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On Periodic Solutions of a Class of Second Order Nonautonomous Systems with Nonhomogeneous Potentials Indefinite in Sign.

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ABSTRACT - In this paper some nonautonomous second order systems of the type

$$\ddot{x}(t) + b(t)(V_1'(x(t))) + V_2'(x(t) + \ldots + V_m'(x(t))) = 0 \tag{1}$$

as well as of the type

$$\ddot{x}(t) - A(t)x(t) + b(t)(V_1'(x(t)) + V_2'(x(t)) + \ldots + V_m'(x(t))) = 0\,, \tag{2}$$

where b is a real T-periodic continuous function, $V_1, \ldots, V_m$ are homogeneous with respective degrees $\beta_1 < \beta_2 < \ldots < \beta_m$ and A is a continuous T-periodic $N \times N$ matrix valued function, are considered. For system (1), an existence result is stated under the assumptions that b has a negative mean in $[0, T]$ and has a not identically zero positive part b^+ satisfying some further conditions, the homogeneous degrees β_i verify the conditions $\beta_1 > 2$, $\beta_{m-1} < 6$, $\beta_m \geqslant 8$ and V_m has a positive definite Hessian outside of the origin. For system (2), the same existence result is stated in case that A is symmetric and positive definite in $[0, T]$, under the weaker condition that the mean of b in $[0, T]$ is different from zero. The techniques of the proofs are based on a finite dimensional approximation of the problem, a suitable truncation argument of the potential $V_1 + V_2 + \ldots + V_m$ and the use of some Morse index estimates for critical points of Mountain Pass type.

Introduction.

The problem of existence of periodic solutions to nonautonomous second order systems where the potential V changes sign was dealt in case that V is homogeneous in the space-variable in [13], [14], [4]. In the

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following, some results were obtained without the homogeneity assumption in [10], [11], [2], [3], [6], [7]. This paper deals with the case that V has a superquadratic growth in the space-variable as well as the case that V contains a further quadratic term.

As for the superquadratic case, in [10], [11] the authors of the present paper considered a system of the type

$$\ddot{x}(t) + b(t)\, V'(x(t)) = 0 \tag{V}$$

where b is a T-periodic continuous real function with a non-zero mean and V has a superquadratic behaviour either near zero and near to infinity. The existence of a non trivial T-periodic solution to (V) was proved in case that a suitable assumption connecting the maximum of the negative part of b and the «homogeneity gap» for V at infinity is satisfied.

In the present paper no condition on the negative part b^- of b is assumed, only one requires that the mean of b in $[0, T]$ is negative and its positive part $b^+ = b + b^-$ is not identically zero. In this case, under some further assumptions on b^+, an existence result for T-periodic solutions to (V) is stated if $V = V_1 + V_2 + \ldots + V_m$, where V_i $(i = 1, \ldots, m)$ is homogeneous with degree $\beta_i (2 < \beta_1 < \ldots < \beta_{m-1} < 6, \beta_m = \max_{i=1,\ldots,m} \beta_i \geq 8)$ and V_m has a positive Hessian outside of the origin. For example V can be choosen in a wide class of polynomia in the $|x|$-variable (see Remark 5).

As for the case that a further quadratic positive term is added to the superquadratic part, the same kind of existence result is stated here, under the weaker assumption that b has a non-zero mean in $[0, T]$.

The proofs of these two results are both based on two basic tools: a finite dimensional approximation of the problem and a suitable truncation argument in such a way that the «truncation potential» is homogeneous at infinity (thus the Palais-Smale condition is satisfied by the corresponding functional). The reduction to the original problem is then guaranteed by the Montain Pass nature of the approximated solutions (in particular the use of the well known estimates for the Morse index of a critical point of Montain Pass type, (see [8], [9], [12]) and an appropriate use of the Gagliardo-Nirenberg inequality.

1. – The results.

Let us consider the following two Hamiltonian systems

$$\ddot{x}(t) + b(t)(V_1'(x(t))) + V_2'(x(t)) + \ldots + V_m'(x(t))) = 0\,, \tag{H_1}$$

$$\ddot{x}(t) - A(t)\,x(t) + b(t)(V_1'(x(t)) + V_2'(x(t)) + \ldots + V_m'(x(t))) = 0\,, \tag{H_2}$$

where b is a continuous T-periodic real function ($T > 0$), $V_1, V_2, \dots, V_m$ belong to $C^2(\mathbb{R}^N)(N \geq 2)$ and A is a continuous T-periodic $N \times N$ matrix valued function.

We are interested in the study of T-periodic solutions of (H_1) and (H_2), in case that $V_1, V_2, \dots, V_m$ are homogeneous with different degrees and b changes its sign.

The following two results can be stated.

THEOREM 1. *Let $V_1, V_2, \dots, V_m$ be positively homogeneous with respective degrees $\beta_1 < \beta_2 < \dots < \beta_m$, with $\beta_1 > 2, \beta_{m-1} < 6, \beta_m \geq 8$, and $V_i(x) > 0$ for $i = 1, \dots, m$, $x \neq 0$, and let V_m satisfy the assumption*

$$(\mathrm{V_m}) \qquad V_m''(x) \quad \text{is positive definite for } x \neq 0\,.$$

Let b satisfy the following conditions:

(b_1) $\int_0^T b(t) < 0$;

(b_2) $b^+(t) \overset{\text{def}}{=} \max(b(t), 0)$ *belongs to* $H^1([0, T])$ *and* $(b^+)^{-1}$ *belongs to* $L^{3/2}([0, T])$;

(b_3) *the set*

$$Z^+ = \{t \in [0, T]: b(t) = 0 \quad \text{and } \exists \varepsilon = \varepsilon(t) > 0$$
$$\text{s.t.}\, b(s_1)b(s_2) > 0 \quad \text{for } s_1 \in (t - \varepsilon, t),\, s_2 \in (t, t + \varepsilon)\}$$

is not empty and finite.

Then there exists at least one non-zero T-periodic solution of (H_1).

THEOREM 2. *Let $V_1, V_2, \dots, V_m$ satisfy the same assumptions of Theorem* 1, *let b satisfy* (b_2), (b_3) *and*

(b_4) $\int_0^T b(t) \neq 0$

and let A verify the condition

(A) *$A(t)$ is a positive definite symmetric $N \times N$ matrix for any $t \in \mathbb{R}$.*

Then there exists at least one non-zero T-periodic solution of (H_2).

REMARK 1. Note that the statement of Theorem 1 still holds in case that $m = 1$. In this case, which can be obtained as an easy consequence of Proposition 1 below (see also [10]), it is enough to take $\beta = \beta_1 = \ldots = \beta_m > 2$, ($V_m$) and ($b_2$) can be omitted and ($b_3$) can be replaced by the weaker condition (13).

The case $m = 1$ for problem (H_2) was dealt in [7], where the conditions on $V = V_1 = \ldots = V_m$ and on b are weakened exactly as in the previous case. See also [2] for other results in this framework.

REMARK 2. We note that the class of functions b such that (b_1), (b_2), (b_3) hold is indeed not empty. Actually it is easy to check that any regular function b satisfying (b_1), (b_3) and such that, for any $t_0 \in Z^+$,

$$b^+(t) \sim (t - t_0)^\alpha \qquad \text{as } t \to t_0$$

with $\alpha \in (1/2, 2/3)$, verifies condition (b_2) too.

REMARK 3. Let us note that no upper bound is imposed on the negative part b^- of b, i.e. $b^-(t) = b^+(t) - b(t)$.

REMARK 4. Assumption (b_3) does not esclude that b has an infinite numbers of zeroes t such that b is negative in the neighbourhood of t.

REMARK 5. Let us point out that a large class of polynomia in the $|x|$-variable satisfies the assumption of the potential $V = V_1 + V_2 + \ldots + V_m$ in Theorems 1, 2. More precisely V can be chosen as $V(x) = P(|x|)\ \forall x \in \mathbb{R}^N$, where

$$P(s) = a_3 s^3 + a_4 s^4 + a_5 s^5 + a_m s^m \qquad s \in \mathbb{R}$$

with an arbitrary integer number $m \geq 8$ and arbitrary numbers $a_3 \geq 0$, $a_4 \geq 0$, $a_5 \geq 0$, $a_m > 0$, $a_3^2 + a_4^2 + a_5^2 > 0$. Obviously, taking into account Remark 1, one can also choose

$$P(s) = a_m s^m, \qquad m \geq 3, \qquad a_m > 0.$$

2. - The proofs of the theorems.

PROOF OF THEOREM 1. First of all we need some lemmas in order to modify the potential $V_1 + V_2 + \ldots + V_m$ by a suitable potential which is homogeneous at infinity: this can be done by using a suitable truncation argument.

LEMMA 1. *For any $R > 0$ there exists some $\chi_R \in C^2(\mathbb{R}_+)$ such that*

(1) $\chi_R(t) = 1$ *for* $t \in [0, R]$, $\chi_R(t) = 0$ *for* $t \in [2R, +\infty)$, $0 \leq \chi_R(t) \leq 1$ $\forall t \in \mathbb{R}_+$;

(2) $\quad |\chi'_R(t)| \leqslant \lambda_1/R \quad \forall t \in \mathbb{R}_+$, *for some* $\lambda_1 > 0$;

(3) $\quad |\chi''_R(t)| \leqslant \lambda_2/R^2 \quad \forall t \in \mathbb{R}_+$, *for some* $\lambda_2 > 0$.

PROOF. Let P be an arbitrarily fixed polynomium (e.g. of degree 5) on $\mathbb{R}^N$, such that

$$P(0) = 1, \qquad P(1) = 0$$

$$P'(0) = P'(1) = 0,$$

$$P''(0) = P''(1) = 0,$$

and let $Q_R(t) = P((t - R)/R)$ for $t \in (R, 2R)$. Then it is easy to check that the function

$$\chi_R(t) = \begin{cases} 1 & \text{for } t \in [0, R], \\ Q_R(t) & \text{for } t \in (R, 2R), \\ 0 & \text{for } t \in [2R, +\infty), \end{cases}$$

belongs to $C^2(\mathbb{R}_+)$ and satisfies (1), (2), (3).

Now let us put

$$\overline{V}(x) = V_1(x) + \ldots + V_{m-1}(x), \qquad \forall x \in \mathbb{R}^N$$

and define, for any $R \geqslant 1$, the following «truncated» potential V_R

(4) $$V_R(x) = \chi_R(|x|)\overline{V}(x) + V_m(x), \qquad \forall x \in \mathbb{R}^N.$$

LEMMA 2. *The function V_R defined by* (4) *belongs to $C^2(\mathbb{R}^N)$ and verifies the following properties:*

(5) $\quad V_R(x) \geqslant a_1 |x|^{\beta_1} \quad \forall x \in \mathbb{R}^N$, *for some* $a_1 > 0$,

(6) $\quad V_R(x) \leqslant a_1' |x|^{\beta_m} + a_1'' \quad \forall x \in \mathbb{R}^N$, *for some* $a_1', a_1'' > 0$,

(7) $\quad |V_R'(x)| \leqslant a_2 |x|^{\beta_1 - 1} \quad \forall x \in \mathbb{R}^N, \quad |x| \leqslant 1$, *for some* $a_2 > 0$,

(8) $\quad V_R'(x)x = \beta_m V_R(x) \quad \forall x \in \mathbb{R}^N, \ |x| \geqslant 2R$,

(9) $\quad |V_R''(x)| \leqslant a_3 |x|^{\beta_m - 2} \quad \forall x \in \mathbb{R}^N, \ |x| \geqslant 2R$, *for some* $a_3 > 0$,

(10) $\quad V_R''(x)\xi\xi \geqslant a_4 |x|^{\beta_m - 2} - a_5 \quad \forall x, \xi \in \mathbb{R}^N, \ |\xi| = 1$,

for some $a_4, a_5 > 0$.

PROOF. Conditions (5), (6), (7) are easy consequences of (1), the positivity and the β_i-homogeneity of V_i $(i = 1, \dots, m)$ and the facts that $\beta_1 = \min \beta_i$, $\beta_m = \max \beta_i$. Conditions (8), (9) simply derive from the facts that $V_R(x) = V_m(x)$ for $|x| \geqslant 2R$ and the β_m-homogeneity of V_m. Let us prove (11) now. Actually, taking into account (1), (2), (3) and the β_m-homogeneity of V_m, a direct calculation yields

(11) $V_R''(x)\xi\xi \geqslant V_m''(x)\xi\xi + \varphi(x)$ $\forall x, \xi \in \mathbb{R}^N$, $|\xi| = 1$ where $\varphi \in C^0(\mathbb{R}^N)$ does not depend on R and verifies

$$\varphi(x)/|x|^{\beta_m - 2} \to 0 \quad \text{as } |x| \to +\infty, \tag{12}$$

so (11), (12), ($\mathrm{V_m}$) and still the β_m-homogeneity of V_m imply (10).

Another basic element for the proof of Theorem 1 is the use of a previous result obtained by the authors in [10].

PROPOSITION 1 (see Theorem 2.1 of [10]). *Let $\tilde{b}$ be a T-periodic continuous real function satisfying* (b_1) *and*

$$\exists t_0 \in [0, T]: \tilde{b}(t_0) > 0 \tag{13}$$

Let $\tilde{V} \in C^2(\mathbb{R}^N)$ be such that

($\tilde{\mathrm{V}}_1$) $\exists \tilde{a}_1, \tilde{r} > 0, \tilde{\beta}_1 > 2: \tilde{V}(x) \geqslant \tilde{a}_1 |x|^{\tilde{\beta}_1}$ $\forall x \in \mathbb{R}^N$, $|x| \leqslant \tilde{r}$,

($\tilde{\mathrm{V}}_2$) $\exists \tilde{a}_2 > 0: |\tilde{V}'(x)| \leqslant \tilde{a}_2 |x|^{\tilde{\beta}_1 - 1}$ $\forall x \in \mathbb{R}^N$, $|x| \leqslant \tilde{r}$,

($\tilde{\mathrm{V}}_3$) $\exists \tilde{\beta}_2 > 2: \tilde{R} > 0: \tilde{V}'(x)x \geqslant \tilde{\beta}_2 \tilde{V}(x) > 0$ $\forall x \in \mathbb{R}^N$, $|x| \geqslant \tilde{R}$.

Further, putting

$$\tilde{B}^- = \max\{\tilde{b}^-(t): t \in \mathbb{R}\}, \tilde{b}^-(t) = -\min\{\tilde{b}(t), 0\}, \qquad \forall t \in \mathbb{R},$$

let there exists two numbers $\tilde{c} \geqslant 0$, $\tilde{d} > 0$ such that

$$\tilde{B}^-(\tilde{V}'(x)x - \tilde{\beta}_2 \tilde{V}(x)) \leqslant \tilde{c}|x|^2 \qquad \forall x \in \mathbb{R}^N, \qquad |x| \geqslant \tilde{R}, \tag{14}$$

$$\tilde{c} < \frac{2(\tilde{\beta}_2 - 2)\pi^2}{(1 + 4\pi^2)T^2}, \tag{15}$$

$$\tilde{B}^-(|\tilde{V}''(x)| - \tilde{d}|x|^{\tilde{\beta}_2 - 2}) \leqslant 0, \qquad \forall x \in \mathbb{R}^N, \qquad |x| \geqslant \tilde{R}. \tag{16}$$

Then there exists at least one non-zero T-periodic solution of the system

$$(\tilde{\mathrm{H}}) \qquad \ddot{x}(t) + \tilde{b}(t)\tilde{V}'(x(t)) = 0\,.$$

Actually in case that $(\tilde{V}_1)$ holds in the whole space, we can prove two important estimates for the solution x given by Proposition 1. This can be achieved by looking at the very way of constructing this solution, namely by extending to the whole real line a critical point u of Mountain Pass type (in the sense of Ekeland and Hofer, see [8], [9], [13]) for the functional

$$\tilde{f}(v) = \frac{1}{2}\int_0^T |\dot{v}|^2 - \int_0^T \tilde{b}(t)\tilde{V}(v)\,,$$

$$\forall v \in H_T^1 = \{v \in H^1([0, T]; \mathbb{R}^N) : v(0) = v(T)\}.$$

Indeed the «Mountain Pass nature» of u was not completely clarified in [10], so we give some details about this point.

PROPOSITION 2. *Let $\tilde{b}$, $\tilde{V}$ satisfy the same assumptions of Proposition 1 with $\tilde{r} = +\infty$ in $(\tilde{\mathrm{V}}_1)$. Then there exists a solution $\tilde{x}$ of $(\tilde{\mathrm{H}})$ generated by a non-zero critical point $\tilde{u}$ of $\tilde{f}$ such that*

$$(17) \qquad \tilde{f}(\tilde{u}) \leqslant \tilde{k}$$

where $\tilde{k}$ is a positive number only depending on $\tilde{a}_1$, $\tilde{\beta}_1$, $\tilde{b}$ but not on the particular choice of $\tilde{V}$. Moreover one can choose $\tilde{u}$ in such a way that

$$(18) \qquad i(\tilde{u}) \leqslant 1$$

where $i(\tilde{u})$ is the Morse index of $\tilde{u}$ with respect to $\tilde{f}$.

PROOF. If one looks at Proposition 2.2 of [10], one checks that the functional $\tilde{f}$ has the following properties:

(19) $\tilde{f} \in C^1(H_T^1)$ satitsfies the Palais-Smale condition

$$(20) \qquad \tilde{f}(0) = 0,$$

(21) $\exists \varrho > 0$ such that $\tilde{f}(v) > 0$ $\forall v \in H_T^1 \setminus \{0\}$, with $\|v\|_{H_T^1} = \varrho$,

(22) $\exists u_- \in H_T^1$ only depending on $\tilde{a}_1$, $\tilde{\beta}_1$, but not on the particular choice of $\tilde{V}$ such that $\operatorname{supp} u_- \subsetneq \operatorname{supp} \tilde{b}^+$, $\tilde{f}(u_-) < 0$.

Now let us show that property (21) can be reinforced by the following one:

(21′) $\exists \varrho' > 0$ such that $\tilde{f}(v) \geq c > 0$, $\forall v \in H_T^1$ with $\|v\|_{H_T^1} = \varrho'$.

Indeed, we claim that, if one chooses $\varrho' = \varrho$ appearing in (21), then one gets (21′). In fact, if it was false, then there would exist some sequence $\{u_n\} \subset H_T^1$ such that

$$\|u_n\|_{H_T^1} = \varrho > 0\,, \tag{23}$$

$$\tilde{f}(u_n) \to 0\,, \tag{24}$$

By passing to a subsequence, let $\{u_n\}$ converge to some $\bar{u}$ weakly in H_T^1, so uniformly on the interval $[0, T]$, in such a way that

$$\int_0^T \tilde{b}(t)\,\tilde{V}(u_n) \to \int_0^T \tilde{b}(t)\,\tilde{V}(\bar{u})\,. \tag{25}$$

On the other hand,

$$\int_0^T |\dot{\bar{u}}|^2 \leq \underline{\lim} \int_0^T |\dot{u}_n|^2\,, \tag{26}$$

so (24), (25), (26) yield

$$\tilde{f}(\bar{u}) \leq \underline{\lim}\ \frac{1}{2} \int_0^T |\dot{u}_n|^2 - \int_0^T \tilde{b}(t)\,\tilde{V}(\bar{u}) \leq \lim \tilde{f}(u_n) = 0\,. \tag{27}$$

Moreover

$$\|\bar{u}\| \leq \varrho$$

which, together with (21), (27), yields

$$\bar{u} = 0$$

then, by (24), (25),

$$\int_0^T |\dot{u}_n|^2 \to 0.$$

Therefore $\{u_n\}$ strongly converges to $\bar{u} = 0$ in H_T^1 which contradicts (23).

Since $\tilde{f}$ verifies (19), (20), (21′), (23), then, by the Mountain Pass theorem by Ambrosetti and Rabinowitz [1], there exists a point $\tilde{u} \in$ $\in H_T^1 \setminus \{0\}$ such that

$$\tilde{f}(\tilde{u}) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \tilde{f}(\gamma(t)) \tag{28}$$

where

$$\Gamma = \{\gamma \in C^0([0, T]; H_T^1) : \gamma(0) = 0, \gamma(1) = u_-\}. \tag{29}$$

Moreover, by some well known results by Ekeland and Hofer (see [8], [9], [12]) one can choose $\tilde{u}$ in such a way that (18) holds.

Let us prove (17) now. Indeed (28), (29) imply

$$\tilde{f}(\tilde{u}) \leq \sup_{\lambda \in \mathrm{R}} \tilde{f}(\lambda u_-), \tag{30}$$

so, using $(\tilde{\mathrm{V}}_1)$ with $\tilde{r} = +\infty$ and taking into account that $\operatorname{supp} u_- \subsetneq \operatorname{supp} \tilde{b}^+$, (30) yields

$$\tilde{f}(\tilde{u}) \leq \sup_{\lambda \in \mathrm{R}} \left\{ \frac{\lambda^2}{2} \int_0^T |\dot{u}_-|^2 - \tilde{a}_1 \lambda_1^{\tilde{\beta}_1} \left(\max_{t \in \operatorname{supp} u_-} \tilde{b}^+(t) \right) \int_0^T |u_-|^{\tilde{\beta}_1} \right\} \doteq \tilde{k} \tag{31}$$

where $\tilde{k}$ is finite, as $\tilde{\beta}_1 > 2$, and only depends on $\tilde{a}_1, \tilde{\beta}_1, \tilde{b}$, as u_- only depends on $\tilde{a}_1, \tilde{\beta}_1, \tilde{b}$.

REMARK 6. By looking at the arguments given in the proofs of Proposition 2.2 of [10] and Proposition 2 of the present paper, one can note that condition (21′) is satisfied under conditions (b_1), (13), $(\tilde{\mathrm{V}}_1)$, $(\tilde{\mathrm{V}}_2)$ only. This remark will be very important for the proof of the final step of the proof of Theorem 1.

Now let us define a suitable sequence of finite-dimensional approximations of problem (H_1).

First of all, let us observe that it is not restrictive to suppose $b(0) = b(T) = 0$. Indeed, if it was not case, one could consider the function $\bar{b}(t) = b(t + \bar{t})$ where $\bar{t}$ is a zero of b in $[0, T]$, then consider the problem

$$\ddot{x}(t) + \bar{b}(t)(V_1'(x) + V_2'(x) + \ldots + V_m'(x)) = 0. \tag{$\overline{\mathrm{H}}_1$}$$

Thus, if $\bar{x}$ is a non-zero T-periodic solution to $(\overline{\mathrm{H}}_1)$, then $x(t) = \bar{x}(t - \bar{t})$ is a non-zero T-periodic solution to (H_1).

Then let us suppose $b(0) = b(T) = 0$. Let us consider now the open disjoint intervals of $[0, T]$, say I_k, for $k = 1, \ldots, s$, where b is strictly positive (recall that s is finite as (b_3) holds).

Observe that condition (b_2) implies, for any $k = 1, \dots, s$, the property

$$(32) \qquad b_k^+(t) = \begin{cases} b^+(t) & \text{if } t \in I_k\,, \\ 0 & \text{if } t \in [0, T] \setminus I_k, \end{cases} \qquad \textit{belongs to } H^1([0, T]).$$

At this point let us fix a sequence $\{E_n\}$ of finite-dimensional subspaces of H_T^1 such that

$$(33) \qquad H_T^1 = \overline{\bigcup_{n \in \mathbb{N}} E_n}$$

and such that, defining the $\mathbb{R}^N$-valued functions $\vec{b}^{\,+}$, $\vec{b}_k^{\,+}$ as

$$\vec{b}^{\,+} = \vec{b}^{\,+}(t) = (b^+(t), \dots, b^+(t)),$$

$$\vec{b}_k^{\,+} = \vec{b}_k^{\,+}(t) = (b_k^+(t), \dots, b_k^+(t)),$$

one has

$$(34) \qquad \vec{b}^{\,+}, \vec{b}_k^{\,+} \quad (k = 1, \dots, s), \quad u_- \quad \text{belong to } E_n \ \forall n.$$

The idea is to obtain a non-zero critical point u of the functional

$$f(v) = \frac{1}{2}\int_0^T |\dot{v}|^2 - \int_0^T b(t)(V_1(v) + V_2(v) + \dots + V_m(v)) \qquad \forall v \in H_T^1$$

as a non-zero H^1-limit of a sequence u_n of critical points for the restriction f_n of f to each subspace E_n.

The critical point u_n is found through a truncation argument connected with the choice of the potential V_R. The estimates on u_n (which yield the convergence of $\{u_n\}$ to a critical point u of f) are obtained by an appropriate use of the mentioned estimates for the index of Mountain Pass type critical points.

Finally the non-triviality of u is got as a consequence of the Mountain-Pass nature of u_n and the behaviour itself of $V_1 + V_2 + \dots + V_m$ at the origin. Then the usual argument will give a non-zero T-periodic solution x by T-periodically extending u on the whole real line.

Now let us proceed by steps.

STEP 1. For any $R \geqslant 1$ and for any $n \in \mathbb{N}$, there exists a critical point u_n^R for the functional

$$f_n^R(v) = \frac{1}{2}\int_0^T |\dot v|^2 - \int_0^T b(t)\,V_R(v), \qquad \forall v \in E_n, \tag{35}$$

such that

$$f_n^R(u_n^R) = \inf_{\gamma \in \Gamma_n} \max_{t \in [0,1]} f_n^R((\gamma(t)) \tag{36}$$

where

$$\Gamma_n = \{\gamma \in C^0([0,1]; E_n) \colon \gamma(0) = 0\,,\ \gamma(1) = u_-\}. \tag{37}$$

PROOF. First of all, observing that Proposition 1 holds as well replacing the whole space H_T^1 by an arbitrary closed subspace E containing u_-, $\vec{b}^{\,+}$, $\vec{b}_k^{\,+}$, in particular $E = E_n$, one applies Proposition 1 with the choices $\tilde b = b$, $\tilde V = V_R$. Indeed (13) is satisfied thanks to (b_3), while $(\tilde{\mathrm{V}}_1)$, $(\tilde{\mathrm{V}}_2)$, $(\tilde{\mathrm{V}}_3)$ are obvious consequences of (5), (6), (7), (8). Finally (8), (9), (10) easily yield (14), (15), (16) with $\tilde c = 0$, $\tilde d = a_3$.

STEP 2. There exists a constant number $k_1 > 0$ such that

$$f_n^R(u_n^R) \leqslant k_1\,. \tag{38}$$

Moreover one can choose u_n^R in such a way that

$$i_n^R(u_n^R) \leqslant 1\,. \tag{39}$$

PROOF. One can apply Proposition 2, with $\tilde b = b$, $\tilde V = V_R$ and H_T^1 replaced by E_n. Finally one can note that the number $\tilde k$ appearing in (31) can be choosen independent not only of R, but of n too, due to the fact that u_- belongs to E_n for any n.

STEP 3. There exists a constant number $k_2 > 0$ such that

$$\|u_n^R\|_{L^2(\operatorname{supp} b^+)} \leqslant k_2\,, \qquad \forall R \geqslant 1 \quad \forall n \in \mathbb{N}\,. \tag{40}$$

PROOF. Let us fix $k \in \{1, \dots, s\}$ and the related interval I_k where b is positive. By (39) we know, in particular, that the form

$$Q_R^n(v,v) = \int_0^T |\dot v|^2 - \int_0^T b(t)\,V_R''(u_n^R)vv\,, \qquad v \in E_n\,,$$

cannot be negative definite on the whole N-dimensional (recall that $N \geq 2$) subspace E_n^k of E_n defined as

$$E_n^k = \{v \in E_n : v(t) = \lambda(b_k^+(t)),\ t \in [0, T],\ \lambda \in \mathbb{R}^N\}$$

(note that $E_n^k \not\equiv \{0\}$ as (34) holds, so $\vec{b}_k^+$ belongs to E_n^k).

Therefore there exists some $\lambda_k = \lambda_k(R, n)$ with $|\lambda_k| = \|\vec{b}_k\|_{E_n}^{-1}$ ($\|\cdot\|_{E_n}$ being the H^{-1}-norm endowed in E_n) such that

$$Q_R^k(\lambda_k b_k^+, \lambda_k b_k^+) \geq 0\,. \tag{41}$$

Actually (41) implies, thanks to (10), that

$$\int_{I_k} (b^+(t))^3 |u_n^R(t)|^{\beta_m - 2} \leq \text{const} \|\vec{b}_k^+\|_{E_n}^2 + \text{const}\,, \qquad \forall R \geq 1\,,$$

then, as $\vec{b}_k^+$ belongs to H_T^1 and (33) holds, one gets

$$\int_{I_k} (b^+(t))^3 |u_n^R(t)|^{\beta_m - 2} \leq \text{const}\,, \qquad \forall R \geq 1\,, \quad \forall n \in \mathbb{N}\,. \tag{42}$$

Finally a simple use of the Hölder inequality and (b_2), which takes into account that $\beta_m \geq 8$, yields, from (42), the relation

$$\int_{I_k} |u_n^R(t)|^2 \leq \text{const}\,, \qquad \forall R \geq 1, \quad \forall n \in \mathbb{N} \tag{43}$$

so (40) follows by (43), as $\operatorname{supp} b^+ = \overline{\bigcup_{k=1}^s I_k}$.

STEP 4. For any $n \in \mathbb{N}$, there exists a constant number $c_n > 0$ such that

$$\|u_n^R\|_{H_T^1} \leq c_n \qquad \forall R \geq 1\,. \tag{44}$$

PROOF. By (38) one deduces

$$\frac{1}{2} \int_0^T |\dot{u}_n^R|^2 \leq k_1 + \sum_{k=1}^s \int_{I_k} b^+(t)\, V_R(u_n^R), \qquad \forall R \geq 1\,, \quad \forall n \in \mathbb{N}\,,$$

so, by (6)

$$\frac{1}{2} \int_0^T |\dot{u}_n^R|^2 \leq \text{const}\left(1 + \sum_{k=1}^s \int_{I_k} b^+(t) |u_n^R(t)|^{\beta_m}\right), \quad \forall R \geq 1, \forall n \in \mathbb{N}. \tag{45}$$

On the other side, putting

$$c_n^R = \frac{1}{T}\int_0^T u_n^R$$

one easily checks

$$(46) \qquad |c_n^R| \leqslant \min_{t\in[0,T]} |u_n^R(t)| + \text{const}\left(\int_0^T |\dot u_n^R|^2\right)^{1/2}, \qquad \forall R \geqslant 1,\ \forall n \in \mathbb{N}.$$

Actually an easy argument shows that (42) implies

$$(47) \qquad \min_{t\in[0,T]} |u_n^R(t)| \leqslant \text{const}, \qquad \forall R \geqslant 1, \quad \forall n \in \mathbb{N},$$

so (45), (46), (47) yield

$$(48) \qquad \|u_n^R\|_{H_T^1}^2 \leqslant \text{const}\left(1 + \sum_{k=1}^{s} \int_{I_k} b^+(t)\,|u_n^R(t)|^{\beta_m}\right), \qquad \forall R \geqslant 1,\ \forall n \in \mathbb{N}.$$

Let us consider now the space

$$E_n^+ = \{v\colon \operatorname{supp} b^+ \to \mathbb{R}^N : \exists\, \tilde v \in E_n \text{ s.t. } \tilde v(t) = v(t) \qquad \forall t \in \operatorname{supp} b^+\}.$$

Obviously the dimension of E_n^+ is finite, then the L^{β_m}-norm is equivalent to the L^2-norm in E_n^+. Therefore for some $k_n > 0$

$$(49) \qquad \int_{I_k} b^+(t)\,|u_n^R(t)|^{\beta_m} \leqslant k_n \|u_n^R\|_{L^2(\operatorname{supp} b^+)}^{\beta_m} \qquad \forall k = 1, \dots, s,\ \forall R \geqslant 1,$$

so (44) follows from (48), (49), (40).

STEP 5. For sufficiently large $R \geqslant 1$ $u_n = u_n^R$ is a non-zero critical point of the functional

$$(50) \qquad f_n(v) = \frac{1}{2}\int_0^T |\dot v|^2 - \int_0^T b(t)(V_1(v) + V_2(v) + \ldots + V_m(v)), \quad \forall v \in E_n.$$

PROOF. Property (44) implies that, for some $\tilde c_n > 0$, independent of $R \geqslant 1$, one has

$$\sup_{t\in[0,T]} |u_n^R(t)| \leqslant \tilde c_n, \qquad \forall R \geqslant 1.$$

Then for any fixed $n \in \mathbb{N}$, it is sufficient to choose $R > R_n = \max(1, \tilde c_n)$

in order to get, by definition of V_R,

$$V_R(u_n^R(t)) = V_1(u_n^R(t)) + V_2(u_n^R(t)) + \dots + V_m(u_n^R(t)), \qquad \forall t \in [0, T],$$

then $u_n = u_n^R$, which is a non-zero critical point of f_n^R, is a non-zero critical point of f_n too.

STEP 6. There exists a constant number $c > 0$ such that

$$\|u_n\|_{H_T^1} \leq c, \qquad \forall n \in \mathbb{N}. \tag{51}$$

PROOF. Putting $u_n = \tilde{u}_n + u_n^0$ with $u_n^0 = 1/T \int_0^T u_n$, one has, by (46), (47),

$$|u_n^0| \leq \text{const}\left(1 + \left(\int |\dot{u}_n|^2\right)^{1/2}\right). \tag{52}$$

On the other side, by criticality of u_n for f_n and the β_i-homogeneity of V_i, one has

$$\int_0^T |\dot{u}_n|^2 = \sum_{i=1}^m \beta_i \int_0^T b(t) V_i(u_n) \tag{53}$$

while, by (38),

$$\frac{1}{2} \int_0^T |\dot{u}_n|^2 \leq k_1 + \sum_{i=1}^m \int_0^T b(t) V_i(u_n). \tag{54}$$

Then (53), (54) yield

$$\left(\frac{1}{2} - \frac{1}{\beta_m}\right) \int_0^T |\dot{u}_n|^2 \leq k_1 + \sum_{i=1}^{m-1} \left(1 - \frac{\beta_i}{\beta_m}\right) \int_0^T b(t) V_i(v) \leq$$

$$\leq k_i + \sum_{i=1}^{m-1} \left(1 - \frac{\beta_i}{\beta_m}\right) \int_{\text{supp}b^+} b^+(t) V_i(v)$$

thus, by the β_i-homogeneity of V_i, with $\beta_{m-1} = \max_{i=1,\dots,m_1} \beta_i$, one deduces

$$\int_0^T |\dot{u}_n|^2 \leq \text{const}\left(1 + \int_{\text{supp}b^+} |u_n|^{\beta_{m-1}}\right). \tag{55}$$

At this point (52), (55) yield

$$\|u_n\|^2_{H^1_T} \leqslant \text{const}\left(1 + \int\limits_{\text{supp}\, b^+} |u_n|^{\beta_{m-1}}\right). \tag{56}$$

Now let us apply the Gagliardo-Niremberg inequality (see e.g. [5] pag. 147)

$$\|v\|_{L^p(I)} \leqslant \text{const}\, \|v\|^{1-a}_{L^q(I)}\, \|v\|^a_{W^{1,r}(I)}, \qquad \forall v \in W^{1,r}(I),$$

where $I = \text{supp}\, b^+$, $p = \beta_{m-1}$, $q = r = 2$ $a = 1/2 - 1/\beta_{m-1}$.
Then from (56) one gets

$$\|u_n\|^2_{H^1_T} \leqslant \text{const}\, \|u_n\|^{(\beta_{m-1}-2)/2}_{H^1(\text{supp}\, b^+)} + \text{const} \leqslant \tag{57}$$

$$\leqslant \text{const}\, \|u_n\|^{(\beta_{m-1}-2)/2}_{H^1_T} + \text{const}$$

which yields (51), as $\beta_{m-1} < 6$.

STEP 7 (Conclusion). There exists a non-zero critical point u of f, so a non-zero T-periodic solution x of (H_1).

PROOF. The point u is obtained by passing to the weak-limit in the H^1-norm on the sequence $\{u_n\}$. The density property of $\{E_n\}$ given by (33) and the fact that, in particular, $\{u_n\}$ uniformly converges to u easily yield the criticality of u for the functional f.

Finally let us show now that u is different from zero. Indeed one has to note that $\tilde{V} = V_1 + V_2 + \ldots + V_m$ satisfies assumptions $(\tilde{V}_1)$, $(\tilde{V}_2)$ of Proposition 2 (take $\tilde{\beta}_1 = \beta_1$, $\tilde{r} \leqslant 1$) and $\tilde{b} = b$ satisfies (b_1) and (13) by (b_3). Therefore, recalling Remark 6, one can state that, for some positive numbers ϱ, c, one has

$$f(v) \geqslant c\,, \qquad \forall v \in H^1_T\,, \quad \|v\| = \varrho\,.$$

On the other side the construction of $u_n = u_n^R$, given by (36), (37) (recall that u_- does not depend on n, R but only on b and a_1, and that $f(u_-) < 0$, so $\varrho < \|u_-\|$), yields

$$f(u_n) = f_n(u_n) \geqslant c > 0 \qquad \forall n \in \mathbb{N}\,. \tag{58}$$

By (58) it follows that $\{u_n\}$ cannot uniformly converge to $u \equiv 0$, as it would imply

$$f(u_n) = \frac{1}{2}\int_0^T |\dot{u}_n| - \int_0^T b(t)(V_1(u_n) + V_2(u_n) + \ldots + V_m(u_n)) =$$

$$= \sum_{i=1}^{m}\left(\frac{\beta_i}{2} - 1\right)\int_0^T b(t)\,V_i(u_n) \to 0 \qquad \text{as } n \to +\infty$$

which contradicts (58). Therefore $u \not\equiv 0$ and Theorem 1 is completely proved. ■

REMARK 7. Note that the argument used for the proof of Theorem 1 enables to state an H^1-estimate of the solution u deduced by taking the limit as $n \to \infty$ in (57). Indeed the same kind of H^1-estimate can be obtained by the same argument also if V is given by a single β-homogeneous term with $\beta \geqslant 8$, satisfying $(\mathrm{V_m})$. In this case, of course, one does not need the truncation and finite dimensional approximation arguments, since one can directly prove the L^2-estimate given by Step 3 (due to the Mountain Pass nature of the critical point): however this estimate is based on the properties $\beta \geqslant 8$ and $(\mathrm{V_m})$, which can on the other side omitted for the only existence result.

REMARK 8. If one looks at the proof of Theorem 3 (in particular the passage (42)$\Rightarrow$(43)) one needs indeed, for $(b^+)^{-1}$, just the property

$$(b^+)^{-1} \in L^{6/(\beta_m - 4)} \tag{59}$$

which is compatible with the property $b^+ \in H^1$ exactly in the case

$$\beta_m > 7\,. \tag{60}$$

Actually, we have decided to present the statement of Theorem 3 under the stronger assumptions

$$b_+^{-1} \in L^{3/2}\,, \qquad \beta_m \geqslant 8$$

(instead of (59), (60)) in order to simplify the exposition and having in mind the polynomial example (see Remark 5).

PROOF OF THEOREM 2. It is easy to observe that the same kind of arguments carried on for the proof of Theorem 1 can be used, replacing the functional f by the functional f_A defined as

$$f_A(v) = \frac{1}{2}\int_0^T (|\dot v|^2 + A(t)vv) - \\ - \int_0^T b(t)(V_1(v) + V_2(v) + \ldots + V_m(v)), \qquad \forall v \in H_T^1 .$$

Indeed one has to observe that the position

$$|||v||| = \left(\int_0^T |\dot v|^2 + \int_0^T A(t)vv\right)^{1/2}, \qquad \forall v \in H_T^1 ,$$

defines a norm on H_T^1 which is equivalent to the standard H^1-norm, due to assumption (A).

Then one can argue as in the proof of Theorem 1, by replacing the term $\int_0^T |\dot v|^2$ with the term $\int_0^T |\dot v|^2 + \int_0^T A(t)vv$ without any problem. Moreover condition (b_1) can be weakened: the presence of the integral term containing $A(t)$ enables to state the same kind of results expressed by Proposition 1, Proposition 2, under the weaker assumption (b_4) in place of (b_1) (indeed (b_4) is necessary in order to be sure that the Palais-Smale condition is satisfied by the «truncated» functional f^R, see Lemma 2.1 of [10]). Actually (b_1) was only used in the proof of Proposition 2.2 of [10] which does not need any assumption on the sign of the mean of b if the potential has a positive quadratic term (indeed condition (I_1') of Proposition 2.2 in [10] is quite easy to check in this case). ■

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