

$$\text{(6.4)} \quad W(M(z_n)v(n_{j1}; z_n), M(z_n)v(n_{j2}; z_n), M(z_n)v(n_{j3}; z_n)) \to 0$$
$$\text{as } n \to +\infty, \ j = 1, \dots, 20,$$

where we denote by $m_k = m_k(z)$ the $k$-th row vector of $M(z)$, $k = = 1, 2, 3$. By (6.2), we can choose a subsequence of $\{z_n\}_{n \in \mathrm{N}}$, denoted by the same symbol $\{z_n\}_{n \in \mathrm{N}}$ again, so that $M(z_n)$ and $v(l; z_n)$, $l = 1, \dots, 6$, converge as $n \to +\infty$. Therefore, from (6.3), we deduce that $\lim_{n \to +\infty} M(z_n)$ is a $3 \times 6$ matrix whose row vectors are linearly independent and that $\lim_{n \to +\infty} v(l; z_n)$, $l = 1, \dots, 6$, are linearly independent 6-dimensional vectors. However, it follows from (6.4) that the rank of

$\lim_{n \to +\infty} M(z_n) v(l; z_n)$, $l = 1, \dots, 6$, is smaller than 3. This is a contradiction (see Remark 6.4).

## 7. – Estimates for some Jacobian.

The purpose of this section is to calculate the rank of $\mathbb{J} = \mathbb{J}(\omega_1)$ by making use of Lemma 6.3. The main result of this section is Lemma 7.4, which will be employed in §8 in order to prove (5.5)-(5.6).

Let $r$ and $T$ be positive constants. Write $\Omega_j = \Omega_j(r, T)$, $j = 1, 2, 3, 4$, as the sets of all vectors of the following forms respectively:

$$\omega_1 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1), \quad \omega_2 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2),$$

$$\omega_3 = (x_4, \eta_4, t_4, \eta_3, t_3), \quad \omega_4 = (x_4, \eta_4, t_4),$$

where $x_4 \in \Omega$, $|\eta_j| \leqq r$, $\eta_j \in \mathbb{R}^3$, $0 \leqq t_j \leqq T$, $j = 1, \dots, 4$. We can regard $(x_j, \xi_j)$, $j = 0, \dots, 3$, in (5.9) as functions of $\omega_{j+1} \in \Omega_{j+1}$, $j = 0, \dots, 3$, respectively. By $x_{i,j}$, $\eta_{i,j}$ we denote the $j$-th component of $x_i$, $\eta_i \in \mathbb{R}^3$ respectively, $j = 1, 2, 3$, that is,

$$x_i = (x_{i,1}, x_{i,2}, x_{i,3}), \qquad \eta_i = (\eta_{i,1}, \eta_{i,2}, \eta_{i,3}).$$

Let $c_j \in \mathbb{R}^m$, $j = 1, \dots, N$, and $\varepsilon > 0$. By $U(c_1, \dots, c_N; \varepsilon)$ we denote the set of all $x \in \mathbb{R}^m$ whose distance from the subspace spanned by $c_j$, $j = 1, \dots, N$, is greater than or equal to $\varepsilon$.

In the present section we demonstrate Lemmas 7.1-7.4. By making use of Lemma 7.$j$, we prove Lemma 7.$j+1$, $j = 1, 2, 3$.

LEMMA 7.1. (i) $\operatorname{ess\,sup}_{\omega_4 \in \Omega_4} \|\partial(x_3)/\partial(\eta_4, t_4)\| < +\infty$, *where* $\|\cdot\|$ *is the norm of matrices defined in Proof of Lemma* 6.1.

(ii) Let $\varepsilon > 0$. *There exists a Lebesgue-measurable, integer-valued function* $G = G(\omega_4)$ *which satisfies the following conditions for a.e.* $\omega_4 \in \mathbf{\Omega}_4(\varepsilon) \equiv \{\omega_4 = (x_4, \eta_4, t_4) \in \Omega_4;\ |\eta_4| \geqq \varepsilon\}$:

(1) $G(\omega_4) \in \{1, \dots, 4\}$,

(2) $c_{7.1} \leqq |a(G(\omega_4), \omega_4)|$,

*where* $c_{7.1}$ *is a positive constant dependent on* $\varepsilon$ *but independent of* $\omega_4$. *By* $a(k, \omega_4)$ *we denote the k-th column vector of the Jacobian matrix* $\partial(x_3)/\partial(\eta_4, t_4)$, $k = 1, \dots, 4$.

REMARK. Lemma 7.1 deals with the rank of the Jacobian matrix

$\partial(x_3)/\partial(\eta_4, t_4)$. This lemma shows that we can choose one column vector from $\partial(x_3)/\partial(\eta_4, t_4)$ for $\omega_4 \in \mathbf{\Omega}_4(\varepsilon)$ almost everywhere so that the essential infimum (in $\mathbf{\Omega}_4(\varepsilon)$) of the norm of that column vector is positive.

PROOF OF LEMMA 7.1. We can obtain (i) easily. Let us prove (ii). We introduce the following column vectors:

$$\alpha_1 = {}^t(0, 0, 0, 1, 0, 0), \quad \alpha_2 = {}^t(0, 0, 0, 0, 1, 0), \quad \alpha_3 = {}^t(0, 0, 0, 0, 0, 1),$$

$$\alpha_4 = {}^t(-\eta_{4,1}, -\eta_{4,2}, -\eta_{4,3}, \partial\phi(x_4)/\partial x_{4,1}, \partial\phi(x_4)/\partial x_{4,2}, \partial\phi(x_4)/\partial x_{4,3}),$$

where the superscript $^t$ denotes the transposition. We have already introduced the following column vectors:

$$a(k, \omega_4) = (\partial/\partial\eta_{4,k})\,{}^t(x_{3,1}, x_{3,2}, x_{3,3}), \qquad k = 1, 2, 3,$$

$$a(4, \omega_4) = (\partial/\partial t_4)\,{}^t(x_{3,1}, x_{3,2}, x_{3,3}).$$

We can easily obtain

$$\text{(7.1)} \qquad \partial x(-t, X, \Xi)/\partial t = -\Lambda(X, \Xi)\, x(-t, X, \Xi), \qquad \text{for any } (t, X, \Xi),$$

where $\Lambda(X, \Xi) \equiv \Xi \cdot \nabla_X - \nabla_X \phi(X) \cdot \nabla_\Xi$. This equality will play a very important role. By making use of (7.1), we see that

$$a(4, \omega_4) = -\Lambda(x_4, \eta_4)\,{}^t(x_{3,1}, x_{3,2}, x_{3,3}).$$

Applying this and considering the definitions of the above column vectors, we have

$$\text{(7.2)} \qquad a(k, \omega_4) = (\partial(x_3)/\partial(x_4, \eta_4))\,\alpha_k, \qquad k = 1, \dots, 4,$$

where if $1 \leqq i, j \leqq 3$, then the $(i, j)$ component of $\partial(x_3)/\partial(x_4, \eta_4)$ is equal to $\partial x_{3,i}/\partial x_{4,j}$. If $1 \leqq i \leqq 3$ and $4 \leqq j \leqq 6$, then the $(i, j)$ component of $\partial(x_3)/\partial(x_4, \eta_4)$ is equal to $\partial x_{3,i}/\partial \eta_{4,j-3}$.

We easily deduce that

$$\text{(7.3)} \qquad \alpha_i \cdot \alpha_j = \delta_{ij}, \qquad i, j = 1, 2, 3,$$

where $\delta_{ij}$ denotes the Kronecker's delta. It follows immediately from $|\eta_4| \geqq \varepsilon$ that

$$\text{(7.4)} \qquad \alpha_4 \in U(\alpha_1, \alpha_2, \alpha_3; \varepsilon).$$

Hence we see that $\alpha_j$, $j = 1, \dots, 4$, are 6-dimensional vectors uniformly linearly independent for almost all $\omega_4 \in \mathbf{\Omega}_4(\varepsilon)$. By Assumption $\phi$, we easily see that $\alpha_j, j = 1, \ldots, 4$, are uniformly bounded in $\mathbf{\Omega}_4(\varepsilon)$. Applying these results, Lemma 6.3, (i) with $(t, X, \Xi) = (t_4, x_4, \eta_4)$, and Lemma 6.3, (ii) with $(v(j), n, M) = (\alpha_j, 4, \partial(x_3)/\partial(x_4, \eta_4))$ to (7.2), we obtain Lemma 7.1.

LEMMA 7.2. (i) $\operatorname*{ess\,sup}_{\omega_3 \in \Omega_3} |\partial(x_2)/\partial(\eta_4, t_4, \eta_3, t_3)| < +\infty$.

(ii) *Let $\varepsilon > 0$. There exist Lebesgue-measurable, integer-valued functions $H_j = H_j(\omega_3)$, $j = 1, 2$, such that for a.e.*

$$\omega_3 \in \mathbf{\Omega}_3(\varepsilon) \equiv \{\omega_3 = (x_4, \eta_4, t_4, \eta_3, t_3) \in \Omega_3\ ;$$

$$\omega_4 = (x_4, \eta_4, t_4) \in \mathbf{\Omega}_4(\varepsilon),\ \eta_3 \in U(a(G(\omega_4), \omega_4); \varepsilon)\},$$

(1) $H_j(\omega_3) \in \{1, \ldots, 8\}$, $j = 1, 2$,

(2) $H_1(\omega_3) \neq H_2(\omega_3)$,

(3) $b(H_j(\omega_3), \omega_3)$, $j = 1, 2$, *are uniformly linearly independent for a.e.* $\omega_3 \in \mathbf{\Omega}_3(\varepsilon)$,

*where $a(\cdot, \omega_4)$ and $G = G(\omega_4)$ are those in Lemma 7.1. By $b(k, \omega_3)$ we denote the k-th column vector of $\partial(x_2)/\partial(\eta_4, t_4, \eta_3, t_3)$, $k = 1, \ldots, 8$.*

REMARK. Lemma 7.2 deals with the rank of the Jacobian matrix $\partial(x_2)/\partial(\eta_4, t_4, \eta_3, t_3)$. This lemma shows that we can choose two column vectors from the Jacobian matrix $\partial(x_2)/\partial(\eta_4, t_4, \eta_3, t_3)$ for $\omega_3 \in \mathbf{\Omega}_3(\varepsilon)$ almost everywhere so that those column vectors are uniformly linearly independent for almost all $\omega_3 \in \mathbf{\Omega}_3(\varepsilon)$.

PROOF OF LEMMA 7.2. We can obtain (i) easily. Let us prove (ii). We introduce the following column vectors:

$$\beta_k = {}^t(\partial x_{3,1}/\partial \eta_{4,k}, \partial x_{3,2}/\partial \eta_{4,k}, \partial x_{3,3}/\partial \eta_{4,k}, 0, 0, 0), \qquad k = 1, 2, 3,$$

$$\beta_4 = {}^t(\partial x_{3,1}/\partial t_4, \partial x_{3,2}/\partial t_4, \partial x_{3,3}/\partial t_4, 0, 0, 0),$$

$$\beta_5 = {}^t(0, 0, 0, 1, 0, 0), \quad \beta_6 = {}^t(0, 0, 0, 0, 1, 0), \quad \beta_7 = {}^t(0, 0, 0, 0, 0, 1),$$

$$\beta_8 = {}^t(-\eta_{3,1}, -\eta_{3,2}, -\eta_{3,3}, \partial\phi(x_3)/\partial x_{3,1}, \partial\phi(x_3)/\partial x_{3,2}, \partial\phi(x_3)/\partial x_{3,3}),$$

$$\beta_9 = {}^t(\gamma_1, \gamma_2, \gamma_3, 0, 0, 0),$$

where $\gamma_j$ is the $j$-th component of $a(G(\omega_4), \omega_4), j = 1, 2, 3$. We note that for a.e. $\omega_4 \in \mathbf{\Omega}_4(\varepsilon)$ there exists $k \in \{1, \ldots, 4\}$ such that $\beta_k = \beta_9$. We have already introduced the following column vectors:

$$b(k, \omega_3) = (\partial/\partial\eta_{4,k})^t(x_{2,1}, x_{2,2}, x_{2,3}), \quad k = 1, 2, 3,$$

$$b(4, \omega_3) = (\partial/\partial t_4)^t(x_{2,1}, x_{2,2}, x_{2,3}),$$

$$b(k, \omega_3) = (\partial/\partial\eta_{3,k-4})^t(x_{2,1}, x_{2,2}, x_{2,3}), \quad k = 5, 6, 7,$$

$$b(8, \omega_3) = (\partial/\partial t_3)^t(x_{2,1}, x_{2,2}, x_{2,3}).$$

By making use of (7.1), we see that

$$b(8, \omega_3) = -\Lambda(x_3, \eta_3)^t(x_{2,1}, x_{2,2}, x_{2,3}).$$

Applying this and considering the definitions of the above column vectors, we have

$$(7.5) \qquad b(k, \omega_3) = (\partial(x_2)/\partial(x_3, \eta_3))\beta_k, \quad k = 1, \dots, 8,$$

where if $1 \leqq i, j \leqq 3$, then the $(i, j)$ component of $\partial(x_2)/\partial(x_3, \eta_3)$ is equal to $\partial x_{2,i}/\partial x_{3,j}$. If $1 \leqq i \leqq 3$ and $4 \leqq j \leqq 6$, then the $(i, j)$ component of $\partial(x_2)/\partial(x_3, \eta_3)$ is equal to $\partial x_{2,i}/\partial\eta_{3,j-3}$.

We easily deduce that

$$(7.6) \qquad \beta_i \cdot \beta_j = \delta_{ij}, \qquad i = 5, 6, 7, 9, \quad j = 5, 6, 7.$$

It follows immediately from $\eta_3 \in U(a(G(\omega_4), \omega_4); \varepsilon)$ that

$$(7.7) \qquad \beta_8 \in U(\beta_5, \beta_6, \beta_7, \beta_9; \varepsilon).$$

Making use of Lemma 7.1, (ii), and (7.6)-(7.7), we see that $\beta_j$, $j = 5, \dots, 9$, are 6-dimensional vectors uniformly linearly independent for almost all $\omega_3 \in \mathbf{\Omega}_3(\varepsilon)$. By Assumption $\phi$ and Lemma 7.1, (i), we easily see that $\beta_j$, $j = 5, \dots, 9$, are essentially bounded in $\mathbf{\Omega}_3(\varepsilon)$. Applying these results, Lemma 6.3, (i) with $(t, X, \Xi) = (t_3, x_3, \eta_3)$, and Lemma 6.3, (ii) with $(v(j), n, M) = (\beta_{j+4}, 5, \partial(x_2)/\partial(x_3, \eta_3))$, we see that we can choose two vectors from $b(k, \omega_3)$, $k = 5, \dots, 9$, for $\omega_3 \in \mathbf{\Omega}_3(\varepsilon)$ almost everywhere so that those vectors are uniformly linearly independent for almost all $\omega_3 \in \mathbf{\Omega}_3(\varepsilon)$, where $b(9, \omega_3) \equiv (\partial(x_2)/\partial(x_3, \eta_3))\beta_9$. Recalling the definition of $\beta_9$, we obtain Lemma 7.2.

LEMMA 7.3. (i) $\operatorname*{ess\,sup}_{\omega_2 \in \Omega_2} |\partial(x_1)/\partial(\eta_4, t_4, \eta_3, t_3, \eta_2, t_2)| < +\infty$.

(ii) *Let* $\varepsilon > 0$. *There exist Lebesgue-measurable, integer-valued functions* $I_j = I_j(\omega_2)$, $j = 1, 2, 3$, *such that for a.e.*

$$\omega_2 \in \mathbf{\Omega}_2(\varepsilon) \equiv \{\omega_2 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2) \in \Omega_2;$$

$\omega_3 = (x_4, \eta_4, t_4, \eta_3, t_3) \in \mathbf{\Omega}_3(\varepsilon), \eta_2 \in U(b(H_1(\omega_3), \omega_3), b(H_2(\omega_3), \omega_3); \varepsilon)\}$,

(1) $I_j(\omega_2) \in \{1, \dots, 12\}$ *for* $j = 1, 2, 3$,

(2) $I_i(\omega_2) \neq I_j(\omega_2)$ *if* $i \neq j$,

(3) $c(I_j(\omega_2), \omega_2)$, $j = 1, 2, 3$, *are uniformly linearly independent for a.e.* $\omega_2 \in \mathbf{\Omega}_2(\varepsilon)$,

*where* $b(\cdot, \omega_3)$ *and* $H_j = H_j(\omega_3)$, $j = 1, 2$, *are those in Lemma* 7.2. *By* $c(k, \omega_2)$ *we denote the k-th column vector of* $\partial(x_1)/\partial(\eta_4, t_4, \eta_3, t_3, \eta_2, t_2)$, $k = 1, \dots, 12$.

Remark. Lemma 7.3 deals with the rank of the Jacobian matrix $\partial(x_1)/\partial(\eta_4, t_4, \eta_3, t_3, \eta_2, t_2)$. This lemma shows that we can choose 3 column vectors from the Jacobian matrix $\partial(x_1)/\partial(\eta_4, t_4, \eta_3, t_3, \eta_2, t_2)$ for $\omega_2 \in \mathbf{\Omega}_2(\varepsilon)$ almost everywhere so that those column vectors are uniformly linearly independent for almost all $\omega_2 \in \mathbf{\Omega}_2(\varepsilon)$. By making use of Lemma 7.2, we can obtain Lemma 7.3 in the same way as that in obtaining Lemma 7.2 from Lemma 7.1. Hence we will omit the proof.

Noting that if $\omega_j \in \mathbf{\Omega}_j(\varepsilon)$, then $\omega_{j+1} \in \mathbf{\Omega}_{j+1}(\varepsilon)$, $j = 2, 3$, we decompose $\Omega_1 = \Omega_1(r, T)$ into four disjoint subsets as follows: $\Omega_1 \equiv N(\varepsilon) \cup \left(\bigcup_{j=2}^{4} S_j(\varepsilon)\right)$, where

$$S_4(\varepsilon) \equiv \{\omega_1 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1) \in \Omega_1; \\ \omega_4 = (x_4, \eta_4, t_4) \notin \mathbf{\Omega}_4(\varepsilon)\},$$

$$S_3(\varepsilon) \equiv \{\omega_1 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1) \in \Omega_1; \\ \omega_4 = (x_4, \eta_4, t_4) \in \mathbf{\Omega}_4(\varepsilon), \omega_3 = (x_4, \eta_4, t_4, \eta_3, t_3) \notin \mathbf{\Omega}_3(\varepsilon)\},$$

$$S_2(\varepsilon) \equiv \{\omega_1 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1) \in \Omega_1; \\ \omega_3 = (x_4, \eta_4, t_4, \eta_3, t_3) \in \mathbf{\Omega}_3(\varepsilon), \omega_2 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2) \notin \mathbf{\Omega}_2(\varepsilon)\},$$

$$N(\varepsilon) \equiv \{\omega_1 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1) \in \Omega_1; \\ \omega_2 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2) \in \mathbf{\Omega}_2(\varepsilon)\}.$$

Write $\zeta(i)$, $i = 1, \dots, 15$, as $\eta_{j,k}$, $j = 1, \dots, 4$, $k = 1, 2, 3$, $t_j$, $j = 2, 3, 4$, respectively.

Lemma 7.4. (i) *For any* $\varepsilon > 0$,

$$\operatorname{meas}\{\eta_j \in \mathbb{R}^3; \omega_1 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1) \in S_j(\varepsilon)\} \leqq c\varepsilon^{j-1}, \\ j = 2, 3, 4,$$

*where c depends on r but is independent of T and* $\varepsilon$.

(ii) *Let* $\varepsilon > 0$. *There exist Lebesgue-measurable, integer-valued functions* $K_i = K_i(\omega_1)$, $i = 1, \dots, 6$, *such that for a.e.* $\omega_1 \in N(\varepsilon)$,

(1) $K_i(\omega_1) \in \{1, \dots, 15\}$,

(2) $K_i(\omega_1) \neq K_j(\omega_1)$ *if* $i \neq j$,

(3) $\operatorname*{ess\,inf}_{\omega_1 \in N(\varepsilon)} |\det(\mathbb{K}(\omega_1))| > 0$,

where $\mathbb{K}(\omega_1) \equiv \partial(x_0, \xi_0)/\partial(\zeta(K_1(\omega_1)), \dots, \zeta(K_6(\omega_1)))$.

PROOF. Recalling the definitions of $S_j(\varepsilon)$, $j = 2, 3, 4$, and that of $U(\cdot; \varepsilon)$, we easily obtain (i). Making use of Lemma 7.3, (ii), we can choose three column vectors from $\partial(x_1, \eta_1)/\partial(\eta_4, t_4, \eta_3, t_3, \eta_2, t_2)$ for $\omega_1 \in N(\varepsilon)$ almost everywhere so that the three vectors are uniformly linearly independent for almost all $\omega_1 \in N(\varepsilon)$. We easily see that these three vectors and the three column vectors of $\partial(x_1, \eta_1)/\partial(\eta_1)$ are uniformly linearly independent for almost all $\omega_1 \in N(\varepsilon)$. Consequently, we can choose six column vectors from $F \equiv$ $\equiv \partial(x_1, \eta_1)/\partial(\eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1)$ for $\omega_1 \in N(\varepsilon)$ almost everywhere so that those six vectors are uniformly linearly independent for almost all $\omega_1 \in N(\varepsilon)$. By making use of Assumption $\phi$, we easily see that $F$ is uniformly bounded in $\Omega_1(r, T)$. Applying these results and Lemma 6.3,(i) with $(t, X, \Xi) = (t_1, x_1, \eta_1)$ to the following equality:

$$\partial(x_0, \xi_0)/\partial(\eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1) = J(t_1, x_1, \eta_1) F,$$

and performing calculations similar to those in proving Lemma 6.3,(ii), we can obtain (ii).

REMARK 7.5. It follows from Lemma 7.4 that we can choose 6 column vectors from $\mathrm{J} = \mathrm{J}(\omega_1)$ for $\omega_1 \in N(\varepsilon)$ almost everywhere so that those column vectors are uniformly linearly independent for almost all $\omega_1 \in N(\varepsilon)$. We have achieved the purpose mentioned in Remark 5.1, (i).

## 8. - Proof of (5.5)-(5.6).

Lemma 3.2 follows immediately from (5.5)-(5.6), as already mentioned in §5. Decompose $\mathbb{G}(\mu, T)$ as follows (recall (5.7)-(5.8)):

$$\mathbb{G}(\mu, T) = \sum_{k=1}^{4} \mathbb{G}_{k, \varepsilon}(\mu, T),$$

$$(\mathbb{G}_{k, \varepsilon}(\mu, T) u(\cdot, \cdot))(x_4, \xi_4) \equiv \int_{\omega_1 \in \Omega_1(\boldsymbol{R}, T)} \psi_{k, \varepsilon} G u(x_0, \xi_0) dt d\eta, k = 1, \dots, 4,$$

where $\psi_{k,\varepsilon}$, $k=1,\dots,4$, denote the characteristic functions of $N(\varepsilon)$, $S_k(\varepsilon)$, $k=2, 3, 4$, respectively. $\boldsymbol{R}$ is the constant defined in §5. The lemma below implies that $\mathbb{G}(\mu, T)$ is decomposed into the principal part $\mathbb{G}_{1,\varepsilon}(\mu, T)$ and the negligible parts $\mathbb{G}_{k,\varepsilon}(\mu, T)$, $k=2, 3, 4$.

LEMMA 8.1 (i) *For* $k=2, 3, 4$, *and for each* $0 \leqq \theta < 1$,

$$\sup_{\mu\in\delta(\theta),\, 1\leqq T<+\infty} |||\mathbb{G}_{k,\varepsilon}(\mu, T)||| \leqq c\varepsilon^{k-1}.$$

(ii) *Let* $1\leqq T<+\infty$, $0\leqq\theta<1$, *and* $\varepsilon>0$. *Then,*

(1) $\mathbb{G}_{1,\varepsilon}(\mu, T)\in \mathbb{C}(L^2(\Omega\times\mathbb{R}^3))$ *if* $\mu\in\delta(\theta)$,

(2) $|||\mathbb{G}_{1,\varepsilon}(\mu, T)|||\to 0$ as $|\mu|\to+\infty$, $\mu\in\delta(\theta)$.

PROOF. Making use of Lemma 7.4, (i), and (5.10), we obtain (i).

We will make use of Lemma 7.4, (ii) in order to pick out the integration kernel of $\mathbb{G}_{1,\varepsilon}(\mu, T)$. Decompose $\Omega_1=\Omega_1(\boldsymbol{R}, T)$ into 3 603 600 (${}_{15}P_6 = 3\,603\,600$) measurable disjoint subsets as follows: $\Omega_1 = \bigcup_{l\in H}\Omega_{1,l}$, where $\Omega_{1,l}\equiv\{\omega_1\in\Omega_1; (K_1(\omega_1),\dots,K_6(\omega_1))=l\}$, $l\in H\equiv\{(k_1,\dots,k_6)\in\{1,\dots,15\}^6;\ k_i\neq k_j \text{ if } i\neq j\}$. $K_j(\omega_1)$, $j=1,\dots,6$, are those in Lemma 7.4. Next we decompose $I\equiv(\mathbb{G}_{1,\varepsilon}(\mu, T)u(\cdot,\cdot))(x_4, \xi_4)$ as follows: $I=\sum_{l\in H} I_l$, where

$$I_l \equiv \int_{\omega_1\in\Omega_{1,l}} \psi_{1,\varepsilon} Gu(x_0, \xi_0)\,dt\,d\eta\,.$$

Let $l\in H$ be fixed. Let us change the variables of $I_l$. Removing 6 components which correspond to $\zeta(K_1(\omega_1)),\dots,\zeta(K_6(\omega_1))$, $l = (K_1(\omega_1),\dots,K_6(\omega_1))$, from the 16-dimensional vector $(\eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1)$, we obtain a new 10-dimensional vector $\lambda_l$ such that

$$d\eta\,dt = d\lambda_l d\zeta(K_1(\omega_1))\cdots d\zeta(K_6(\omega_1))\,,$$

where $\zeta(\cdot)$ is that in Lemma 7.4. Changing the variables as follows: $(\zeta(K_1(\omega_1)),\dots,\zeta(K_6(\omega_1)))\to(x_0, \xi_0)$, we deduce that

$$d\eta\,dt = |\det(\mathbb{K}(\omega_1))|^{-1} d\lambda_l dx_0 d\xi_0\,,$$

where $\mathbb{K}(\omega_1)$ is that in Lemma 7.4, (ii). Substituting this in $I_l$, we can pick out the integration kernel as follows:

$$I_l = \int\limits_{(x_0,\, \xi_0) \in \Omega \times \mathbb{R}^3} \overline{G}_l u(x_0, \xi_0)\, dx_0\, d\xi_0 ,$$

$$\overline{G}_l = \overline{G}_l(\varepsilon, \mu, T; x_4, \xi_4, x_0, \xi_0) \equiv \int \psi_{1,\varepsilon} G\, |\det(\mathbb{K}(\omega_1))|^{-1} d\lambda_l .$$

Let us obtain estimates for $\overline{G}_l$. Noting that $G$ contains $k_4(x_4, \xi_4, \eta_4)$, and making use of (5.11), we see that there exists a compact set $M \subseteq \mathbb{R}^3$ such that for any $\mu \in \delta(\theta)$, $0 \leqq \theta < 1$, $1 \leqq T < +\infty$, and $\varepsilon > 0$, (cf. [14, (8.4)]),

$$\text{(8.1)} \qquad \operatorname{supp} \overline{G}_l(\varepsilon, \mu, T; \cdot, \cdot, \cdot, \cdot) \subseteq \Omega_{x_4} \times M_{\xi_4} \times \Omega_{x_0} \times M_{\xi_0} .$$

Making use of Lemma 7.4, (ii), (3) and (5.10), we conclude that for any $\varepsilon > 0$ and $0 \leqq \theta < 1$, (cf. [14, (8.5)]),

$$\text{(8.2)} \qquad \sup_{x_j,\, \xi_j,\, j = 0,4,\, \mu \in \delta(\theta),\, 1 \leqq T < +\infty} |\overline{G}_l(\varepsilon, \mu, T; x_4, \xi_4, x_0, \xi_0)| < +\infty .$$

Recalling that $G$ contains $\exp(-\mu t_j)$, $j = 1, \ldots, 4$, and applying the Riemann-Lebesgue theorem, we see that for any $x_j, \xi_j$, $j = 0, 4$, $1 \leqq T < +\infty$, $0 \leqq \theta < 1$, and $\varepsilon > 0$, (cf. [14, (8.6)]),

$$\text{(8.3)} \qquad |\overline{G}_l(\varepsilon, \mu, T; x_4, \xi_4, x_0, \xi_0)| \to 0 \quad \text{as } |\mu| \to +\infty,\ \mu \in \delta(\theta) .$$

Making use of the fact that $\Omega$ is compact, we obtain (ii) from (8.1)-(8.3).

REMARK 8.2. The fact that $\Omega$ is compact plays an essential role in obtaining (ii) from (8.1)-(8.3). If $\Omega$ is not compact, then we cannot obtain (ii) from (8.1)-(8.3).

PROOF (5.5)-(5.6). (5.6) follows from Lemma 8.1, (i), (ii), (2) immediately. Making use of Lemma 8.1, (i), (ii), (1), we see that $\mathbb{G}(\mu, T)$ can be approximated with the compact operator $\mathbb{G}_{1,\varepsilon}(\mu, T)$ in $\mathbb{B}(L^2(\Omega \times \mathbb{R}^3))$ uniformly for $\mu \in \delta(\theta)$ and $T \in [1, +\infty)$. Making use of this result and [12, p. 200, Theorem VI.12, (a)], we obtain (5.5).

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