Rendiconti
del
Seminario Matematico
della
Università di Padova

A. Sankaranarayanan

K. Srinivas

On a method of Balasubramanian and Ramachandra (on the abelian group problem)

Rendiconti del Seminario Matematico della Università di Padova, tome 97 (1997), p. 135-161

<http://www.numdam.org/item?id=RSMUP_1997__97__135_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 97 (1997)

On a Method of Balasubramanian and Ramachandra (on the Abelian Group Problem).

A. SANKARANARAYANAN - K. SRINIVAS (*)

ABSTRACT - Let a_n denote the number of non-isomorphic abelian groups of order n. We consider

$$A(x) = \sum_{n \leq x} a_n = \sum_{j=1}^{10} C_j x^{1/j} + E(x)$$

where $E(x)$ is the error term. We study $E(x)$ through the general method of Balasubramanian and Ramachandra.

1. - Introduction.

In [3] R. Balasubramanian and K. Ramachandra developed a general method of proving Ω results and also Ω_+, Ω_- results. They applied it to some arithmetical questions and in particular to the question of number of abelian groups of order $\leq x$. However, their method is not widely known and certainly it deserves to be known widely. Since their proofs are somewhat sketchy, we wish to explain their method with special reference to the abelian group problem.

Let a_n denote the number of non-isomorphic abelian groups of order n. By standard arguments, we get the Dirichlet series identity

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{k=1}^{\infty} \zeta(ks)$$

(*) Indirizzo degli AA.: School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India.
E-mail address: sank@tifrvax.tifr.res.in - srini@moduli.math.tifr.res.in
AMS Subject Classification: 11 M XX-11 M 06.

valid for $\sigma > 1$. For $x \geqslant 100$ put

$$A(x) = \sum_{n \leqslant x} a_n . \tag{1.1}$$

An approximation to $A(x)$ is

$$\sum_{j=1}^{10} C_j x^{1/j}$$

where $C_j x^{1/j}$ is the residue at $s = 1/j$ of

$$\left(\frac{x^s}{s} \prod_{k=1}^{\infty} \zeta(ks) \right) .$$

It is the aim of this paper to study the function

$$E(x) = A(x) - \sum_{j=1}^{10} C_j x^{1/j} . \tag{1.2}$$

It is known that (see [7])

$$E(x) \ll x^{97/381} (\log x)^{35} .$$

From the corollary of the following Theorem 1.1 and the mean-square upper bound for $E(x)$ due to D. R. Heath-Brown (see § 6), one can even conjecture that

$$E(x) \ll x^{1/6} (\log x)^{100}$$

which is still far away. In (1.2) any constant greater than or equal to 6 will work in place of 10.

Let

$$M = \max_{T \leqslant u \leqslant X^{100}} \left| \frac{E(u)}{u^{1/6}} \right| \tag{1.3}$$

where $X = T^{200}$ and $T \geqslant T_0$ (T_0 is a large positive constant).

Throughout this paper, $A_1, A_2, A_3, \ldots$ denote positive constants. η is a small positive constant and l is a large positive constant chosen such that $\eta l \leqslant 1/100$. ε is a small positive constant. Let $s_0 = 1/6 + it$ and $s_1 = 1/10 + it_1$ where t_1 is a fixed number such that $T \leqslant t_1 \leqslant 2T$. We fix $X = T^{200}$. We prove

THEOREM 1.1. *We have*

$$\int_T^{\infty} \left| \frac{E(u)}{u^{1/6}} \right|^2 e^{-u/X} \frac{du}{u} \gg (\log T)^2 \tag{1.4}$$

where the implied constant is effective.

COROLLARY. *We have*

$$M \gg \sqrt{\log T}\,. \tag{1.5}$$

PROOF OF THE COROLLARY. From the Theorem 1.1, we have

$$(\log T)^2 \ll \int_T^{X^{100}} \left| \frac{E(u)}{u^{1/6}} \right|^2 e^{-u/X} \frac{du}{u} + \int_{X^{100}}^{\infty} \left| \frac{E(u)}{u^{1/6}} \right|^2 e^{-u/X} \frac{du}{u} \ll$$

$$\ll \int_T^{X^{100}} \left| \frac{E(u)}{u^{1/6}} \right|^2 e^{-u/X} \frac{du}{u} + O\left(\frac{1}{T}\right)$$

and this implies that

$$(\log T)^2 \ll \int_T^{X^{100}} \left| \frac{E(u)}{u^{1/6}} \right|^2 \frac{du}{u} \ll M^2 \log T$$

which proves the corollary.

THEOREM 1.2. *There exist effective positive constants A_1 and A_2 such that for all $X \geqslant 10$*

$$\min_{X \leqslant u \leqslant X^{A_1}} \left(\frac{E(u)}{u^{1/10}} \right) < -\exp\left(A_2 \left(\frac{\log X}{\log\log X} \right)^{1/2} \right)$$

and

$$\max_{X \leqslant u \leqslant X^{A_1}} \left(\frac{E(u)}{u^{1/10}} \right) > \exp\left(A_2 \left(\frac{\log X}{\log\log X} \right)^{1/2} \right).$$

REMARK. In Theorem 1.1 C_j may be taken to be any fixed positive constants whereas in Theorem 1.2, we have to take C_j to be the constants coming from the residues at $s = 1/j$.

2. - Main tools.

We make use of the following two theorems in proving Theorems 1.1 and 1.2.

THEOREM 2.1. *Let* $G(s) = 1 + \sum_{n=2}^{\infty} b_n/n^s$ *be analytic in* $(\sigma \geq 1/2, T \leq t \leq 2T)$ *and there* $\max |G(s)| < \exp(T^{A_3})$. *The series* $G(s)$ *is assumed to converge for at least one complex number* s *and we have also assumed that* b_n'*s are complex numbers with* $|b_n| \leq (nT)^{A_4}$. *Then, we have*

$$\frac{1}{T}\int_T^{2T} |G(1/2+it)|^2 dt \gg \sum_{n \leq T/100+1} \frac{|b_n|^2}{n}\left(1 - \frac{\log n}{\log T} + \frac{1}{\log\log T}\right)$$

where the implied constant is effective and depends on A_3 *and* A_4.

REMARK. This theorem is due to K. Ramachandra (see [10]).In fact he has improved this theorem (see [11]). For our purpose this one is more than sufficient.

THEOREM 2.2. *If* $\{c_n\}$ *is a sequence of complex numbers such that* $\sum_{n=1}^{\infty} n|c_n|^2$ *is convergent, then*

$$\int_0^T \left|\sum_{n=1}^{\infty} c_n n^{-it}\right|^2 dt = \sum_{n=1}^{\infty} |c_n|^2 (T + O(n))$$

where the implied constant is effective.

REMARK. This theorem is due to H. L. Montgomery and R. C. Vaughan [8]. For a simpler proof see [9].

3. - Some lemmas.

LEMMA 3.1. *We have*

$$a_n \ll_\varepsilon n^\varepsilon .$$

PROOF. The proof is well-known to experts. For the sake of completeness we indicate the proof. Since $F(s)$ is an infinite product of zeta-functions and each of them has an Euler product, we first note that a_n's are multiplicative. Now,

$$a_{p^m} = \sum_{m = l_1 + 2l_2 + 3l_3 + \dots} 1 \tag{3.1.1}$$

where $l_j \geqslant 0$ for $j = 1, 2, 3, \dots$ in the above sum. Writing $a(m)$ for a_{p^m}, we get the generating function of $a(m)$ from (3.1.1) to be

$$\sum_{m=0}^{\infty} a(m) x^m = (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\dots \tag{3.1.2}$$

for $0 < x < 1$. Hence

$$a(m) x^m \leqslant \exp\left\{ - \sum_{n_1=1}^{\infty} \log(1 - x^{n_1}) \right\} = \tag{3.1.3}$$

$$= \exp\left\{ \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{x^{n_1 n_2}}{n_2} \right\} = \exp\left\{ \sum_{n_2=1}^{\infty} \frac{x^{n_2}}{n_2(1 - x^{n_2})} \right\}.$$

Since,

$$1 - x^{n_2} = (1-x)(1 + x + \dots + x^{n_2 - 1}) \geqslant (1-x) n_2 x^{n_2 - 1},$$

we have

$$x^{n_2} \leqslant \frac{x(1 - x^{n_2})}{n_2(1-x)}. \tag{3.1.4}$$

From (3.1.3) and (3.1.4), we get

$$a(m)x^m \leqslant \exp\left\{ \sum_{n_2=1}^{\infty} \frac{x}{n_2^2(1-x)} \right\} = \exp\left(\frac{\pi^2}{6} \frac{x}{(1-x)} \right). \tag{3.1.5}$$

From (3.1.5), by choosing $x = 1 - 1/N$, $N = \sqrt{m}$, we get

$$a(m) \leqslant \exp\left(\frac{\pi^2(N-1)}{6} - m \log(1 - 1/N) \right) \leqslant \tag{3.1.6}$$

$$\leqslant \exp\left(\frac{\pi^2 N}{6} + \sum_{r=1}^{\infty} \frac{m}{rN^r} \right) \leqslant \exp\{A_5(N + m/N)\} = \exp(2A_5 \sqrt{m}).$$

Now, if $n = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}$, then

$$\frac{a_n}{n^{\varepsilon}} = \prod_{p^{m_i} \| n,\, p \leqslant 2^{1/\varepsilon}} \frac{a_{p^{m_i}}}{p^{m_i \varepsilon}} \prod_{p^{m_i} \| n,\, p > 2^{1/\varepsilon}} \frac{a_{p^{m_i}}}{p^{m_i \varepsilon}} \leqslant \tag{3.1.7}$$

$$\leqslant \prod_{p^{m_i} \| n,\, p \leqslant 2^{1/\varepsilon}} \frac{a_{p^{m_i}}}{p^{m_i \varepsilon}} \leqslant A_6 ,$$

since the other factor with $p > 2^{1/\varepsilon}$ in (3.1.7) is bounded. Here A_6 depends on ε. This proves the lemma.

LEMMA 3.2. *For $x > 0$, we have*

$$e^{-1/x} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} x^w \Gamma(w) dw .$$

PROOF. The proof is well known (see pp. 33 of [14]).

LEMMA 3.3. *For* $\operatorname{Re} s \geqslant 1/11$ *and* $T/2 \leqslant t \leqslant 5T/2$, *we have*

$$F(s) \ll T^{10} .$$

PROOF. First we note that $\zeta(s) \ll (|t|^{1-\sigma} + 1) \log |t|$ for $|t| \geqslant 2$, $\sigma \geqslant 1/11$. Therefore,

$$|F(s)| = \left| \prod_{k=1}^{\infty} \zeta(ks) \right| \ll \prod_{k=1}^{12} |\zeta(ks)| \ll \prod_{k=1}^{12} (T^{(1-k\sigma)} \log T) \ll T^{10}$$

which proves the lemma.

LEMMA 3.4. *Let $G(s) = 1 + \sum_{n=2}^{\infty} b_n / n^s$ be absolutely convergent in* $\operatorname{Re} s > A_7$ *and there* $\max |G(s)| < T^{A_8}$. *$G(s)$ may have poles only on the real line. We assume that b_n's are complex numbers with $|b_n| \leqslant \leqslant (nT)^{A_9}$. Then, for $T \leqslant t \leqslant 2T$, we have*

$$G(s_0) = \sum_{n=1}^{\infty} \frac{b_n}{n^{s_0}} e^{-n/Y} + O(T^{-A_8/7})$$

where $Y = T^{48 A_8}$.

PROOF. First of all, we note that for $\alpha_1 \leqslant \sigma \leqslant \alpha_2$, $|t| \geqslant 1$, we have

$$|\Gamma(\sigma + it)| \ll e^{-A_{10} |t|} \tag{3.4.1}$$

where A_{10} depends on α_1 and α_2. With $w = u + iv$, from Lemma 3.2, we have

$$\sum_{n=1}^{\infty} \frac{b_n}{n^{s_0}} e^{-n/Y} = \frac{1}{2\pi i} \int_{\operatorname{Re} w = A_7 + 1} G(s_0 + w) Y^w \Gamma(w)\, dw. \tag{3.4.2}$$

In (r.h.s) of (3.4.2),we break off the portion $|v| \geqslant (\log T)^2$ which contributes an error $O(Y^{A_7+1} e^{-A_{11}(\log T)^2})$. In the remaining portion,we move the line of integration to $\operatorname{Re}(s_0 + w) = 1/7$. We notice that there is only one pole inside the rectangle $(-1/42 \leqslant u \leqslant A_7 + 1,\ |v| \leqslant \leqslant (\log T)^2)$ at $w = 0$ which comes from the Γ function. The residue at $w = 0$ is $G(s_0)$. The horizontal portions contribute an error

$$\frac{1}{2\pi i} \int_{-1/42}^{A_7+1} G(\sigma_0 + u + i(t + (\log T)^2))\, Y^{u + i(\log T)^2} \Gamma(u + i(\log T)^2)\, du = \tag{3.4.3}$$

$$= O(T^{A_8} Y^{A_7+1} e^{-A_{12}(\log T)^2}).$$

The vertical portion is

$$\frac{1}{2\pi} \int_{|v| \leqslant (\log T)^2} G(1/7 + i(t + v))\, Y^{-1/42 + iv} \Gamma(-1/42 + iv)\, dv = \tag{3.4.4}$$

$$= O\left(T^{A_8} Y^{-1/42} \int_0^{\infty} |\Gamma(-1/42 + iv)|\, dv\right) =$$

$$= O\left(T^{A_8} Y^{-1/42} \left(\int_0^{\infty} \frac{|\Gamma(41/42 + iv)|}{|-1/42 + iv|}\, dv \right)\right) =$$

$$= O\left(T^{A_8} Y^{-1/42} \left\{ \int_0^1 dv + \int_1^{\infty} e^{-A_{13} v}\, dv \right\}\right) =$$

$$= O(T^{A_8} Y^{-1/42}).$$

Since $Y = T^{48A_8}$, the lemma follows from (3.4.2) to (3.4.4).

COROLLARY. *We have, with* $X = T^{200}$,

$$F(s_0) = \sum_{n=1}^{\infty} \frac{a_n}{n^{s_0}} e^{-n/X} + O(T^{-1/2}).$$

PROOF In the Lemma 3.4, we can take $A_8 = 4$ by the proof of Lemma 3.3. By Lemma 3.1, we have $a_n \ll_\varepsilon n^\varepsilon$. Hence, by fixing $X = T^{200} > Y = T^{48A_8}$, the corollary follows.

LEMMA 3.5. *If $T \leqslant y \leqslant 2T$ and $5/6 < \beta < 1$, then*

$$\int_{T^\beta}^{T} \left| \sum_{n \leqslant y} \frac{a_n}{n^{s_0}} e^{-n/X} \right|^2 \frac{dt}{t^2} \ll 1$$

PROOF. From Theorem 2.2, we get,

$$\int_{T^\beta}^{T} \left| \sum_{n \leqslant y} \frac{a_n}{n^{s_0}} e^{-n/X} \right|^2 \frac{dt}{t^2} \ll \frac{1}{T^{2\beta}} \int_{T^\beta}^{T} \left| \sum_{n \leqslant y} \frac{a_n}{n^{s_0}} e^{-n/X} \right|^2 dt \ll$$

$$\frac{1}{T^{2\beta}} \sum_{n \leqslant y} \frac{|a_n|^2}{n^{1/3}} e^{-2n/X} (T - T^\beta + O(n)) \ll T^{1 - 2\beta + \frac{2}{3} + \varepsilon}$$

since $a_n \ll_\varepsilon n^\varepsilon$ and $e^{-2n/X} \ll 1$. By choosing $\beta \geqslant 5/6 + \varepsilon/2$ (for ε a small positive constant $< 1/3$), the lemma follows.

LEMMA 3.6. *We have*

$$\frac{1}{T} \int_{T}^{2T} \left| \left(\prod_{k=1}^{3} \zeta(1 - ks_0) \right) \left(\prod_{k=4}^{6} \zeta(ks_0) \right) \right|^2 dt \gg \log T$$

where the implied constant is effective.

PROOF. First of all, we note that

$$\left| \left(\prod_{k=1}^{3} \zeta(1 - ks_0) \right) \left(\prod_{k=4}^{6} \zeta(ks_0) \right) \right|^2 = \tag{3.6.1}$$

$$= \left| \left(\prod_{k=1}^{3} \zeta(1 - k\overline{s_0}) \right) \left(\prod_{k=4}^{6} \zeta(ks_0) \right) \right|^2 .$$

We notice that,we can define $k_n^{(1)}, k_n^{(2)}, \ldots,$ in the following way with $\operatorname{Re} w$ sufficiently large,

$$\zeta(5/6 + w) = \sum_{n=1}^{\infty} k_n^{(1)} n^{-w}, \qquad \zeta(2/3 + 2w) = \sum_{n=1}^{\infty} k_n^{(2)} n^{-w},$$

$$\zeta(1/2 + 3w) = \sum_{n=1}^{\infty} k_n^{(3)} n^{-w}, \qquad \zeta(2/3 + 4w) = \sum_{n=1}^{\infty} k_n^{(4)} n^{-w},$$

$$\zeta(5/6 + 5w) = \sum_{n=1}^{\infty} k_n^{(5)} n^{-w}, \quad \text{and} \quad \zeta(1 + 6w) = \sum_{n=1}^{\infty} k_n^{(6)} n^{-w} .$$

Hence, we find that $k_1^{(j)} = 1$ for $j = 1, 2, 3, 4, 5, 6$; $k_n^{(j)} \geq 0$ for every n and for every $j = 1, 2, 3, 4, 5, 6$; and

$$k_{n^3}^{(3)} = \frac{1}{\sqrt{n}} \quad \text{for every } n. \tag{3.6.2}$$

For $\operatorname{Re} w$ large, let h_n be defined by

$$\left(\prod_{k=1}^{3} \zeta\left(1 - \frac{k}{6} + kw\right)\right)\left(\prod_{k=4}^{6} \zeta\left(\frac{k}{6} + kw\right)\right) = \sum_{n=1}^{\infty} \frac{h_n}{n^w}.$$

From (3.6.2), we find that $h_{n^3} \geq \frac{1}{\sqrt{n}}$ and hence by Theorem 2.1, we have

$$\frac{1}{T}\int_T^{2T} \left|\left(\prod_{k=1}^{3} \zeta(1 - ks_0)\right)\left(\prod_{k=4}^{6} \zeta(ks_0)\right)\right|^2 dt \gg$$

$$\gg \sum_{n \leq T/100} |h_n|^2 \left(1 - \frac{\log n}{\log T} + \frac{1}{\log\log T}\right) \gg$$

$$\gg \sum_{n^3 \leq T/1000} |h_{n^3}|^2 \left(1 - \frac{\log n^3}{\log T} + \frac{1}{\log\log T}\right) \gg$$

$$\gg \sum_{n \leq T^{1/5}} \frac{1}{n} \left(1 - \frac{\log n^3}{\log T} + \frac{1}{\log\log T}\right) \gg \log T$$

which proves the lemma.

LEMMA 3.7. *Let* $1/2 < \beta < 1$. *We have*

$$\int_{T^\beta}^{T} |F(s_0)|^2 \frac{dt}{t^2} \gg (\log T)^2 .$$

PROOF. We have

$$F(s_0) = \prod_{k=1}^{\infty} \zeta(ks_0) = \left(\prod_{k=1}^{6} \zeta(ks_0)\right) H(s_0) \tag{3.7.1}$$

where $H(s_0) = \prod_{k=7}^{\infty} \zeta(ks_0)$. We notice that $H(s_0) \gg 1$. Also from the functional equation of zeta-function, we have

$$\zeta(ks_0) = \chi(ks_0)\,\zeta(1 - ks_0) \tag{3.7.2}$$

with

$$\chi(ks_0) \sim |t|^{1/2 - k/6} \tag{3.7.3}$$

and hence, we get

$$F(s_0) = H(s_0)\left(\prod_{k=1}^{3} (\chi(ks_0)\zeta(1-ks_0))\right)\left(\prod_{k=4}^{6} \zeta(ks_0)\right). \tag{3.7.4}$$

Therefore, we obtain from Lemma 3.6

$$\frac{1}{T}\int_{T}^{2T} |F(s_0)|^2 dt \gg \tag{3.7.5}$$

$$\gg T^{2\left((3/2 - 1/6 \sum_{k=1}^{3} k\right)} \cdot \frac{1}{T}\int_{T}^{2T} \left| \left(\prod_{k=1}^{3} \zeta(1-ks_0)\right)\left(\prod_{k=4}^{6} \zeta(ks_0)\right) \right|^2 dt \gg T\log T.$$

Now, from (3.7.5), we have

$$\int_{U}^{2U} |F(s_0)|^2 \frac{dt}{t^2} \geqslant \frac{1}{4U^2}\int_{U}^{2U} |F(s_0)|^2 dt \gg \log U. \tag{3.7.6}$$

Since, $U \geqslant T^{\beta}$ and there are at least $\gg (\log T)$ intervals of the type $(U, 2U)$ in (T^{β}, T), the lemma now follows from (3.7.6).

LEMMA 3.8. *For $T \leqslant y \leqslant 2T$, we have*

$$\int_{T^{\beta}}^{T} \left| \int_{y}^{\infty} u^{-s_0} e^{-u/X} dA(u) \right|^2 \frac{dt}{t^2} \ll \int_{T}^{\infty} \left| \frac{E(u)}{u^{1/6}} \right|^2 e^{-u/X} \frac{du}{u}$$

for all β with $5/6 < \beta < 1$.

PROOF. First, we note that, we can write

$$A(u) = \sum_{j=1}^{10} C_j u^{1/j} + E(u) \tag{3.8.1}$$

where C_j may be taken to be any fixed positive constants. Therefore, we get

$$\text{(3.8.2)} \qquad \int_y^\infty u^{-s_0} e^{-u/X} dA(u) =$$

$$= \sum_{j=1}^{10} \frac{C_j}{j} \int_y^\infty u^{1/j - s_0 - 1} e^{-u/X} du + \int_y^\infty u^{-s_0} e^{-u/X} dE(u) =$$

$$= \sum_{j=1}^{10} \frac{C_j}{j} \int_y^\infty u^{1/j - s_0 - 1} e^{-u/X} du + y^{-s_0} E(y) e^{-y/X} +$$

$$+ s \int_y^\infty E(u) u^{-s_0 - 1} e^{-u/X} du + \int_y^\infty \frac{E(u) u^{-s_0} e^{-u/X}}{X} du = J_1 + J_2 + J_3 + J_4. \text{ (say)}$$

By considering the real part and the imaginary part separately, from the second Mean-Value Theorem of calculus, it follows that

$$\text{(3.8.3)} \qquad \operatorname{Re} J_1 = \sum_{j=1}^{10} \frac{C_j}{j} \operatorname{Re}\left(\int_y^\infty u^{1/j - s_0 - 1} e^{-u/X} du \right) =$$

$$= \sum_{j=1}^{10} \frac{C_j}{j} e^{-y/X} \operatorname{Re}\left(\int_y^{\xi_j} u^{1/j - s_0 - 1} du \right) =$$

$$= O\left(\frac{e^{-T/X} \left(\sum_{j=1}^{10} T^{(1/j - 1/6)} \right)}{T} \right) = O(T^{-1/6})$$

and hence we have $J_1 = O(T^{-1/6})$. Therefore we have

$$\text{(3.8.4)} \qquad \int_{T^\beta}^T |J_1|^2 \frac{dt}{t^2} = O(T^{-1/3}).$$

We choose our y from $[T, 2T]$ in such a way that $|y^{-s_0} E(y) e^{-y/X}|$ is minimum.Therefore we obtain

$$\text{(3.8.5)} \qquad |y^{-s_0} E(y) e^{-y/X}|^2 \leqslant$$

$$\leqslant \frac{1}{T} \int_T^{2T} \left| \frac{E(u)}{u^{1/6}} \right|^2 e^{-2u/X} du \ll \int_T^\infty \left| \frac{E(u)}{u^{1/6}} \right|^2 e^{-2u/X} \frac{du}{u}$$

and hence we have

$$\int_{T^\beta}^{T} |J_2|^2 \frac{dt}{t^2} \ll \int_{T}^{\infty} \left| \frac{E(u)}{u^{1/6}} \right|^2 e^{-u/X} \frac{du}{u}. \tag{3.8.6}$$

Now,

$$\int_{T^\beta}^{T} |J_3|^2 \frac{dt}{t^2} \ll \int_{T^\beta}^{T} \left| \int_0^1 \sum_{n=0}^{\infty} \frac{E(y+n+u)}{(y+n+u)^{s+1}} e^{-(y+n+u)/X} du \right|^2 dt \ll \tag{3.8.7}$$

$$\ll \int_0^1 \left(\int_{T^\beta}^{T} \left| \sum_{n=0}^{\infty} \frac{E(y+n+u)}{(y+n+u)^{s+1}} e^{-(y+n+u)/X} \right|^2 dt \right) du \ll$$

$$\ll \int_y^{\infty} \left| \frac{E(u)}{u^{1/6}} \right|^2 \frac{e^{-2u/X}}{u^2} (T - T^\beta + O(u))\, du \ll \int_T^{\infty} \left| \frac{E(u)}{u^{1/6}} \right|^2 \frac{e^{-u/X}}{u} du.$$

(In the second step we have used $\left| \int_0^1 f du \right|^2 \leqslant \int_0^1 |f|^2 du$, the third step by Theorem 2.2 and the fourth step by the inequality $u \geqslant y \geqslant T$).
Now,

$$\int_{T^\beta}^{T} |J_4|^2 \frac{dt}{t^2} \leqslant \frac{1}{X^2 T^{2\beta}} \int_{T^\beta}^{T} \left| \int_y^{\infty} u^{-s_0} E(u) e^{-u/X} du \right|^2 dt = \tag{3.8.8}$$

$$= \frac{1}{X^2 T^{2\beta}} \int_0^1 \left(\int_{T^\beta}^{T} \left| \sum_{n=0}^{\infty} \frac{E(y+n+u)}{(y+n+u)^s} e^{-\frac{(y+n+u)}{X}} \right|^2 dt \right) du =$$

$$= \frac{1}{X^2 T^{2\beta}} \int_y^{\infty} \left| \frac{E(u)}{u^{1/6}} \right|^2 e^{-2u/X} (T - T^\beta + O(u))\, du \ll$$

$$\ll \frac{1}{T^{2\beta}} \int_T^{\infty} \left| \frac{E(u)}{u^{1/6}} \right|^2 \frac{e^{-u/X}}{u} du.$$

(Here also, we have used as before, Theorem 2.2, $u \geqslant y \geqslant T$ and $e^{-a} \leqslant \leqslant a^{-2}$ for $a > 0$.)

Since,

$$\int_{T^\beta}^{T} |J_1 + J_2 + J_3 + J_4|^2 \frac{dt}{t^2} \ll \sum_{k=1}^{4} \int_{T\beta}^{T} |J_k|^2 \frac{dt}{t^2},$$

the lemma follows from (3.8.4), (3.8.6), (3.8.7) and (3.8.8).

LEMMA 3.9. *We have*

$$|F(s_1)| = \max_{T \leqslant t \leqslant 2T} |F(1/10 + it)| \geqslant T \exp\left\{ \frac{3}{4\sqrt{5}} \left(\frac{\log T}{\log\log T} \right)^{1/2} \right\}.$$

REMARK. This follows from a theorem of R. Balasubramanian and K. Ramachandra (see[2]). The constant $3/4$ appearing in the r.h.s of the inequality is due to R. Balasubramanian (see [1]).

PROOF. Let $s = 1/10 + it$. Then

$$|F(s)| = \left| \left(\prod_{k=1}^{5} \chi(ks)\zeta(1-ks) \right) \left(\prod_{k=6}^{10} \zeta(ks) \right) \left(\prod_{k=11}^{\infty} \zeta(ks) \right) \right| \sim \tag{3.9.1}$$

$$\sim t \left| \left(\prod_{k=1}^{5} \zeta(1-k\bar{s}) \right) \left(\prod_{k=6}^{10} \zeta(ks) \right) \left(\prod_{k=11}^{\infty} \zeta(ks) \right) \right| \sim t |G_1(s) H_1(s)|$$

where for $\mathrm{Re}\, w$ large, we define

$$G_1(w) = \left(\prod_{k=1}^{5} \zeta\left(1 - \frac{k}{10} + kw\right) \right) \left(\prod_{k=6}^{10} \zeta\left(\frac{k}{10} + kw\right) \right) \tag{3.9.2}$$

and

$$H_1(s) = \left(\prod_{k=11}^{\infty} \zeta(ks) \right). \tag{3.9.3}$$

First of all, we note that $H_1(s) \gg 1$. Let e_n, $e_j(n)$ be defined by

$$G_1(w) = \sum_{n=1}^{\infty} \frac{e_n}{n^w} \tag{3.9.4}$$

and

$$G_1^j(w) = \sum_{n=1}^{\infty} \frac{e_j(n)}{n^w}. \tag{3.9.5}$$

As in Lemma 3.6, clearly we have $e_n \geqslant 0$,

$$e_n = \sum_{n_1 n_2^2 \dots n_{10}^{10} = n} (n_1^{9/10} n_2^{4/5} n_3^{7/10} n_4^{3/5} n_5^{1/2} n_6^{3/5} n_7^{7/10} n_8^{4/5} n_9^{9/10} n_{10})^{-1} \leqslant \tag{3.9.6}$$

$$\leqslant \sum_{n_1 n_2^2 \dots n_{10}^{10} = n} 1 \leqslant \sum_{n_1 n_2 n_3 \dots n_{10} = n} 1 \leqslant d_{10}(n) \leqslant n^{\varepsilon}$$

and hence

$$\sum_{n=1}^{\infty} \frac{e_n}{n^2} < A_{14} . \tag{3.9.7}$$

Therefore, we have

$$\frac{e_j(n)}{n^2} \leqslant \sum_{n=1}^{\infty} \frac{e_j(n)}{n^2} = \left(\sum_{n=1}^{\infty} \frac{e_n}{n^2} \right)^j \leqslant A_{14}^j . \tag{3.9.8}$$

From (3.9.8) we get

$$e_j(n) \leqslant n^2 A_{14}^j \leqslant (nT)^{A_{15}} \tag{3.9.9}$$

for $1 \leqslant j \leqslant A_{16} \log T$. Also we have,for $1 \leqslant j \leqslant A_{16} \log T$,

$$\max_{T \leqslant t \leqslant 2T} |G_1^j(s)| \ll T^{\frac{j}{10} \sum_{k=1}^{5} k} \cdot T^{j \sum_{k=6}^{10} \left(1 - \frac{k}{10}\right)} (\log T)^{10} \ll T^{3j} \ll \exp(T^{A_{17}}). \tag{3.9.10}$$

Therefore from (3.9.7), (3.9.9) and (3.9.10), conditions for Theorem 2.1 are satisfied and hence we can apply this theorem to $G_1^j(s)$. Also, we note that $e_j(n^5) \geqslant d_j(n)/\sqrt{n}$.

Now, from (3.9.1),

$$\frac{1}{T} \int_T^{2T} |F(s)|^{2j} dt \gg T^{2j} \cdot \frac{1}{T} \int_T^{2T} |G_1^j(s)|^2 dt \gg \tag{3.9.11}$$

$$\gg T^{2j} \left(\sum_{n \leqslant T/100} |e_j(n)|^2 \left(1 - \frac{\log n}{\log T} + \frac{1}{\log\log T}\right)\right) \gg$$

$$\gg T^{2j} \left(\sum_{n \leqslant T^{1/5}} \frac{|d_j(n)|^2}{n} \left(1 - \frac{\log n^5}{\log T} + \frac{1}{\log\log T}\right)\right).$$

From (3.9.11), it follows that

(3.9.12) $$|F(1/10 + it_1)| \gg$$

$$\gg T\left(\sum_{n \leqslant T^{1/5}} \frac{|d_j(n)|^2}{n}\left(1 - \frac{\log n^5}{\log T} + \frac{1}{\log\log T}\right)\right)^{1/2j} \gg$$

$$\gg \frac{T}{(\log\log T)^{1/2j}}\left(\sum_{n \leqslant T^{1/5}} \frac{|d_j(n)|^2}{n}\right)^{1/2j}.$$

From [1], since the maximum over $1 \leqslant j \leqslant A_{16}\log T$ of the second factor of (r.h.s) of (3.9.12) occurs only when $j \geqslant (\log T)^{\varepsilon}$ where ε is a small positive constant and the maximum is $\geqslant \exp\left\{\frac{3}{4\sqrt{5}}\left(\frac{\log T}{\log\log T}\right)^{1/2}\right\}$, the lemma follows.

LEMMA 3.10. *For $T \leqslant y_1 \leqslant 2T$, we have*

$$\sum_{n \leqslant y_1} a_n n^{-s_1} e^{-n/X} \ll_{\varepsilon} T^{9/10+\varepsilon}.$$

PROOF. Follows from the fact that $a_n \ll_{\varepsilon} n^{\varepsilon}$ and $e^{-n/X} \ll 1$ for $n \leqslant y_1$ and $X = T^{200}$.

LEMMA 3.11. *We have*

$$\int_T^{\infty} \left|\frac{E(u)}{u^{1/10}}\right| e^{-u/2X}\frac{du}{u} \gg \exp\left\{\frac{3}{4\sqrt{5}}\left(\frac{\log T}{\log\log T}\right)^{1/2}\right\}.$$

REMARK. This is the key theorem to deduce the Ω_+ and Ω_- results, namely Theorem 1.2

PROOF. First of all, we note that as before the corollary to the Lemma 3.4, we can have

(3.11.1) $$F(s_1) = \sum_{n=1}^{\infty} a_n n^{-s_1} e^{-n/X} + O(T^{-1/2}).$$

Therefore from Lemmas 3.9 and 3.10, we have

(3.11.2) $$\sum_{n > y_1} a_n n^{-s_1} e^{-n/X} \gg T\exp\left\{\frac{3}{4\sqrt{5}}\left(\frac{\log T}{\log\log T}\right)^{1/2}\right\}.$$

Now as before, we have

$$(3.11.3)\qquad \sum_{n>y_1} a_n n^{-s_1} e^{-n/X} = \int_{y_1}^{\infty} u^{-s_1} e^{-u/X} dA(u) =$$

$$= \int_{y_1}^{\infty} u^{-s_1} e^{-u/X} d\left(\sum_{j=1}^{10} C_j u^{1/j} + E(u) \right) =$$

$$= \sum_{j=1}^{10} \frac{C_j}{j} \int_{y_1}^{\infty} u^{1/j - s_1 - 1} e^{-u/X} du + \int_{y_1}^{\infty} u^{-s_1} e^{-u/X} dE(u) =$$

$$= \sum_{j=1}^{10} \frac{C_j}{j} \int_{y_1}^{\infty} u^{1/j - s_1 - 1} e^{-u/X} du + y_1^{-s_1} E(y_1) e^{-y_1/X} +$$

$$+ s_1 \int_{y_1}^{\infty} E(u) u^{-s_1 - 1} e^{-u/X} du + \int_{y_1}^{\infty} \frac{E(u) u^{-s_1} e^{-u/X}}{X} du = K_1 + K_2 + K_3 + K_4 .$$

Using the second Mean-value theorem of calculus, as before we get

$$(3.11.4)\qquad K_1 = \sum_{j=1}^{10} \frac{C_j}{j} \int_{y_1}^{\infty} u^{1/j - s_1 - 1} e^{-u/X} du = O\left(\frac{1}{T} \right).$$

We choose our y_1 such that $|y_1^{-s_1} E(y_1) e^{-y_1/X}|$ is minimum. Hence we obtain

$$(3.11.5)\qquad K_2 \leqslant |y_1^{-s_1} E(y_1) e^{-y_1/X}| \leqslant$$

$$\leqslant \frac{1}{T} \int_T^{2T} \left| \frac{E(u)}{u^{1/10}} \right| e^{-u/X} du \ll \int_T^{\infty} \left| \frac{E(u)}{u^{1/10}} \right| e^{-u/X} \frac{du}{u} .$$

Now, since $T \leqslant t_1 \leqslant 2T$, we have

$$(3.11.6)\qquad K_3 = s_1 \int_{y_1}^{\infty} E(u) u^{-s_1 - 1} e^{-u/X} du \ll T \int_T^{\infty} \left| \frac{E(u)}{u^{1/10}} \right| e^{-u/X} \frac{du}{u} .$$

Also, we have

$$(3.11.7)\qquad K_4 = \frac{1}{X}\int_{y_1}^{\infty} E(u)\,u^{-s_1}e^{-u/X}\,du \ll \frac{1}{X}\int_{T}^{\infty}\left|\frac{E(u)}{u^{1/10}}\right| e^{-u/X}\,du \ll$$

$$\ll \int_{T}^{\infty}\left|\frac{E(u)}{u^{1/10}}\right| e^{-u/2X}\,\frac{du}{u}\,.$$

From (3.11.2) to (3.11.7), we obtain

$$T\int_{T}^{\infty}\left|\frac{E(u)}{u^{1/10}}\right| e^{-u/2X}\,\frac{du}{u} \gg T\exp\left\{\frac{3}{4\sqrt{5}}\left(\frac{\log T}{\log\log T}\right)^{1/2}\right\}$$

and hence the lemma.

LEMMA 3.12. *For $T \leqslant U \leqslant X^{100}$, we have*

$$\frac{1}{U}\int_{U}^{2U}\left|\frac{E(u)}{u^{1/10}}\right| du \gg \exp\left(A_{18}\left(\frac{\log T}{\log\log T}\right)^{1/2}\right)$$

where A_{18} is a certain fixed positive constant $< 3/(4\sqrt{5})$.

PROOF. First of all, we note that

$$(3.12.1)\qquad \int_{X^{100}}^{\infty}\left|\frac{E(u)}{u^{1/10}}\right| e^{-u/2X}\,\frac{du}{u} \ll e^{-X^{99}/4}\int_{X^{100}}^{\infty}\frac{u}{u^{1/10}}\,\frac{X^2}{u^2}\,\frac{du}{u} \ll e^{-X^{50}},$$

since, $E(u) = O(u)$ and $e^{-a} \ll a^{-2}$ for $a > 0$. Since, there can be $\ll \log T$ intervals of the type $(U, 2U)$ in the interval $[T, X^{100}]$, we have

$$(3.12.2)\qquad (\log T)\int_{U}^{2U}\left|\frac{E(u)}{u^{1/10}}\right| e^{-u/2X}\,\frac{du}{u} \gg \exp\left(\frac{3}{4\sqrt{5}}\left(\frac{\log T}{\log\log T}\right)^{1/2}\right)$$

from Lemma 3.11. Since $T^{20000} \geqslant U \geqslant T$, $e^{-u/2X} \ll 1$, the lemma now follows from (3.12.2).

4. - Proof of the Theorem 1.1.

From Lemmas 3.7 and 3.4, for $T \leqslant y \leqslant 2T$, we get

$$(\log T)^2 \ll \int_{T^\beta}^{T} |F(s_0)|^2 \frac{dt}{t^2} \ll \int_{T^\beta}^{T} \left| \sum_{n=1}^{\infty} \frac{a_n}{n^{s_0}} e^{-n/X} + O(T^{-1/2}) \right|^2 \frac{dt}{t^2} \ll$$

$$\ll \int_{T^\beta}^{T} \left\{ \left| \sum_{n \leqslant y} \frac{a_n}{n^{s_0}} e^{-n/X} \right|^2 + \left| \sum_{n > y} \frac{a_n}{n^{s_0}} e^{-n/X} \right|^2 + O(T^{-1}) \right\} \frac{dt}{t^2} .$$

Note that, we have used $|a+b+c|^2 \leqslant 9(|a|^2 + |b|^2 + |c|^2)$. From Lemma 3.5 the above inequality implies that

$$(\log T)^2 \ll \int_{T^\beta}^{T} \left| \sum_{n > y} \frac{a_n}{n^{s_0}} e^{-n/X} \right|^2 \frac{dt}{t^2} \ll$$

$$\ll \int_{T^\beta}^{T} \left| \int_{y}^{\infty} u^{-s_0} e^{-u/X} dA(u) \right|^2 \frac{dt}{t^2} \ll \int_{T}^{\infty} \left| \frac{E(u)}{u^{1/6}} \right|^2 e^{-u/X} \frac{du}{u}$$

(the last step follows by Lemma 3.8) and hence the Theorem 1.1.

5. - Proof of the Theorem 1.2.

Let $0 \leqslant \alpha_i \leqslant \eta U$, $T \leqslant U \leqslant X^{100}$. We fix $\eta = 1/3000$ and $l = 20$.We define a set R to be

$$R = \left\{ u \,|\, U - \eta l U \leqslant u \leqslant 2U + \eta l U \text{ with } \left| \frac{E(u)}{u^{1/10}} \right| \leqslant \right.$$
$$\left. \leqslant \varepsilon \exp\left(A_{18} \left(\frac{\log U}{\log\log U} \right)^{1/2} \right) \right\} \tag{5.1}$$

where A_{18} is the same as in Lemma 3.12. Let

$$Z_1 = U - \alpha_1 - \alpha_2 - \ldots - \alpha_l ,$$

$$Z_2 = 2U + \alpha_1 + \alpha_2 + \ldots + \alpha_l ,$$

$$Z = \frac{F(s)}{s(s+9/10)} \{ Z_2^{(s+9/10)} - Z_1^{(s+9/10)} \} .$$

From Perron's formula,we have

$$(5.2)\qquad \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F(s)\frac{u^s}{s}ds =$$

$$= \frac{A(u+0)+A(u-0)}{2} = A(u) \text{ or } A(u)+O(u^{\varepsilon})$$

according as u is not an integer or u is an integer.

Of course, we have

$$(5.3)\qquad A(u) = \sum_{j=1}^{10} C_j u^{1/j} + E(u)$$

where $C_j u^{1/j}$ is the residue at $s = 1/j$ of $\left(\frac{u^s}{s}\prod_{j=1}^{\infty}\zeta(js)\right)$. Hence from (5.2) and (5.3), we have

$$E(u) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F(s)\frac{u^s}{s}ds - \sum_{j=1}^{10} C_j u^{1/j}$$

or

$$(5.4)\qquad \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F(s)\frac{u^s}{s}ds - \sum_{j=1}^{10} C_j u^{1/j} + O(u^{\varepsilon})$$

according as u is not an integer or u is an integer.

Let R^C denote the complement set of R. From Lemma 3.12, we have

$$(5.5)\qquad \exp\left(A_{18}\left(\frac{\log U}{\log\log U}\right)^{1/2}\right) \leqslant \frac{1}{U}\int_{U}^{2U}\left|\frac{E(u)}{u^{1/10}}\right| du \leqslant$$

$$\leqslant \frac{1}{U(\eta U)^l}\int_0^{\eta U}\cdots\int_0^{\eta U}\int_{Z_1}^{Z_2}\left|\frac{E(u)}{u^{1/10}}\right| du\, da_1\ldots da_l\,.$$

From (5.5),it follows that there exists a positive constant $A_{19}(= A_{18} -$

$-\varepsilon$) such that

$$\exp\left(A_{19}\left(\frac{\log U}{\log\log U}\right)^{1/2}\right) \leqslant \tag{5.6}$$

$$\leqslant \frac{1}{U(\eta U)^l}\int_0^{\eta U}\cdots\int_0^{\eta U}\int_{R^C}\left|\frac{E(u)}{u^{1/10}}\right| du\, da_1\dots da_l .$$

If there is a sign change in the integrand of the r.h.s of (5.6), we are through. On the contrary, if we assume that there is no sign change in the integrand of r.h.s of (5.6), then definitely we can add our set R to the integral of r.h.s of (5.6) and hence we will be having the following inequality,

$$\exp\left(A_{20}\left(\frac{\log U}{\log\log U}\right)^{1/2}\right) \leqslant \tag{5.7}$$

$$\leqslant \frac{1}{U(\eta U)^l}\left|\int_0^{\eta U}\cdots\int_0^{\eta U}\int_{Z_1}^{Z_2}\frac{E(u)}{u^{1/10}} du\, da_1\dots da_l\right|.$$

From (5.4), we have

$$\frac{1}{U(\eta U)^l}\int_0^{\eta U}\cdots\int_0^{\eta U}\int_{Z_1}^{Z_2}\frac{E(u)}{u^{1/10}} du\, da_1\dots da_l = \tag{5.8}$$

$$= \frac{1}{U(\eta U)^l}\int_0^{\eta U}\cdots\int_0^{\eta U}\int_{Z_1}^{Z_2}\left\{\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}F(s)\frac{u^{s-1/10}}{s}ds - \sum_{j=1}^{10}C_j u^{1/j-1/10} + O(u^{\varepsilon-1/10})\right\}\cdot$$

$$\cdot du\, da_1\dots da_l = K_5 + K_6 + K_7 \quad \text{(say)},$$

$$K_7 = 0 \tag{5.9}$$

since the term $O(u^{\varepsilon-1/10})$ is zero unless u is a positive integer.

Now,

$$K_6 = -\frac{1}{U(\eta U)^l}\int_0^{\eta U}\cdots\int_0^{\eta U}\int_{Z_1}^{Z_2}\left(\sum_{j=1}^{10}C_j u^{(1/j-1/10)}\right)du\, da_1\dots da_l = \tag{5.10}$$

$$= -\frac{1}{U(\eta U)^l}\int_0^{\eta U}\cdots\int_0^{\eta U}\left(\sum_{j=1}^{10}\frac{C_j}{(1/j+9/10)}\{Z_2^{(1/j+9/10)} - Z_1^{(1/j+9/10)}\}\right)da_1\dots da_l .$$

Now,

$$(5.11)\quad K_5 = \frac{1}{U(\eta U)^l} \int_0^{\eta U} \cdots \int_0^{\eta U} \int_{Z_1}^{Z_2} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F(s) \frac{u^{s-1/10}}{s} \, ds \, du \, da_1 \ldots da_l =$$

$$= \frac{1}{U(\eta U)^l} \int_0^{\eta U} \cdots \int_0^{\eta U} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} Z \, ds \, da_1 \ldots da_l =$$

$$= \frac{1}{U(\eta U)^l} \int_0^{\eta U} \cdots \int_0^{\eta U} \frac{1}{2\pi i} \left\{ \int_{2-iX^{1000}}^{2+iX^{1000}} + \int_{\sigma=2,\ |t|>X^{1000}} \right\} Z \, ds \, da_1 \ldots da_l .$$

We note that,

$$(5.12)\quad \frac{1}{U(\eta U)^l} \int_0^{\eta U} \cdots \int_0^{\eta U} \frac{1}{2\pi i} \int_{\sigma=2,\ |t|>X^{1000}} Z \, ds \, da_1 \ldots da_l =$$

$$= O\left(\frac{(\eta U)^l}{U(\eta U)^l} \frac{((l+2)U)^3}{X^{1000}} \right) = O\left(\frac{U^2}{X^{1000}} \right).$$

In the first integral appearing in the r.h.s of (5.11), we move the line of integration to $\sigma = 21/220$. By Cauchy's theorem, since we have the poles of $F(s)$ at $s = 1/j$ for $j = 1, 2, 3 \ldots 10$, we get

$$(5.13)\quad \frac{1}{U(\eta U)^l} \int_0^{\eta U} \cdots \int_0^{\eta U} \frac{1}{2\pi i} \int_{2-iX^{1000}}^{2+iX^{1000}} Z \, ds \, da_1 \ldots da_l =$$

$$= \frac{1}{U(\eta U)^l} \int_0^{\eta U} \cdots \int_0^{\eta U} \left(\sum_{j=1}^{10} \frac{C_j}{(1/j + 9/10)} \{ Z_1^{(1/j+9/10)} - Z_2^{(1/j+9/10)} \} \right) da_1 \ldots da_l -$$

$$- \frac{1}{U(\eta U)^l} \int_0^{\eta U} \cdots \int_0^{\eta U} \frac{1}{2\pi i} \int_{2-iX^{1000}}^{21/220-iX^{1000}} Z \, ds \, da_1 \ldots da_l -$$

$$- \frac{1}{U(\eta U)^l} \int_0^{\eta U} \cdots \int_0^{\eta U} \frac{1}{2\pi i} \int_{21/220-iX^{1000}}^{21/220+iX^{1000}} Z \, ds \, da_1 \ldots da_l -$$

$$- \frac{1}{U(\eta U)^l} \int_0^{\eta U} \cdots \int_0^{\eta U} \frac{1}{2\pi i} \int_{21/220+iX^{1000}}^{2+iX^{1000}} Z \, ds \, da_1 \ldots da_l .$$

We note that as before, we can have

$$F(s) \ll (|t| + 2)^{10} \tag{5.14}$$

for $\sigma \geqslant 21/220$.

From (5.14), we see that

$$\frac{1}{U(\eta U)^l} \int_0^{\eta U} \cdots \int_0^{\eta U} \frac{1}{2\pi i} \int_{21/220 - iX^{1000}}^{21/220 + iX^{1000}} Z\, ds\, da_1 \ldots da_l = \tag{5.15}$$

$$= \frac{1}{U(\eta U)^l} \frac{1}{2\pi i} \int_{21/220 - iX^{1000}}^{21/220 + iX^{1000}} \cdot$$

$$\cdot \frac{F(s)((2U + l\eta U)^{s + 9/10 + l} - (U - l\eta U)^{s + 9/10 + l})ds}{s(s + 9/10)(s + 9/10 + 1)\ldots(s + 9/10 + l)} =$$

$$= O\left(\frac{U^{l + 9/10 + 21/220}}{U(\eta U)^l} \cdot \right.$$

$$\left. \cdot \left\{ \int_0^{10} \frac{(t + 2)^{10}\, dt}{(21/220)(21/220 + 9/10)\ldots(21/220 + 9/10 + l)} + \int_{10}^{X^{1000}} \frac{t^{11}\, dt}{t^{l+2}} \right\} \right) =$$

$$= O\left(\frac{1}{U^{1/220}} \right),$$

$$\frac{1}{U(\eta U)^l} \int_0^{\eta U} \cdots \int_0^{\eta U} \frac{1}{2\pi i} \int_{21/220 + iX^{1000}}^{2 + iX^{1000}} Z\, ds\, da_1 \ldots da_l = \tag{5.16}$$

$$= \frac{1}{U(\eta U)^l} \frac{1}{2\pi i} \int_{21/220 + iX^{1000}}^{2 + iX^{1000}} \cdot$$

$$\cdot \frac{F(s)((2U + l\eta U)^{s + 9/10 + l} - (U - l\eta U)^{s + 9/10 + l})\, ds}{s(s + 9/10)(s + 9/10 + 1)\ldots(s + 9/10 + l)} =$$

$$= O\left(\frac{U^{l+3} X^{10000}}{U(\eta U)^l X^{1000(l+2)}} \right) = O\left(\frac{U^2 X^{10000}}{X^{1000(l+2)}} \right)$$

and similarly to (5.16),we obtain,

$$(5.17)\qquad \frac{1}{U(\eta U)^l}\int_0^{\eta U}\cdots\int_0^{\eta U}\frac{1}{2\pi i}\int_{2-iX^{1000}}^{21/220-iX^{1000}} Z\,ds\,da_1\ldots da_l = O\left(\frac{U^2X^{10000}}{X^{1000(l+2)}}\right).$$

We note that $T \leqslant U \leqslant X^{100}$; $X = T^{200}$; $\eta = 1/3000$; $l = 20$. Hence, from (5.7) to (5.13) and (5.15) to (5.17), we obtain that

$$\exp\left(A_{20}\left(\frac{\log U}{\log\log U}\right)^{1/2}\right) \ll \frac{1}{U^{1/220}}$$

which is a contradiction. This proves the theorem.

6. - Concluding remarks.

In [12] E. Richert has proved the following two theorems.

THEOREM 6.1. *For any sequence of complex numbers g_n, suppose there holds*

$$\sum_{n \leqslant x} g_n = x + O(x^{1/2-\varepsilon})$$

and suppose there exists a constant A_{21} such that

$$\sum_{mn \leqslant x} g_m g_n = x\log x + A_{21}x + O(x^{\alpha})$$

then, $\alpha \geqslant 1/4$.

THEOREM 6.2. *Let l be a natural number $\geqslant 3$. Let g_n be a sequence of complex numbers for which there holds*

$$\sum_{n \leqslant x} g_n = x + O(x^{1/2-1/2l+\varepsilon})$$

for every $\varepsilon > 0$. Suppose that

$$\sum_{n_1 n_2 \ldots n_l \leqslant x} g_{n_1}\cdots g_{n_l} = xP_{l-1}(\log x) + O(x^{\alpha})$$

is true, where P_{l-1} is a polinomial of degree $l-1$, then there holds $\alpha \geqslant 1/2 - 1/2l$.

In fact, the method of Balasubramanian and Ramachandra can be applied to much more general situations. As a trivial extension of the method described in the present paper, we have the following application. If we define, for $l \geqslant 2$

$$A_l(u) = \sum_{n_1 n_2 \dots n_l \leqslant u} a_{n_1} \dots a_{n_l} = M_l(u) + E_l(u) \tag{6.1}$$

where $M_l(u)$ is the main term arising from the poles of $F^l(s)$ and $E_l(u)$ is the error term. As in the previous method we write

$$F^l(s) = \left\{ \prod_{k=1}^{j_1} \chi(ks) \right\}^l \left\{ \left(\prod_{k=1}^{j_1} \zeta(1-ks) \right) \left(\prod_{k=j_1+1}^{\infty} \zeta(ks) \right) \right\}^l . \tag{6.2}$$

We fix $\sigma = 1/D_1$. Here j_1 and D_1 are positive integers to be chosen such that

$$1 - \frac{k}{D_1} \geqslant \frac{1}{2} \quad \text{for every } k = 1, 2, \dots, j_1 ; \tag{6.3}$$

$$\frac{k}{D_1} \geqslant \frac{1}{2} \quad \text{for every } k = j_1 + 1, j_1 + 2, \dots ; \tag{6.4}$$

and for each fixed l, there exist positive integers j_1 and D_1 satisfying the following inequality

$$\frac{l j_1}{2} - \frac{l j_1 (j_1 + 1)}{2 D_1} \geqslant \frac{1}{2} . \tag{6.5}$$

Similarly, there exist positive integers j_2 and D_2 satisfying

$$1 - \frac{k}{D_2} \geqslant \frac{1}{2} \quad \text{for every } k = 1, 2, \dots, j_2 ; \tag{6.6}$$

$$\frac{k}{D_2} \geqslant \frac{1}{2} \quad \text{for every } k = j_2 + 1, j_2 + 2, \dots ; \tag{6.7}$$

and for each fixed l,

$$\frac{l j_2}{2} - \frac{l j_2 (j_2 + 1)}{2 D_2} \geqslant 1 . \tag{6.8}$$

Note that here j_1, D_1, j_2 and D_2 depend on l. Now, we can obtain, THEOREM 6.3. *There exist effective positive constants A_{22}, A_{23} and A_{24} such that*

$$\max_{T \leq u \leq X^{100}} \left| \frac{E_l(u)}{u^{1/D_1}} \right| \gg (\log T)^{A_{22}} ;$$

$$\min_{X \leq u \leq X^{A_{23}}} \left(\frac{E_l(u)}{u^{1/D_2}} \right) < -\exp\left\{ A_{24} \left(\frac{\log X}{\log\log X} \right)^{1/2} \right\};$$

$$\max_{X \leq u \leq X^{A_{23}}} \left(\frac{E_l(u)}{u^{1/D_2}} \right) > \exp\left\{ A_{24} \left(\frac{\log X}{\log\log X} \right)^{1/2} \right\};$$

Here A_{22} and A_{24} depend on l.

The method of Balasubramanian and Ramachandra for the solution of Ω problems and $\Omega_\pm$ problems can be applied in fairly general situations and gives explicit effective theorems like Theorem 1.1 and Lemma 3.11. These give respectively the explicit effective Ω theorems like the corollary to Theorem 1.1 and explicit $\Omega_\pm$ theorems like Theorem 1.2 respectively. However the $\Omega_\pm$ results are at present weaker than what are generally expected.

In conclusion we would like to mention an important result of D. R. Heath-Brown [5] namely

$$\int_1^X (E(x))^2 \, dx \ll X^{4/3} (\log X)^{89}$$

for $X \geqslant 2$. This result is however not connected with the general method of Balasubramanian and Ramachandra. It must be mentioned that the result

$$\int_X^{2X} |E(x) x^{-1/6}|^2 \, dx \ll X (\log X)^2$$

for $X \geqslant 2$, (which is clearly equivalent to the above mentioned result of D.R. Heath-Brown, in fact with 89 replaced by 2) was announced (without proof) earlier by R. Balasubramanian and K. Ramachandra at the end of their paper [3] but could not be substantiated. However Heath-Brown's proof is given in detail in his paper and is perfectly reliable.

In conclusion we would also like to add another remark about a paper of Aleksandar Ivic [6] written a few years after the paper [3] of Balasubramanian and Ramachandra. In his paper A.Ivic proves the result

$$\int_1^X (E(x))^2 \, dx = \Omega(X^{4/3} \log X)$$

by a different method. However the method of Balasubramanian and Ramachandra is more powerful and yields much more as was evident in their paper [3] and as is explained out in the present paper. Regarding the method of [3] A. Ivic comments as follows: «A good technique of this type has been recently introduced by Balasubramanian and Ramachandra in a very general context. ....».

Acknowledgements. The authors wish to thank Professor R. Balasubramanian and Professor K. Ramachandra for having fruitful discussions with them. They are also thankful to the referee for pointing out some typographical errors and making valuable suggestions. The second author wishes to thank The National Board for Higher Mathematics for the financial support.

REFERENCES

[1] R. Balasubramanian, *On the frequency of Titchmarsh phenomenon for $\zeta(s)$-IV*, Hardy-Ramanujan. J., **9** (1986), pp. 1-10.

[2] R. Balasubramanian - K. Ramachandra, *On the frequency of Titchmarsh phenomenon for $\zeta(s)$-III*, Proc. Indian Acad. Sci., **86A** (1977), pp. 341-351.

[3] R. Balasubramanian - K. Ramachandra, *Some problems of analytic number theory-III*, Hardy-Ramanujan. J., 4 (1981), pp. 13-40.

[4] R. Balasubramanian, K. Ramachandra - M. V. Subbarao, *On the error function in the asymptotic formula for the counting function of K-full numbers*, Acta. Arith., **50** (1988), pp. 107-118.

[5] D. R. Heath-Brown, *The Number of Abelian groups of order at most x*, Asterisque (1991), pp. 198-200, 153-163 (Journees Arithmetiques de Lumines 17-21 Juillet (1989)).

[6] A. Ivic, *The general divisor problem*, J. Number Theory, **27** (1987), pp. 73-91.

[7] G. Kolesnik, *On the number of Abelian groups of a given order*, J. Riene Angew. Math., **329** (1981), pp. 164-175.

[8] H. L. Montgomery - R. C. Vaughan, *Hilbert's inequality*, J. London Math. Soc., 8 (2) (1974), pp. 73-82.

[9] K. Ramachandra, *Some remarks on a theorem of Montgomery and Vaughan*, J. Number Theory. **11** (1979), pp. 465-471.

[10] K. RAMACHANDRA, *Progress towards a conjecture of the mean-value of Titchmarsh series-I, Recent Progress in Analytic Number Theory 1*, edited by H. HALBERSTAM and C. HOOLEY, Academic Press (1981), pp. 303-318.

[11] K. RAMACHANDRA, *Progress towards a conjecture of the mean-value of Titchmarsh series-II*, Hardy-Ramanujan. J., 4 (1981), pp. 1-12.

[12] H. E. RICHERT, *Zur Multiplicativen Zahlentheorie*, J. Reine Angew. Math., **206** (1961), pp. 31-38.

[13] A. SCHINZEL, *On an analytic problem considered by Sierpinski and Ramanujan*, to appear.

[14] E. C. TITCHMARSH, *The Theory of the Riemann Zeta-Function*, Second edition, revised by D. R. HEATH-BROWN, Oxford University Press (1986).

Manoscritto pervenuto in redazione il 7 marzo 1995
e, in forma revisionata, il 9 ottobre 1995.