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Extension of Formal Power Series Solutions.

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Introduction.

In the classical Cauchy problem for a linear partial differential equation with initial data on a hypersurface, smooth initial data together with the equation allow to compute the Taylor series of a smooth solution at any given point of the hypersurface.

This leads to the notion of a formally non-characteristic hypersurface for a system of linear partial differential equations, that was considered in [1], [2], [10].

This remark suggests further generalizations of the Cauchy problem, where the assumption that the initial data are given on a formally noncharacteristic initial manifold is dropped and we allow formal solutions (in the sense of Whitney) of the given system on any closed subset as initial data.

The problem is then to find classical smooth solutions of the system, whose restrictions in the sense of Whitney are the given initial data.

A similar question for Whitney functions of finite order was studied in [5].

In this paper we take up an extreme case of the generalized Cauchy problem, where the initial manifold reduces to a point.

The initial data are then a (vector valued) formal power series φ at a point $x_0 \in \mathbb{R}^n$, satisfying a system $P(x, D)\varphi = 0$ at x_0 in the sense of formal power series, and we try to find a smooth function u , defined on a neighbourhood of x_0 , having φ as its Taylor

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series at x_0 and satisfying the system $P(x, D)u = 0$ on a Schwartz-regular open domain $D \subset U$ with $x_0 \in \overline{D}$.

Most of the results will be obtained for systems of linear partial differential operators with constant coefficients.

1. - Evolution pairs.

Let Ω be an open set in \mathbb{R}^n .

If F is a closed subset of Ω , and we denote by $\mathfrak{J}(F, \Omega)$ the ideal of smooth functions in Ω which vanish of infinite order on F , then the space W_F of Whitney functions on F is defined by the exact sequence

$$0 \rightarrow \mathfrak{J}(F, \Omega) \rightarrow \mathcal{E}(\Omega) \rightarrow W_F \rightarrow 0.$$

The quotient topology of $\mathcal{E}(\Omega)/\mathfrak{J}(F, \Omega)$ makes W_F a Fréchet space. We note that W_F and its Fréchet topology are independent of the choice of the open neighbourhood Ω of F in \mathbb{R}^n .

For $f \in W_F$ and $x_0 \in F$ we set

$$\tau_{x_0}(f) = \sum_{\alpha \in \mathbb{N}^n} \frac{\partial^\alpha f(x_0)}{\alpha!} (x - x_0)^\alpha$$

for the Taylor series of f at x_0 , and

$$\tau_{x_0}^m(f) = \sum_{|\alpha| \leq m} \frac{\partial^\alpha f(x_0)}{\alpha!} (x - x_0)^\alpha$$

for its Taylor polynomial of degree m at x_0 .

For any compact subset K of F and any integer $m \geq 0$ we set

$$\begin{aligned} ||| f |||_m^K &= \|f\|_m^K + |f|_m^K = \\ &= \sup_K \sum_{|\alpha| \leq m} |\partial^\alpha f(x)| + \sup_{\substack{x, y \in K \\ x \neq y}} \sum_{|\alpha| \leq m} [|x - y|^{|\alpha| - m} |\partial_y^\alpha(f(y) - \tau_x^m(f)(y))|] \end{aligned}$$

if K contains at least two distinct points; if $K = \{x_0\}$, then we set $||| f |||_m^{\{x_0\}} = \|f\|_m^{\{x_0\}}$.

These seminorms $||| \cdot |||_m^K$ define the Fréchet space structure of W_F (Whitney extension theorem, cf. [13]).

If F is regular in the sense of Schwartz (cf. [12]), for instance if F is convex, then the topology of W_F can be defined in an equivalent way by the seminorms $\| \cdot \|_m^K$.

The strong dual of W_F is the space \mathcal{E}'_F of distributions with compact support contained in F .

Let $A(x, D)$ be an $a_1 \times a_0$ matrix of linear partial differential operators with smooth coefficients in Ω . It defines a continuous linear map

$$A(x, D): \mathcal{E}^{a_0}(\Omega) \rightarrow \mathcal{E}^{a_1}(\Omega).$$

Let F be a closed subset of Ω . Since

$$A(x, D)(\mathcal{Y}(F, \Omega)^{a_0}) \subset \mathcal{Y}(F, \Omega)^{a_1},$$

we obtain, by passing to the quotients, a continuous linear map

$$A(x, D): W_F^{a_0} \rightarrow W_F^{a_1}.$$

Denote by $S_A(F)$ the space

$$S_A(F) = \{u \in W_F^{a_0} : A(x, D)u = 0\}.$$

Let $F_1 \subset F_2$ be a pair of closed subsets of Ω .

We have a natural restriction map

$$(1.1) \quad r_{F_1}^{F_2}: W_{F_2}^{a_i} \ni u \mapsto u|_{F_1} \in W_{F_1}^{a_i}$$

which commutes with $A(x, D)$.

Therefore we can consider the (generalized) Cauchy problem:

$$\begin{cases} \text{given } \varphi \in W_{F_1}^{a_0}, \text{ which satisfies } A(x, D)\varphi = 0, \\ \text{find } u \in W_{F_2}^{a_0} \text{ such that } A(x, D)u = 0 \text{ and } u|_{F_1} = \varphi. \end{cases}$$

DEFINITION 1.1. We say that (F_1, F_2) is an evolution pair for $A(x, D)$ if the restriction map (1.1) induces an epimorphism

$$(1.2) \quad S_A(F_2) \rightarrow S_A(F_1).$$

When (1.2) is injective, we say that (F_1, F_2) is a causality pair, and when (1.2) is an isomorphism we say that (F_1, F_2) is a hyperbolic pair.

Let us remark that the problem of causality has been largely studied by many authors (see, for example, [3], [8], [10]).

In the case of constant coefficients when F_1 is affine linear and F_2 is a neighbourhood of F_1 the condition that $A(D)$ be non-characteristic in the conormal directions of F_1 is necessary and sufficient for causality.

We introduce the notion of local evolution:

DEFINITION 1.2. *The pair (F_1, F_2) is a local evolution pair for $A(x, D)$ if for every $f \in W_{F_1}^{a_0}$ with $A(x, D)f = 0$ in F_1 , we can find a neighbourhood U of F_1 in \mathbb{R}^n and a function $\tilde{f} \in W_{F_2 \cap U}^{a_0}$ such that $A(x, D)\tilde{f} = 0$ in $F_2 \cap U$ and $\tilde{f}|_{F_1} = f$.*

LEMMA 1.3. *Assume that (F_1, F_2) is a local evolution pair for $A(x, D)$. Then, given any compact subset K of F_1 , we can find a neighbourhood V of K in Ω such that $(F_1, F_1 \cup (F_2 \cap V))$ is an evolution pair for $A(x, D)$.*

PROOF. Let $\{V_\nu\}$ be a countable fundamental system of closed neighbourhoods of K in Ω and set $W_\nu = V_\nu \cap F_2$.

For each ν the space

$$\mathcal{F}_\nu = \{(f, g) \in \mathcal{E}(F_1) \times \mathcal{E}(F_1 \cup W_\nu) : g|_{F_1} = f\}$$

is a Fréchet space as a closed subspace of $\mathcal{E}(F_1) \times \mathcal{E}(F_1 \cup W_\nu)$.

The projection map on the first coordinate

$$\mathcal{F}_\nu \ni (f, g) \mapsto \pi_\nu(f, g) = f \in \mathcal{E}(F_1)$$

is linear and continuous, and by assumption we have:

$$\mathcal{E}(F_1) = \bigcup_\nu \pi_\nu(\mathcal{F}_\nu).$$

By Baire's category theorem, for some ν the subspace $\pi_\nu(\mathcal{F}_\nu)$ is a set of second category in $\mathcal{E}(F_1)$ and then it is equal to $\mathcal{E}(F_1)$ by Banach's theorem (cf. [11]). ■

The following theorem shows that hypoellipticity is in general an obstruction to evolution.

THEOREM 1.4. *Assume that $A(x, D)$ is hypoelliptic in Ω .*

Let K be a compact subset of Ω .

If (K, Ω) is a local evolution pair for $A(x, D)$, then the space $S_A(K)$ is finite dimensional.

PROOF. By Lemma 1.3, since (K, Ω) is a local evolution pair for $A(x, D)$, we can find some open subset U of Ω with $K \subset U \subset\subset \Omega$ such that the pair (K, \bar{U}) is of evolution for $A(x, D)$.

This means that the restriction map

$$S_A(\bar{U}) \rightarrow S_A(K)$$

is surjective.

Therefore, by Banach's open mapping theorem, for all $m \in \mathbb{N}$ we can find $m' \in \mathbb{N}$ and a constant $c > 0$ such that

$$\forall \varphi \in S_A(K) \exists \tilde{\varphi} \in S_A(\bar{U}) \text{ s.t.}$$

$$\tilde{\varphi}|_K = \varphi \text{ and } |||\tilde{\varphi}|||_m^U \leq c |||\varphi|||_{m'}^K .$$

Since $A(x, D)$ is hypoelliptic by assumption, for every $V \subset\subset U$ and $l \in \mathbb{N}$ there is a constant $c' > 0$ such that

$$(1.3) \quad |||u|||_l^V \leq c' |||u|||_m^U \quad \forall u \in S_A(\bar{U}) .$$

Let $\phi \in S_A(K)$. Then for the extension $\tilde{\phi}$ we have:

$$|||\phi|||_l^K \leq |||\tilde{\phi}|||_l^V \leq c' |||\tilde{\phi}|||_m^U \leq cc' |||\phi|||_{m'}^K .$$

This implies that the map

$$S_A(K) \rightarrow W_K^{(m')} = \{ \text{Whitney functions on } K \text{ of order } m' \}$$

is injective.

Since the restriction map

$$W_K^{(l)} \rightarrow W_K^{(m')}$$

is compact when $l > m'$ by the Ascoli-Arzelà theorem, it follows that $S_A(K)$ is finite dimensional. ■

REMARK. The hypothesis of hypoellipticity can be replaced by the assumption that we can find an integer m sufficiently large such that every C^m solution of the homogeneous system is also a C^∞ solution (in this case (1.3) is still valid).

2. - Systems of partial differential equations with constant coefficients.

Let \mathcal{F} be a \mathbb{C} -linear space (of generalized) functions on an open subset of \mathbb{R}^n , closed by derivation, i.e. such that

$$\frac{\partial f}{\partial x_j} \in \mathcal{F} \quad \forall j = 1, \dots, n \text{ and } \forall f \in \mathcal{F} .$$

Let $\mathcal{P} = \mathbb{C}[\xi_1, \dots, \xi_n]$ denote the ring of polynomials in n indeterminates with coefficients in \mathbb{C} .

We consider \mathcal{F} as a differential \mathcal{P} -module, by letting the elements

of \mathcal{P} act on \mathcal{F} as linear partial differential operators with constant coefficients.

By this we mean that if $p(\zeta) = \sum_{|\alpha| \leq m} a_\alpha \zeta^\alpha$ is a polynomial in \mathcal{P} , and we set

$$p(D) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} a_\alpha D^\alpha,$$

where $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, with $D_j = (1/i)(\partial/\partial x_j)$, then $p(\zeta)$ acts on $f \in \mathcal{F}$ by

$$p(\zeta) f = fp(\zeta) = p(D) f.$$

Let $A_0(\zeta)$ be an $a_0 \times a_1$ matrix with coefficients in \mathcal{P} .

We consider the system

$$(2.1) \quad \begin{cases} u \in \mathcal{F}^{a_0} \\ {}^t A_0(D) u = f, \end{cases}$$

for a given $f \in \mathcal{F}^{a_1}$.

We consider the \mathcal{P} -module of finite type

$$M = \text{coker } A_0(\zeta) = \mathcal{P}^{a_0} / A_0(\zeta) \mathcal{P}^{a_1}.$$

Then the map $A_0(\zeta): \mathcal{P}^{a_1} \rightarrow \mathcal{P}^{a_0}$ can be inserted in a Hilbert resolution

$$(2.2) \quad 0 \rightarrow \mathcal{P}^{a_d} \xrightarrow{A_{d-1}} \mathcal{P}^{a_{d-1}} \rightarrow \dots \rightarrow \mathcal{P}^{a_1} \xrightarrow{A_0} \mathcal{P}^{a_0} \rightarrow M \rightarrow 0$$

by free \mathcal{P} -modules of finite type.

We note that

$$(2.3) \quad {}^t A_1(D) f = 0$$

is a necessary condition for the solvability of (2.1) and that every necessary condition for the solvability of (2.1) which can be expressed in terms of linear partial differential equations is a consequence of (2.3).

Via the natural isomorphisms

$$\text{Hom}_{\mathcal{P}}(\mathcal{P}^j, \mathcal{F}) \simeq \mathcal{F}^j$$

we obtain the interpretation of the groups $\text{Ext}_{\mathcal{P}}^j(M, \mathcal{F})$:

$$\begin{aligned} \text{Ext}_{\mathcal{P}}^0(M, \mathcal{F}) &\simeq \{ f \in \mathcal{F}^{a_0} : {}^tA_0(D) f = 0 \} = \text{Ker } {}^tA_0(D), \\ \text{Ext}_{\mathcal{P}}^j(M, \mathcal{F}) &\simeq \frac{\text{Ker}({}^tA_j(D): \mathcal{F}^{a_j} \rightarrow \mathcal{F}^{a_j+1})}{\text{Im}({}^tA_{j-1}(D): \mathcal{F}^{a_{j-1}} \rightarrow \mathcal{F}^{a_j})} = \frac{\text{Ker } {}^tA_j(D)}{\text{Im } {}^tA_{j-1}(D)}, \end{aligned}$$

for $j \geq 1$.

This isomorphisms translate statements about the solvability of systems of linear partial differential operators with constant coefficients into statements on the groups $\text{Ext}_{\mathcal{P}}^j(M, \mathcal{F})$ associated to the \mathcal{P} -module $M = \text{coker } A_0$.

More precisely, uniqueness in (2.1) is equivalent to the condition

$$\text{Ext}_{\mathcal{P}}^0(M, \mathcal{F}) = 0,$$

whereas the existence of solutions $u \in \mathcal{F}^{a_0}$ of (2.1) whenever $f \in \mathcal{F}^{a_1}$ satisfies the integrability conditions

$$(2.4) \quad {}^tA_1(D) f = 0$$

translates into

$$\text{Ext}_{\mathcal{P}}^1(M, \mathcal{F}) = 0.$$

The main advantage of this formulation is that it enlightens the algebraic invariance of the problem. In particular, existence and uniqueness in (2.1) are properties of the \mathcal{P} -module M and are independent of its presentation by a particular matrix $A_0(\xi)$.

Several classical problems in the theory of partial differential operators can be translated into the vanishing of the cohomology groups $\text{Ext}_{\mathcal{P}}^j(M, \mathcal{F})$ for different choices of the differential \mathcal{P} -modules \mathcal{F} .

3. - Extension of power series solutions.

Let F be a locally closed subset of \mathbb{R}^n , with $0 \in F$.

We note that the space $W_{\{0\}}$ can be identified with the space $C\{\{x_1, \dots, x_n\}\}$ of formal power series $f(x) = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} x^{\alpha}$, by Borel's lemma (see [6]).

For $f \in W_F$, we set $f|_{\{0\}}$ for the Taylor series of f at 0, which corresponds to f by the natural restriction map

$$W_F \rightarrow W_{\{0\}} \rightarrow 0.$$

Let $A_0(D)$ be an $a_1 \times a_0$ matrix of linear partial differential operators with constant coefficients in \mathbb{R}^n .

We formulate the Cauchy problem for the pair $(\{0\}, F)$ in the following way:

$$\begin{cases} \text{given } f \in W_F^{a_1} \text{ and } \varphi \in W_{\{0\}}^{a_0} \text{ such that } A_0(D)\varphi = f|_{\{0\}}, \\ \text{find } u \in W_F^{a_0} \text{ such that } A_0(D)u = f \text{ and } u|_{\{0\}} = \varphi. \end{cases}$$

Let $\mathfrak{J}(\{0\}, F)$ be the ideal of functions of W_F which vanish of infinite order at $\{0\}$. Then we have an exact sequence of differential \mathcal{P} -modules:

$$(3.1) \quad 0 \rightarrow \mathfrak{J}(\{0\}, F) \rightarrow W_F \rightarrow W_{\{0\}} \rightarrow 0.$$

If $A_1(D)$ is an $a_2 \times a_1$ matrix of partial differential operators with constant coefficients giving a basis for the integrability conditions of $A_0(D)$, a necessary condition in order that the Cauchy problem be solvable is that

$$A_1(D)f = 0.$$

These integrability conditions for f will be therefore assumed in the following.

Besides, by the extendability of Whitney functions, φ extends to an element $\tilde{\varphi} \in W_F^{a_0}$. Then, upon substituting $f - A_0(D)\tilde{\varphi}$ to f , we are reduced to a new Cauchy problem of the form:

$$\begin{cases} \text{given } f \in \mathfrak{J}(\{0\}, F)^{a_1} \text{ satisfying } A_1(D)f = 0, \\ \text{find } u \in \mathfrak{J}(\{0\}, F)^{a_0} \text{ such that } A_0(D)u = f. \end{cases}$$

Because $\mathfrak{J}(\{0\}, F)$ is a \mathcal{P} -module, existence or uniqueness in the Cauchy problem for the pair $(\{0\}, F)$ are properties of the \mathcal{P} -module $M = \text{coker}({}^t A_0(\xi))$.

Therefore we are led to the following:

1) The pair $(\{0\}, F)$ is of *evolution* for M in the Whitney class if and only if

$$\text{Ext}_{\mathcal{P}}^1(M, \mathfrak{J}(\{0\}, F)) = 0.$$

2) The pair $(\{0\}, F)$ is *hyperbolic* for M in the Whitney class if and only if

$$\text{Ext}_{\mathcal{P}}^0(M, \mathfrak{J}(\{0\}, F)) = \text{Ext}_{\mathcal{P}}^1(M, \mathfrak{J}(\{0\}, F)) = 0.$$

3) The pair $(\{0\}, F)$ is of *causality* for M in the Whitney class if

and only if

$$\text{Ext}_{\mathcal{P}}^0(M, \mathfrak{J}(\{0\}, F)) = 0.$$

From the exact sequence (3.1) we deduce, for every \mathcal{P} -module M , the long exact sequence

$$\begin{aligned} 0 \rightarrow \text{Ext}_{\mathcal{P}}^0(M, \mathfrak{J}(\{0\}, F)) &\rightarrow \text{Ext}_{\mathcal{P}}^0(M, W_F) \rightarrow \\ &\rightarrow \text{Ext}_{\mathcal{P}}^0(M, W_{\{0\}}) \rightarrow \text{Ext}_{\mathcal{P}}^1(M, \mathfrak{J}(\{0\}, F)) \rightarrow \\ &\rightarrow \text{Ext}_{\mathcal{P}}^1(M, W_F) \rightarrow \text{Ext}_{\mathcal{P}}^1(M, W_{\{0\}}) \rightarrow \dots \end{aligned}$$

Assume now that F is a convex set. Then we have (cf.[9])

$$\begin{aligned} \text{Ext}_{\mathcal{P}}^j(M, W_{\{0\}}) &= 0 \quad \forall j \geq 1, \\ \text{Ext}_{\mathcal{P}}^j(M, W_F) &= 0 \quad \forall j \geq 1. \end{aligned}$$

Therefore, for a convex F , the long exact sequence above reduces to

$$\begin{aligned} 0 \rightarrow \text{Ext}_{\mathcal{P}}^0(M, \mathfrak{J}(\{0\}, F)) &\rightarrow \text{Ext}_{\mathcal{P}}^0(M, W_F) \rightarrow \\ &\rightarrow \text{Ext}_{\mathcal{P}}^0(M, W_{\{0\}}) \rightarrow \text{Ext}_{\mathcal{P}}^1(M, \mathfrak{J}(\{0\}, F)) \rightarrow 0. \end{aligned}$$

It follows that:

1) The pair $(\{0\}, F)$ is of evolution for M in the Whitney class if and only if the homomorphism

$$(3.2) \quad \text{Ext}_{\mathcal{P}}^0(M, W_F) \rightarrow \text{Ext}_{\mathcal{P}}^0(M, W_{\{0\}})$$

is onto.

2) The pair $(\{0\}, F)$ is hyperbolic for M in the Whitney class if and only if the homomorphism (3.2) is an isomorphism.

3) The pair $(\{0\}, F)$ is of causality for M in the Whitney class if and only if the homomorphism (3.2) is injective.

It is well known that if F is a neighbourhood of 0, then the pair $(\{0\}, F)$ is of causality for $A_0(D)$ if and only if $A_0(D)$ is elliptic.

If $0 \in \partial F$ and $\overset{\circ}{F} \neq \emptyset$ we can say that if the pair $(\{0\}, F)$ is of causality for $A_0(D)$, then $A_0(D)$ is elliptic.

Let us study now evolution.

LEMMA 3.1. *A necessary and sufficient condition in order that*

the pair $(\{0\}, F)$ be of evolution is that the restriction homomorphism (3.2) has a closed image.

PROOF. Necessity is obvious.
Sufficiency follows because

$$\text{Ext}_{\mathcal{P}}^0(M, W_{\mathbb{R}^n}) \rightarrow \text{Ext}_{\mathcal{P}}^0(M, W_{\{0\}})$$

has always a dense image. ■

Using two algebraic lemmas proved in [10], we have:

PROPOSITION 3.2. *A necessary and sufficient condition in order that the pair $(\{0\}, F)$, for a locally closed convex set F with $0 \in F$, be of evolution (resp. hyperbolic, of causality) in the Whitney class for M is that it be of evolution (resp. hyperbolic, of causality) for \mathcal{P}/\wp for every associated prime ideal \wp of M .*

Therefore, there is no loss of generality in restricting our considerations to modules of the form \mathcal{P}/\wp for a prime ideal $\wp \subset \mathcal{P}$.

Having fixed a prime ideal $\wp \subset \mathcal{P}$, we set

$$V = V(\wp) = \{ \zeta \in \mathbb{C}^n : p(\zeta) = 0 \ \forall p \in \wp \}.$$

Identifying $\text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\wp, W_F)$ to the space of Whitney functions $u \in W_F$ such that $p(D)u = 0$ for all $p \in \wp$, we give to $\text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\wp, W_F)$ a natural structure of Fréchet space, by the family of semi-norms

$$\|f\|_m^K = \sup_K \sum_{|\alpha| \leq m} |D^\alpha f(x)|$$

for compact convex subsets K of F , and for $m \in \mathbb{N}$.

When $V(\wp)$ reduces to one point, the pair $(\{0\}, F)$ is trivially hyperbolic for every choice of a connected locally closed $F \ni 0$.

Thus we will assume in the following that $\dim_{\mathbb{C}} V(\wp) \geq 1$.

By Banach's open mapping theorem, if the pair $(\{0\}, F)$ is of evolution for \mathcal{P}/\wp in the Whitney class, then the surjective restriction map

$$\text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\wp, W_F) \rightarrow \text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\wp, W_{\{0\}})$$

is open.

Therefore:

LEMMA 3.3. *If the pair $(\{0\}, F)$ is of evolution for \mathcal{P}/\wp in the Whitney class, then for every compact subset K of F , and every integer $m \geq$*

≥ 0 , we can find an integer $m' \geq 0$ and a constant $c > 0$ such that

$$\forall f \in \text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\mathcal{P}, W_{\{0\}}) \text{ with } \|f\|_m^{\{0\}} \leq 1,$$

$$\exists \tilde{f} \in \text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\mathcal{P}, W_F) \text{ with } \|\tilde{f}\|_m^K \leq c \text{ and } \tilde{f}|_{\{0\}} = f.$$

It follows:

LEMMA 3.4. *If the pair $(\{0\}, F)$ is of evolution for \mathcal{P}/\mathcal{P} in the Whitney class, then for every compact subset K of F and every integer $m \geq 0$, there is an integer $m' \geq 0$ such that for every distribution T with compact support in K such that*

$$(3.3) \quad |T(f)| \leq c \|f\|_m^K \quad \forall f \in \text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\mathcal{P}, W_F)$$

$$(3.4) \quad T(f) = 0 \quad \forall f \in \text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\mathcal{P}, \mathfrak{J}(\{0\}, F)),$$

we can find a distribution T_1 of order m' and support in $\{0\}$, such that

$$T_1(f) = T(f) \quad \forall f \in \text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\mathcal{P}, W_F).$$

Therefore, every continuous linear functional on $\text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\mathcal{P}, W_F)$ which can be carried by $\{0\}$ uniquely extends to a continuous linear functional on $\text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\mathcal{P}, W_{\{0\}})$.

PROOF. Let T be a distribution with compact support in K such that (3.3) and (3.4) hold.

By Lemma 3.3 we know that

$$\forall f \in \text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\mathcal{P}, W_{\{0\}}) \exists \tilde{f} \in \text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\mathcal{P}, W_F) \text{ such that}$$

$$\|\tilde{f}\|_m^K \leq c' \|f\|_m^{\{0\}} \text{ and } \tilde{f}|_{\{0\}} = f.$$

Then, for every $f \in \text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\mathcal{P}, W_F)$, we can find $\tilde{f} \in \text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\mathcal{P}, W_F)$ such that $f - \tilde{f} \in \text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\mathcal{P}, \mathfrak{J}(\{0\}, F))$ and

$$\|\tilde{f}\|_m^K \leq c' \|f|_{\{0\}}\|_m^{\{0\}}.$$

We obtain

$$|T(f)| = |T(\tilde{f})| \leq c \|\tilde{f}\|_m^K \leq cc' \|f|_{\{0\}}\|_m^{\{0\}}$$

and hence, by the Hahn-Banach's theorem, we can find a distribution T_1

of order m' and support in $\{0\}$ such that

$$T_1(f) = T(f) \quad \forall f \in \text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\varphi, W_F).$$

The last part of the statement follows because the restriction map

$$\text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\varphi, W_F) \rightarrow \text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\varphi, W_{\{0\}})$$

always has a dense image. ■

The dual space of $\text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\varphi, W_F)$ is a quotient of the space of distributions with compact support in F and, using the Fourier-Laplace transform and Ehrenpreis fundamental principle, it can be identified to the inductive limit $\mathcal{O}_F(V)$ for $K \subset\subset F$ and $m \in \mathbb{N}$ of

$$\mathcal{O}_K^{(m)}(V) = \left\{ f \in \mathcal{O}(V) : \sup_V \frac{|f(\xi)|}{(1 + |\xi|)^m \exp H_K(\xi)} < +\infty \right\},$$

where $H_K(\xi) = \sup_K \text{Im} \langle x, \xi \rangle$ is the supporting function of K . Here we use $\mathcal{O}(V)$ for the space of holomorphic functions on V (continuous on V and holomorphic at non-singular points of V).

Note that $H_{\{0\}}(\xi) = 0$. In particular, the dual of $\text{Ext}_{\mathcal{P}}^0(\mathcal{P}/\varphi, W_{\{0\}})$ is identified to the space \mathcal{P}/φ of restriction to V of holomorphic polynomials in \mathbb{C}^n .

Therefore, by Lemma 3.4, we have the following:

THEOREM 3.5. *The pair $(\{0\}, F)$ is of evolution for \mathcal{P}/φ in the Whitney class if and only if the following Phragmén-Lindelöf principle holds:*

$$\forall K \subset\subset F \text{ and } m \in \mathbb{N}$$

$$\exists m' \in \mathbb{N}, c > 0 \text{ such that}$$

$$\text{if } f \in \mathcal{P} \text{ satisfies}$$

$$|f(\xi)| \leq (1 + |\xi|)^m \exp H_K(\text{Im } \xi) \quad \forall \xi \in V$$

$$\text{then it also satisfies}$$

$$|f(\xi)| \leq c(1 + |\xi|)^{m'} \quad \forall \xi \in V.$$

PROOF. If the pair $(\{0\}, F)$ is of evolution, since

$$(\text{Ext}_{\mathcal{P}/\wp}^0(\mathcal{P}/\wp, W_F))' \simeq \mathcal{O}_F(V) = \varinjlim_{\substack{K \subset \subset F \\ m \nearrow \infty}} \mathcal{O}_K^{(m)}(V)$$

and

$$(\text{Ext}_{\mathcal{P}/\wp}^0(\mathcal{P}/\wp, W_{\{0\}}))' \simeq \mathcal{O}_{\{0\}}(V) = \varinjlim_{m \nearrow \infty} \mathcal{O}_{\{0\}}^{(m)}(V) \simeq \mathcal{P}/\wp,$$

then, by Lemma 3.4 the Phragmén-Lindelöf principle holds.

Vice versa, let us assume that the Phragmén-Lindelöf principle is valid.

By Lemma 3.1 we need only to prove that the homomorphism (3.2) has a closed image, i.e. that the dual inclusion homomorphism

$$\mathcal{P}/\wp \hookrightarrow \mathcal{O}_F(V)$$

has a closed image.

The family

$$\mathcal{X}_K^{(m)} = \left\{ f \in \mathcal{O}_F(V) : \sup_V \frac{|f(\xi)|}{(1 + |\xi|)^m \exp H_K(\text{Im } \xi)} \leq m \right\}$$

is a fundamental covering of $\mathcal{O}_F(V)$, (cf. [4]), therefore it suffices to prove that $\mathcal{P}/\wp \cap \mathcal{X}_K^{(m)}$ is closed in $\mathcal{X}_K^{(m)}$ for all $m \in \mathbb{N}$, $K \subset \subset F$.

Let us fix $m \in \mathbb{N}$ and $K \subset \subset F$, and let $f \in \mathcal{X}_K^{(m)}$ be such that $f_n \rightarrow f$ in $\mathcal{O}_K^{(m)}(V)$ for a sequence $\{f_n\} \subset \mathcal{P}/\wp \cap \mathcal{X}_K^{(m)}$.

We want to prove that $f \in \mathcal{P}/\wp$.

Indeed, by the Phragmén-Lindelöf principle, there are $m' \in \mathbb{N}$ and $c > 0$ such that

$$\|f_n\|_m^{(0)} \leq mc \quad \forall n \in \mathbb{N}.$$

This means that the sequence $\{f_n\}$ is bounded and hence relatively compact in $\mathcal{O}_{\{0\}}^{(m')}(V)$.

In particular, there is a subsequence $\{f_{n_h}\}$ such that

$$f_{n_h} \rightarrow \tilde{f} \in \mathcal{O}_{\{0\}}(V) \simeq \mathcal{P}/\wp \quad \text{in } \mathcal{O}_{\{0\}}^{(m')}(V)$$

and hence

$$f_{n_h} \rightarrow \tilde{f} \in \mathcal{P}/\wp \quad \text{in } \mathcal{O}_K^{(m')}(V).$$

Since $f_{n_h} \rightarrow f$ in $\mathcal{O}_K^{(m)}(V)$, we finally have $f = \tilde{f} \in \mathcal{P}/\wp$. ■

For $P \in \mathcal{P}/\wp \setminus \{0\}$ we set

$$\deg P = \limsup_{\substack{\zeta \in V \\ |\zeta| \rightarrow \infty}} \frac{\log |P(\zeta)|}{\log(1 + |\zeta|)}$$

and

$$(\mathcal{P}/\wp)^{(m)} = \{P \in \mathcal{P}/\wp : \deg P \leq m\},$$

which is a finite dimensional vector space over \mathbb{C} .

Then, by the Phragmén-Lindelöf principle above we have:

THEOREM 3.6. *A necessary and sufficient condition for the pair $(\{0\}, F)$ to be of evolution for \mathcal{P}/\wp in the Whitney class is that for every compact convex subset K of F and every integer $m \geq 0$ the space $\mathcal{P}/\wp \cap \mathcal{O}_K^{(m)}(V)$ is finite dimensional.*

PROOF. Necessity follows from Theorem 3.5.

Vice versa, let us set $E = \mathcal{P}/\wp \cap \mathcal{O}_K^{(m)}(V)$.

If E is finite dimensional, then

$$\tilde{E} = \left\{ f \in E : \sup_V \frac{|f(\zeta)|}{(1 + |\zeta|)^m \exp H_K(\text{Im } \zeta)} = 1 \right\}$$

is a compact subset of E .

Then $E \subset (\mathcal{P}/\wp)^{(m')}$ for some $m' \in \mathbb{N}$, and \tilde{E} is a compact subset of $(\mathcal{P}/\wp)^{(m')}$.

It follows that

$$\sup \left\{ \frac{|f(\zeta)|}{(1 + |\zeta|)^{m'}} : \zeta \in V, f \in E \right\} < +\infty$$

and it is a maximum. ■

Let $d = \dim_{\mathbb{C}} V \geq 1$. By the preparation lemma (cf. [2]), after a real linear change of coordinates in \mathbb{C}^n we can assume that the following conditions are satisfied:

(1) The projection map $\pi: V \rightarrow \mathbb{C}^d$ given by

$$V \ni (\zeta_1, \dots, \zeta_n) \mapsto (\zeta_1, \dots, \zeta_d) \in \mathbb{C}^d$$

is proper and surjective.

(2) $\mathbb{C}[\zeta_1, \dots, \zeta_d] \cap \wp = \{0\}$.

(3) \mathcal{P}/\wp is integral over $\mathbb{C}[\zeta_1, \dots, \zeta_d]$.

(4) For all $j = 1, \dots, n - d$, we can find an irreducible polynomial $P_j(\zeta_1, \dots, \zeta_d, \zeta_{d+j})$ in $\wp \cap \mathbb{C}[\zeta_1, \dots, \zeta_d, \zeta_{d+j}]$ that is monic with respect to ζ_{d+j} .

(5) $\wp \cap \mathbb{C}[\zeta_1, \dots, \zeta_d, \zeta_{d+1}]$ is a principal ideal generated by P_1 .

(6) Let $\Delta = \Delta(\zeta_1, \dots, \zeta_d)$ be the discriminant of P_1 with respect to ζ_{d+1} . Then there are polynomials $Q_2, \dots, Q_{n-d} \in \mathbb{C}[\zeta_1, \dots, \zeta_{d+1}]$ such that

$$\Delta \zeta_{d+j} - Q_j \in \wp \quad \text{for } j = 2, \dots, n - d.$$

$$(7) \quad |\zeta| \leq c \left(1 + \sum_1^d |\zeta_j|^2 \right)^{m/2} \quad \text{for } \zeta \in V, \text{ with } c, m \geq 0.$$

Moreover, in (7) we can take, by a suitable choice of real coordinates, $m = 1$, and also we can assume that the principal parts of P_1, \dots, P_{n-d} are monic with respect to $\zeta_{d+1}, \dots, \zeta_n$ respectively.

Then we can consider \mathcal{P}/\wp as a $\mathbb{C}[\zeta_1, \dots, \zeta_d]$ -module.

When the pair $(\{0\}, F)$ is of evolution for \mathcal{P}/\wp in the Whitney class, it follows that for every $K \subset\subset F$ the sequence

$$\mathcal{P}/\wp \cap \mathcal{O}_K^{(0)}(V) \subset \mathcal{P}/\wp \cap \mathcal{O}_K^{(1)}(V) \subset \dots \subset \mathcal{P}/\wp \cap \mathcal{O}_K^{(m)}(V) \subset \dots$$

gives a good filtration of \mathcal{P}/\wp as a $\mathbb{C}[\zeta_1, \dots, \zeta_d]$ -module.

According to Hilbert's theorem, we have

$$\dim_{\mathbb{C}} \mathcal{P}/\wp \cap \mathcal{O}_K^{(m)}(V) = a_s m^s + a_{s-1} m^{s-1} + \dots + a_1 m + a_0$$

for some rational numbers a_0, a_1, \dots, a_s with $a_s \neq 0$, and some integer s with $0 \leq s \leq d$, for every m greater or equal to some $m_0 \geq 0$.

Hence we deduce:

THEOREM 3.7. *A necessary and sufficient condition for the pair $(\{0\}, F)$ to be of evolution for \mathcal{P}/\wp in the Whitney class is that for every compact convex subset K of F we can find $k > 0$ such that*

$$\mathcal{P}/\wp \cap \mathcal{O}_K^{(m)}(V) \subset (\mathcal{P}/\wp)^{(km)} \quad \text{for } m \geq 1.$$

From the preparation lemma we have also that for every polynomial $Q \in \mathcal{P}$ there is a unique polynomial $Q' \in \mathbb{C}[\zeta_1, \dots, \zeta_d, \zeta_{d+1}]$, of degree with respect to ζ_{d+1} less than the degree m_1 of P_1 with respect to ζ_{d+1} , such that $Q' - \Delta Q \in \wp$.

Hence we have:

THEOREM 3.8. *A necessary and sufficient condition for the pair*

$(\{0\}, F)$ to be of evolution for \mathcal{P}/\wp in the Whitney class is that for every compact convex subset K of F we can find $k \geq 0$ such that if a polynomial $P \in C[\zeta_1, \dots, \zeta_d, \zeta_{d+1}]$ has degree with respect to ζ_{d+1} less than the degree m_1 of P_1 with respect to ζ_{d+1} , and belongs to $\mathcal{O}_K^{(m)}(V)$, then its degree as an element of \mathcal{P}/\wp does not exceed km .

PROOF. Let us assume that for every compact convex subset K of F we can find $k \geq 0$ such that if a polynomial $P \in C[\zeta_1, \dots, \zeta_d, \zeta_{d+1}]$ has degree with respect to ζ_{d+1} less than the degree m_1 of P_1 with respect to ζ_{d+1} , and belongs to $\mathcal{O}_K^{(m)}(V)$, then its degree as an element of \mathcal{P}/\wp does not exceed km , and let us prove that the pair $(\{0\}, F)$ is of evolution for \mathcal{P}/\wp in the Whitney class.

Let us take $Q \in \mathcal{P}/\wp \cap \mathcal{O}_K^{(m)}(V)$.

By Theorem 3.7 we have to prove that $Q \in (\mathcal{P}/\wp)^{(km)}$ for some $k > 0$.

By the preparation lemma we know that there exists a unique polynomial $Q' \in C[\zeta_1, \dots, \zeta_{d+1}]$ of degree with respect to ζ_{d+1} less than m_1 , such that $Q' - \Delta Q \in \wp$. This means that $Q' = \Delta Q$ on V .

Since $Q \in \mathcal{O}_K^{(m)}(V)$, we have that

$$|Q'(\zeta)| \leq c(1 + |\zeta|)^{(m+\nu)} \exp H_K(\zeta) \quad \text{on } V$$

where ν is the degree of Δ as an element of \mathcal{P}/\wp .

Therefore $Q' \in \mathcal{O}_K^{(m+\nu)}(V)$ and hence, by hypothesis, its degree as an element of \mathcal{P}/\wp is less or equal to $k(m + \nu)$ for some $k > 0$.

This means that ΔQ has degree, as an element of \mathcal{P}/\wp , less or equal to $k(m + \nu)$.

Since the ideal generated by Δ is closed in $\mathcal{O}_K(V)$, we have a closed map of Fréchet spaces

$$\mathcal{O}_K(V) \rightarrow \mathcal{O}_K(V)$$

$$Q \mapsto \Delta Q.$$

This is then a (strong) homomorphism and hence the estimate found for ΔQ yields to a similar estimate for Q .

The vice versa is obvious. ■

EXAMPLE 1. Let us consider the Schrödinger operator

$$L = \frac{\partial}{\partial t} + i\Delta$$

in \mathbb{R}^{n+1} .

The corresponding algebraic variety is

$$V = \{(\tau, \zeta) \in \mathbb{C}^{n+1} : \tau = -(\zeta_1^2 + \dots + \zeta_n^2)\}.$$

Then we can identify $\mathcal{P}/\wp \simeq \mathbb{C}[\zeta_1, \dots, \zeta_n]$ by the map

$$\mathbb{C}[\tau, \zeta_1, \dots, \zeta_n] \rightarrow \mathbb{C}[\zeta_1, \dots, \zeta_n]$$

$$P \mapsto \tilde{P} = P(-(\zeta_1^2 + \dots + \zeta_n^2), \zeta_1, \dots, \zeta_n).$$

For each convex compact subset K of \mathbb{R}^{n+1} , the inequality

$$|P(\tau, \zeta)| \leq c(1 + |(\tau, \zeta)|)^m \exp H_K(\tau, \zeta)$$

becomes

$$|\tilde{P}(\zeta)| \leq c'(1 + |\zeta|)^{2m} \exp H_K(-(\zeta_1^2 + \dots + \zeta_n^2), \zeta_1, \dots, \zeta_n)$$

and hence

$$|\tilde{P}(\xi)| \leq c'(1 + |\xi|)^{2m} \quad \text{for } \xi \in \mathbb{R}^n.$$

It follows that \tilde{P} has degree less or equal to $2m$ and therefore P has degree less or equal to $2m$ as a polynomial in \mathcal{P}/\wp .

Therefore the pair $(\{0\}, \mathbb{R}^{n+1})$ is of evolution for L in the class of Whitney functions.

EXAMPLE 2. Let us consider the heat operator

$$L = \frac{\partial}{\partial t} - \Delta$$

in \mathbb{R}^{n+1} .

The corresponding algebraic variety is

$$V = \{(\tau, \zeta) \in \mathbb{C}^{n+1} : \tau = -i(\zeta_1^2 + \dots + \zeta_n^2)\}.$$

If K is a compact convex set contained in $\mathbb{R}_+^{n+1} = \{(t, x) \in \mathbb{R}^{n+1} : t \geq 0\}$, then $H_K(\tau, \xi) = 0$ for $\xi \in \mathbb{R}^n$ and so we can argue as in the case of the Schrödinger operator.

It follows that the pair $(\{0\}, \mathbb{R}_+^{n+1})$ is of evolution for L in the class of Whitney functions.

EXAMPLE 3. Let us consider the Cauchy-Riemann operator

$$L = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$$

in \mathbb{R}^2 .

The associated algebraic variety is

$$V = \{(\xi, \eta) \in \mathbb{C}^2 : \eta = i\xi\}.$$

If K is a compact convex subset of $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$, then

$$H_K(\xi, \eta) = \sup_{(x, y) \in K} \operatorname{Im}(\xi x + i\xi y) = 0 \quad \text{for } \xi \in \mathbb{R}, \xi \leq 0.$$

It follows that every $P \in \mathcal{P}/\varphi$ which satisfies

$$|P(\xi, \eta)| \leq c(1 + |(\xi, \eta)|)^m \exp H_K(\xi, \eta) \quad \text{on } V$$

also satisfies

$$|P(\xi, i\xi)| \leq c(1 + |\xi|)^m \quad \text{for } \xi \in \mathbb{R}, \xi \leq 0.$$

Therefore P has degree less or equal to m as an element of \mathcal{P}/φ .

It follows that the pair $(\{0\}, \mathbb{R}_+^2)$ is of evolution for L in the class of Whitney functions.

EXAMPLE 4. Let φ be a principal ideal in $\mathbb{C}[\zeta_1, \zeta_2, \zeta_3]$ generated by $P(\zeta) = \zeta_1 \zeta_2 + i\zeta_3^2$, and let $V = V(\varphi)$.

We have that

$$V \cap \mathbb{R}^3 = \{(\xi_1, \xi_2, \xi_3) \in V \cap \mathbb{R}^3 : \xi_3 = 0\}.$$

Let us write $\zeta_j = r_j e^{i\theta_j}$ for $j = 1, 2$. Then for all $(\zeta_1, \zeta_2, \zeta_3)$ in V we have:

$$\zeta_3^2 = -\frac{\zeta_1 \zeta_2}{i} = r_1 r_2 e^{i(\theta_1 + \theta_2 + \pi/2)}$$

and hence

$$\begin{aligned} \zeta_3 &= \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2 + \pi/2)/2} \text{ or} \\ \zeta_3 &= \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2 + \pi/2 + 2\pi)/2} = -\sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2 + \pi/2)/2}. \end{aligned}$$

Therefore for $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in V$ we have that

$$|\operatorname{Im} \zeta|^2 = r_1^2 \sin^2 \theta_1 + r_2^2 \sin^2 \theta_2 + r_1 r_2 \sin^2((\theta_1 + \theta_2 + \pi/2)/2)$$

and hence for $r_1 r_2 > 0$ we obtain:

$$\begin{aligned} |\zeta|^2 |\operatorname{Im} \zeta|^2 &= \\ &= (r_1^2 + r_2^2 + r_1 r_2)(r_1^2 \sin^2 \theta_1 + r_2^2 \sin^2 \theta_2 + r_1 r_2 \sin^2([\theta_1 + \theta_2 + \pi/2]/2)) = \\ &= r_1^2 r_2^2 \left(\frac{r_1}{r_2} + \frac{r_2}{r_1} + 1 \right) \left(\frac{r_1}{r_2} \sin^2 \theta_1 + \frac{r_2}{r_1} \sin^2 \theta_2 + \sin^2([\theta_1 + \theta_2 + \pi/2]/2) \right) \geq \\ &\geq r_1^2 r_2^2 (\sin^2 \theta_1 + \sin^2 \theta_2 + \sin^2([\theta_1 + \theta_2 + \pi/2]/2)) \geq k r_1^2 r_2^2 \quad \text{with } k > 0. \end{aligned}$$

Therefore

$$|\zeta_3| = \sqrt{r_1 r_2} \leq k' |\zeta|^{1/2} |\operatorname{Im} \zeta|^{1/2} \quad \text{on } V, \text{ for } k' > 0.$$

On the other hand we cannot have any inequality of the form

$$|\zeta_3| \leq |\zeta|^\varepsilon \exp A |\operatorname{Im} \zeta| \quad \text{with } \varepsilon < \frac{1}{2} \text{ and } A > 0.$$

Indeed, if we consider points of the form

$$(t, -i, \sqrt{t}) \in V \quad \text{with } t > 0,$$

we would have

$$\sqrt{t} \leq (t^2 + 1 + t)^{\varepsilon/2} e^A \quad \text{with } \varepsilon < \frac{1}{2}$$

which is not true for $t \rightarrow +\infty$.

Let us fix $m \geq 0$ and let us assume that $q \in C[\zeta_1, \zeta_2, \zeta_3]$ satisfies

$$(3.5) \quad |q(\zeta)| \leq c(1 + |\zeta|)^m \exp(A |\operatorname{Im} \zeta|) \quad \forall \zeta \in V,$$

and that $\nu = \deg q > m$.

Since (3.5) is trivial for polynomials of degree less or equal to m , we can assume that q is of the form

$$q = \sum_{j=m+1}^{\nu} q_j$$

for q_j homogeneous of degree j .

By (3.5), for $\zeta = t\theta$ with t real > 0 and $\theta \in V \cap \mathbb{R}^3$ we have that $q_j(\theta) = 0$ for $j = m + 1, \dots, \nu$.

Therefore every q_j can be written, mod \wp , in the form

$$q_j = \zeta_3^{h_j} \tilde{q}_j(\zeta)$$

where $\tilde{q}_j(\zeta)$ is homogeneous of degree $j - h_j \geq 0$ and $\tilde{q}_j(\theta) \neq 0$ for some $\theta \in V \cap \mathbb{R}^3$.

From (3.5) we obtain that

$$\left| \sum_{j=m+1}^{\nu} t^j q_j(\zeta) \right| \leq c(1 + t|\zeta|)^m \exp At |\operatorname{Im} \zeta| \quad \text{for } \zeta \in V \text{ and } t > 0,$$

and hence

$$|q_j(\zeta)| \leq c_j(1 + |\zeta|)^m \exp A_j |\operatorname{Im} \zeta| \quad \forall \zeta \in V$$

for $j = m + 1, \dots, \nu$.

Let $\theta_j \in V \cap \mathbb{R}^3$ be such that $\tilde{q}_j(\theta_j) \neq 0$. Since \tilde{q}_j is homogeneous and $V \cap \mathbb{R}^3 = \{(t, 0, 0) : t \in \mathbb{R}\} \cup \{(0, t, 0) : t \in \mathbb{R}\}$, we can assume $\theta_j = (1, 0, 0)$ or $\theta_j = (0, 1, 0)$.

In the first case we consider points $\zeta \in V$ of the form

$$\zeta_t = (t, -i, \sqrt{t}),$$

whereas in the second case we consider points $\zeta \in V$ of the form

$$\zeta_t = (-i, t, \sqrt{t}).$$

Then we have

$$(3.6) \quad t^{h_j/2} |\tilde{q}_j(\zeta_t)| \leq c_j(1 + (1 + t + t^2)^{1/2})^m \exp A_j \quad \forall t > 0,$$

for $j = m + 1, \dots, \nu$.

Since

$$\tilde{q}_j(\zeta_t) = O(t^{j-h_j}) \quad \text{for } t \rightarrow +\infty,$$

the left hand side of (3.6) is $O(t^{j-h_j/2})$, and the right hand side of (3.6) is $O(t^m)$ for $t \rightarrow +\infty$.

Therefore $j - h_j/2 \leq m$, and since $h_j \leq j$, we have that $j \leq 2m$.

This shows that a polynomial $q \in \mathbb{C}[\zeta_1, \zeta_2, \zeta_3]$ satisfying (3.5) has degree at most $2m \bmod \wp$.

We have thus proved that:

If $\tau = \sum_{\alpha \in \mathbb{N}^3} a_\alpha x^\alpha$ is a formal power series solution of

$$\left(\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} + i \frac{\partial^2}{\partial x_3^2} \right) \tau = 0,$$

then we can find $u \in C^\infty(\mathbb{R}^3)$ solving

$$\left(\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} + i \frac{\partial^2}{\partial x_3^2} \right) u = 0 \quad \text{in } \mathbb{R}^3$$

and

$$\left(\frac{\partial}{\partial x} \right)^\alpha u(0) = a! a_\alpha \quad \forall \alpha \in \mathbb{N}^3.$$

4. - Local extension of formal power series solutions.

Let us give some reduction theorems.

THEOREM 4.1. *The following conditions are equivalent for a \mathcal{P} -module M of finite type:*

- (i) $(\{0\}, \mathbb{R}^n)$ is a local evolution pair for M
- (ii) $(\{0\}, \mathbb{R}^n)$ is an evolution pair for M .

PROOF. (ii) \Rightarrow (i) is trivial.

Let us prove that (i) \Rightarrow (ii).

It suffices to consider the case where $M = \mathcal{P}/\mathfrak{p}$ for a prime ideal \mathfrak{p} in \mathcal{P} .

If (i) holds, by Lemma 1.3 we can find $\varepsilon > 0$ such that the pair $(\{0\}, B_\varepsilon)$ is of evolution, where $B_\varepsilon = \{x \in \mathbb{R}^n : |x| \leq \varepsilon\}$.

By Theorem 3.7 we have that there exists an integer $k > 0$ such that, if $V = V(\mathfrak{p})$ and $p \in \mathcal{P}/\mathfrak{p}$ satisfies

$$|p(\zeta)| \leq c(1 + |\zeta|)^m \exp(\varepsilon |\operatorname{Im} \zeta|) \quad \forall \zeta \in V$$

then

$$\deg p \leq km.$$

We have to prove that for every $R \in \mathbb{R}$ and $m \in \mathbb{N}$ if $p \in \mathcal{P}/\mathfrak{p} \cap \mathcal{O}_{B_R}^{(m)}(V)$ then the degree of p is bounded by some integer depending only on R and m .

For $p \in \mathcal{P}/\mathfrak{p} \cap \mathcal{O}_{B_R}^{(m)}(V)$ we consider

$$\varphi(r) = \sup \left\{ \frac{|p(\zeta)|}{(1 + |\zeta|)^m} : |\operatorname{Im} \zeta| \leq r, \zeta \in V \right\}.$$

This is a semi-algebraic function which is finite for $r \gg 1$ since $p \in \mathcal{O}_{B_R}^{(m)}(V)$.

Therefore by the Tarski-Seidenberg theorem (see [7] th. A.2.5) there are a constant $A > 0$ and a rational number $q \geq 0$ such that

$$\varphi(r) = Ar^q(1 + o(1)) \quad \text{for } r \rightarrow +\infty.$$

Therefore p satisfies an estimate of the form

$$\begin{aligned} |p(\zeta)| &\leq c(1 + |\zeta|)^m(1 + |\operatorname{Im} \zeta|)^q \\ &\leq c'(1 + |\zeta|)^m \exp \varepsilon |\operatorname{Im} \zeta| \quad \forall \zeta \in V. \end{aligned}$$

From the previous remarks we have that p has degree less or equal to $k(m + 1)$.

Then (ii) follows by Theorem 3.7. ■

Let us consider now the case where $0 \in \partial F$.

We shall make the following assumption on F :

ASSUMPTION 1. We can find a semi-algebraic compact convex set K in \mathbb{R}^n with $0 \in K$ such that for some $\varepsilon > 0$ and some neighbourhood U of 0 in \mathbb{R}^n we have:

$$\varepsilon K \subset F \cap U \subset \varepsilon^{-1} K.$$

REMARKS. *a)* This property is satisfied if $U \cap F$ is semi-algebraic for some open neighbourhood U of 0 in \mathbb{R}^n .

b) It is also satisfied if ∂F is of class C^2 in a neighbourhood of 0 and the hessian of any defining function of F is positive definite on the tangent hyperplane to ∂F at 0 .

Indeed, assuming that F is contained in the half space $\{x \in \mathbb{R}^n : x_1 \geq 0\}$, we have in a neighbourhood U of 0 in \mathbb{R}^n :

$$F \cap U = \{x = (x_1, x') \in U \subset \mathbb{R} \times \mathbb{R}^{n-1} : \varphi(x') \leq x_1\}$$

where

$$(4.1) \quad \varphi(x') = q_2(x') + o(|x'|^2)$$

and q_2 is a homogeneous polynomial of second degree with

$$q_2(x') \geq \alpha |x'|^2 \quad \text{for some } \alpha > 0.$$

Then we can take

$$K = \{x \in \mathbb{R}^n : q_2(x') \leq x_1 \leq 1\}.$$

c) More in general, Assumption 1 holds if we have a representation of $F \cap U$ of the form (4.1) with a polynomial $q_2(x')$ that is positive on $\mathbb{R}^{n-1} \setminus \{0\}$ and with an error term that is $o(q_2(x'))$ for $x' \rightarrow 0$.

With the same proof of Theorem 4.1, as $H_K(\zeta)$ is a semi-algebraic function for a semi-algebraic subset K of \mathbb{R}^n , we have:

THEOREM 4.2. *Assume that F satisfies Assumption 1. Then the following conditions are equivalent for a \mathcal{P} -module M of finite type:*

- (i) $(\{0\}, F)$ is a local evolution pair for M
- (ii) $(\{0\}, F)$ is an evolution pair for M .

Assume now that $\{K_n\}$ is an increasing sequence of compact convex semi-algebraic sets with $0 \in \partial K_n$ for every n .

Then we consider the convex set

$$F = \bigcup_n K_n.$$

We note that F is a cone and that $F \setminus \{0\}$ is an open half space if ∂K_n is of class C^1 at 0 for some n .

We construct a Fréchet space $\tilde{\mathcal{E}}(F, \{K_n\})$ by taking the inductive limit

$$\tilde{\mathcal{E}}(F, \{K_n\}) = \varinjlim_{n \rightarrow \infty} \mathcal{E}(nK_n).$$

LEMMA 4.3. *Let K be a compact convex semi-algebraic subset of \mathbb{R}^n , and let $\lambda \in \mathbb{R}$.*

Then, the pair $(\{0\}, K)$ is of evolution (for M) if and only if the pair $(\{0\}, \lambda K)$ is of evolution.

PROOF. Let $(\{0\}, K)$ be of evolution and let us show that $(\{0\}, \lambda K)$ is of evolution, i.e. that the following Phragmén-Lindelöf principle holds:

$\forall m \exists m', c > 0$ such that for every polynomial $p(\zeta)$ with

$$|p(\zeta)| \leq (1 + |\zeta|)^m \exp H_{\lambda K}(\zeta) = (1 + |\zeta|)^m \exp \lambda H_K(\zeta) \quad \forall \zeta \in V,$$

then

$$|p(\xi)| \leq c(1 + |\xi|)^{m'} \quad \forall \xi \in V.$$

Indeed, let

$$\varphi(r) = \sup_V \left\{ \frac{|p(\xi)|}{(1 + |\xi|)^m} : H_K(\xi) \leq r \right\}.$$

This is a semi-algebraic function which is bounded by assumption. Then, by the Tarski-Seidenberg theorem (see [7], th. A.2.5), we have that

$$\varphi(r) = Ar^q(1 + o(1)) \quad \text{for } r \rightarrow +\infty, q \in \mathbb{Q}$$

for some constant $A > 0$.

This implies that

$$\frac{|p(\xi)|}{(1 + |\xi|)^m} \leq A'(H_K(\xi) + 1)^q \quad \forall \xi \in V$$

and hence

$$|p(\xi)| \leq A''(1 + |\xi|)^m \exp H_K(\xi) \quad \forall \xi \in V.$$

Since $(\{0\}, K)$ is of evolution we have, by the Phragmén-Lindelöf principle, that there exist $c > 0$ and $m' \in \mathbb{N}$ such that

$$|p(\xi)| \leq c(1 + |\xi|)^{m'} \quad \forall \xi \in V$$

We have thus proved the thesis. \blacksquare

LEMMA 4.4. *Let $K_1 \subset K_2$ be compact convex subsets of \mathbb{R}^n , and let us assume that the pair $(\{0\}, K_2)$ is of evolution for M . Then the map*

$$\text{Ext}_\varphi^0(M, \mathfrak{J}(\{0\}, K_2)) \rightarrow \text{Ext}_\varphi^0(M, \mathfrak{J}(\{0\}, K_1))$$

has dense image.

PROOF. Let $\varphi \in \mathcal{S}'_{K_1}$ be such that

$$\langle \varphi, u \rangle = 0 \quad \forall u \in \text{Ext}_\varphi^0(M, \mathfrak{J}(\{0\}, K_2))$$

and let us show that

$$\langle \varphi, u \rangle = 0 \quad \forall u \in \text{Ext}_{\varphi}^0(M, \mathfrak{J}(\{0\}, K_1))$$

Let m be the (finite) order of φ .

By assumption the pair $(\{0\}, K_2)$ is of evolution, i.e. the map

$$\text{Ext}_{\varphi}^0(M, W_{K_2}) \rightarrow \text{Ext}_{\varphi}^0(M, W_{\{0\}})$$

is onto.

By Banach's open mapping theorem we have that we can find $c > 0$ and $m' \in \mathbb{N}^*$ such that for all v in $\text{Ext}_{\varphi}^0(M, W_{\{0\}})$ there exists \tilde{v} in $\text{Ext}_{\varphi}^0(M, W_{K_2})$ such that $\tilde{v}|_{\{0\}} = v$ and

$$\|\tilde{v}\|_m = \sum_{|\alpha| \leq m} |D^\alpha \tilde{v}(0)| \leq c \|v\|_{m'}.$$

Then, for $u \in \text{Ext}_{\varphi}^0(M, W_{K_2})$, we have that

$$\begin{aligned} |\langle \varphi, u \rangle| &= |\langle \varphi, (u - \widetilde{u}|_{\{0\}}) \rangle + \langle \varphi, \widetilde{u}|_{\{0\}} \rangle| = \\ &= |\langle \varphi, \widetilde{u}|_{\{0\}} \rangle| \leq c(\varphi) \cdot \| \widetilde{u}|_{\{0\}} \|_m \leq c \cdot c(\varphi) \cdot \|u|_{\{0\}}\|_{m'}. \end{aligned}$$

By the Hahn-Banach theorem we can find $\psi \in \mathcal{E}'_{\{0\}}$ (of order m') such that

$$\langle \varphi, u \rangle = \langle \psi, u \rangle \quad \forall u \in \text{Ext}_{\varphi}^0(M, W_{K_2}).$$

But we know that the map

$$\text{Ext}_{\varphi}^0(M, W_{K_2}) \rightarrow \text{Ext}_{\varphi}^0(M, W_{K_1})$$

has dense image, and hence for every u in $\text{Ext}_{\varphi}^0(M, W_{K_1})$ we can find a sequence $\{v_n\}$ in $\text{Ext}_{\varphi}^0(M, W_{K_2})$ such that $v_n|_{K_1} \rightarrow u$ (with all derivatives).

Then

$$\langle \varphi, u \rangle = \lim_{n \rightarrow \infty} \langle \varphi, v_n \rangle = \lim_{n \rightarrow \infty} \langle \psi, v_n \rangle = \langle \psi, u \rangle.$$

Therefore

$$\langle \varphi, u \rangle = 0 \quad \forall u \in \text{Ext}_{\varphi}^0(M, \mathfrak{J}(\{0\}, K_1)),$$

as we wanted to prove. ■

THEOREM 4.5. *If $(\{0\}, K_n)$ is a local evolution pair for M for every*

n , then the map

$$(4.2) \quad \text{Ext}_{\mathcal{P}}^0(M, \tilde{\delta}(F, \{K_n\})) \rightarrow \text{Ext}_{\mathcal{P}}^0(M, \delta(\{0\}))$$

is surjective. In particular, if G is a locally closed convex set with $0 \in \partial G$ and for each n we have $K_n \subset G$, and $(\{0\}, G)$ is a local evolution pair for M , then the map (4.2) is surjective.

PROOF. By assumption the pair $(\{0\}, K_n)$ is of evolution for every n and hence for all $f_0 \in \text{Ext}_{\mathcal{P}}^0(M, W_{\{0\}})$ we can find $f_n \in \text{Ext}_{\mathcal{P}}^0(M, W_{K_n})$ such that $f_n|_{\{0\}} = f_0$.

Let us construct a sequence $\{g_n\}$ with $g_n \in \text{Ext}_{\mathcal{P}}^0(M, W_{K_n})$, $g_n|_{\{0\}} = f_0$ and

$$\|g_n - g_{n+1}\|_n^{K_n} < 2^{-n}.$$

Let us take $g_1 = f_1$. Let us suppose we already constructed g_n for some $n \geq 1$ and let us construct g_{n+1} :

since $f_{n+1}|_{K_n} - g_n \in \text{Ext}_{\mathcal{P}}^0(M, \mathfrak{J}(\{0\}, K_n))$, by Lemma 4.4 we can find $v_n \in \text{Ext}_{\mathcal{P}}^0(M, \mathfrak{J}(\{0\}, K_{n+1}))$ such that

$$\|f_{n+1} - g_n - v_n\|_n^{K_n} < 2^{-n}$$

and so we can take $g_{n+1} = f_{n+1} - v_n$.

Now we define

$$u_n = g_n + \sum_{j=0}^{\infty} (g_{n+j+1} - g_{n+j}) \quad \text{on } K_n.$$

Then $u_{n+1}|_{K_n} = u_n$ and so it defines an element of $\text{Ext}_{\mathcal{P}}^0(M, \tilde{\delta}(F, \{K_n\}))$ with $u_n|_{\{0\}} = f_0$ for all n . ■

Let us see an application of Theorem 4.5.

EXAMPLE. Let us consider in \mathbb{C}^n the convex set $G = \{\text{Re } \zeta_1 > 0\} \cup \{0\}$. We assert that for every formal power series

$$\tau = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} \zeta^{\alpha}$$

we can find a holomorphic function f on $\{\text{Re } \zeta_1 > 0\}$ such that, for each set

$$B_{s,R} = \{\zeta \in \mathbb{C}^n : |\zeta_1 - R|^s + |\zeta_2|^s + \dots + |\zeta_n|^s \leq R^s\}, \quad R, s \geq 2,$$

we have

$$\lim_{\substack{\zeta \rightarrow 0 \\ \zeta \in B_{s,R}}} D_{\zeta}^{\alpha} f(\zeta) = a! a_{\alpha} \quad \forall \alpha \in \mathbb{N}^n .$$

Indeed, if we consider the sequence of convex compact sets $\{K_m\}_{m \geq 2}$ defined by

$$K_m = \left\{ \zeta \in \mathbb{C}^n : |\operatorname{Re} \zeta_1 - 1|^m + \sum_{j=2}^n |\operatorname{Re} \zeta_j|^m + \sum_{j=1}^n |\operatorname{Im} \zeta_j|^m \leq 1 \right\},$$

we have:

- (i) $0 \in K_m \quad \forall m \geq 2,$
- (ii) $K_m \subset K_{m+1} \quad \forall m \geq 2,$
- (iii) $\bigcup_{m \geq 2} mK_m = G,$
- (iv) $\forall s, R$ we have $B_{s,R} \subset mK_m$ for some $m.$

Therefore the statement above follows from Theorem 4.5 if we show that $(\{0\}, K_m)$ is an evolution pair for every $m,$ for the module \mathcal{P}/\wp where $\mathcal{P} = \mathbb{C}[\zeta_1, \dots, \zeta_{2n}]$ and $\wp = (\zeta_1 + i\zeta_{n+1}, \dots, \zeta_n + i\zeta_{2n}).$

We have:

$$V = V(\wp) = \{ \zeta \in \mathbb{C}^{2n} : \zeta_{n+j} = i\zeta_j \text{ for } j = 1, \dots, n \} \simeq \mathbb{C}_{\zeta_1 \dots \zeta_n}^n$$

and

$$M = \mathcal{P}/\wp \simeq \mathbb{C}[\zeta_1, \dots, \zeta_n].$$

Note that

$$H_{K_m}(\zeta) = \psi(\zeta_1, \dots, \zeta_n) = \operatorname{Im} \zeta_1 + \left[\sum_{j=1}^n (|\operatorname{Re} \zeta_j|^{m'} + |\operatorname{Im} \zeta_j|^{m'}) \right]^{1/m'}$$

with $m' = m/(m - 1).$

We introduce real coordinates η_1, \dots, η_{2n} in $\mathbb{C}_{\zeta_1 \dots \zeta_n}^n$ by:

$$\begin{aligned} \eta_1 &= -\operatorname{Im} \zeta_1, & \eta_2 &= \operatorname{Im} \zeta_2, & \dots, & \eta_n &= \operatorname{Im} \zeta_n, \\ \eta_{n+1} &= \operatorname{Re} \zeta_1, & \dots, & \eta_{2n} &= \operatorname{Re} \zeta_n. \end{aligned}$$

Let us consider a polynomial $p \in \mathbb{C}[\zeta_1, \dots, \zeta_n]$ satisfying

$$|p(\eta)| = |p(\zeta_1, \dots, \zeta_n)| \leq c(1 + |\eta|)^l \exp \psi(\eta) \quad \forall \eta \in \mathbb{R}^{2n} .$$

In particular we have that

$$|p(\eta)| \leq ce^R(1 + |\eta|)^l \quad \text{if } \psi(\eta) \leq R.$$

Note that the inequality $\psi(\eta) \leq R$ means:

$$\|\eta\|_{m'} = \left(\sum_{j=1}^n |\eta_j|^{m'} \right)^{1/m'} \leq R + \eta_1.$$

Let us consider the set

$$A = \left\{ \sum_{j=2}^n |\eta_j|^{m'} \leq \eta_1^{1/(m-1)}, \eta_1 \geq 0 \right\}.$$

Then we have

$$\begin{aligned} (\|\eta\|_{m'})^m &\leq (|\eta_1|^{m/(m-1)} + \eta_1^{1/(m-1)})^{m-1} = \\ &= \sum_{h=0}^{m-1} \binom{m-1}{h} \eta_1^{hm/(m-1) + (m-1-h)/(m-1)} = \sum_{h=0}^{m-1} \binom{m-1}{h} \eta_1^{h+1} \end{aligned}$$

and then, since

$$(R + \eta_1)^m = \sum_{h=0}^m \binom{m}{h} R^{m-h} \eta_1^h,$$

if $R \geq 1$ we have that

$$\psi(\eta) \leq R \quad \text{on } A.$$

Let us now consider the map

$$\phi(t_1, \dots, t_{2n}) = (t_1^{m(m-1)}, t_2^{m-1}, \dots, t_{2n}^{m-1}).$$

The cone

$$W = \{t_1 \geq 0, |t_2|^m + \dots + |t_{2n}|^m \leq t_1^m\}$$

is mapped by ϕ into A and $\phi^*p(t)$ is a polynomial in \mathbb{R}^{2n} satisfying

$$|\phi^*p(t)| \leq c'(1 + |t|)^{m(m-1)l} \quad \text{on } W.$$

Since W contains an open cone, it follows that ϕ^*p has degree in t less or equal to $m(m-1)l$, and hence p has degree less or equal to ml .

It follows that $(\{0\}, K_m)$ is an evolution pair by Theorem 3.7.

We have thus proved the thesis.

REMARK. From the Example above we deduce that for every formal power series solution u in $0 \in \mathbb{R}^3$ of the Hans Lewy equation

$$Lu = \frac{\partial u}{\partial \bar{z}} - iz \frac{\partial u}{\partial t} = 0,$$

where $z \in \mathbb{C}$, $t = \operatorname{Re} w$ for $w \in \mathbb{C}$, we can find a solution $\tilde{u} \in C^\infty(\mathbb{R}^3) \cap \mathcal{A}(\mathbb{R}^3 \setminus \{0\})$ of

$$L\tilde{u} = 0.$$

Indeed, if we consider the manifold in \mathbb{C}^2

$$M = \{(z, w) \in \mathbb{C}^2 : \varrho(z, w) = z\bar{z} - \operatorname{Im} w = 0\}$$

we obtain that if u is a solution of the Hans Lewy equation, then it satisfies the tangential Cauchy-Riemann equations on M .

It follows that we can find a formal power series in 0 , which is a solution of the $\bar{\partial}$ operator in 0 and which is equal to u in $\{0\}$ in the sense of Whitney.

By the preceding Example we can then extend u to a holomorphic function on $\{\operatorname{Im} w > 0\}$, which extends to a Whitney function U , having Taylor series u at 0 , on all sets of the form

$$\{(z, w) \in \mathbb{C}^2 : \operatorname{Im} w \geq \lambda z\bar{z}\}$$

for $0 < \lambda < 1$.

Restricting U to the manifold M we obtain a function $\tilde{u} \in C^\infty(\mathbb{R}^3) \cap \mathcal{A}(\mathbb{R}^3 \setminus \{0\})$ which is a solution of the Hans Lewy equation with $\tilde{u}|_{\{0\}} = u$.

Let us consider the special case where the algebraic variety $V(\varphi)$ is an algebraic curve in \mathbb{C}^n , i.e. when $d = \dim_{\mathbb{C}} V = 1$.

Let \bar{V} be the closure of V in $\mathbb{C}P^n$ and let P_1, \dots, P_s be the intersection of \bar{V} with the hyperplane at infinity.

We can fix neighbourhoods V_1, \dots, V_s of P_1, \dots, P_s in \bar{V} such that $V_i \cap V_j = \emptyset$ for $i \neq j$ and $\pi|_{\check{V}_j} : \check{V}_j \rightarrow \pi(\check{V}_j)$ is an m_j -fold covering, where $\check{V}_j = V_j \setminus \{P_j\}$.

Let K be a convex compact semi-algebraic subset of \mathbb{R}^n .

PROPOSITION 4.6. *The pair $(\{0\}, K)$ is of evolution for \mathcal{P}/φ in the Whitney class if and only if there exists a constant M such that each P_j can be approximated, on each connected component, by points in $E_M = \{\zeta \in V : H_K(\zeta) \leq M\}$.*

PROOF. By passing to the normalization of \bar{V} , we can assume that \check{V}_j is connected for every j . Let us assume that $(\{0\}, K)$ is of evolution and

let us prove that we can approximate each P_j by points in E_M for some $M > 0$.

We argue by contradiction: we assume that P_1 cannot be approximated in E_M for any $M > 0$.

Let

$$\varphi(r) = \inf_V \{H_K(\zeta) : \zeta \in \hat{V}_1, |\zeta| \geq r\}.$$

By assumption $\varphi(r)$ is not bounded for $r \rightarrow +\infty$, so that

$$\varphi(r) = Ar^q(1 + o(1)) \quad \text{for } r \rightarrow +\infty$$

for $A \neq 0$ and $q \in \mathbb{Q}$, $q > 0$.

Therefore

$$(4.3) \quad H_K(\zeta) \geq A' |\zeta|^{q'} \quad \text{for } \zeta \in \hat{V}_1, |\zeta| > R$$

where $A' > 0$ and $q' \in \mathbb{Q}$ with $q' > 0$.

By Weierstrass gap theorem we can find a meromorphic function f on V with zeros in P_2, \dots, P_s and a single pole in P_1 of arbitrarily high order.

Thus we can find polynomials on V with arbitrarily high degree, but bounded in $\hat{V}_2, \dots, \hat{V}_s$.

By (4.3) we can find polynomials p_ν satisfying an estimate of the form

$$|p_\nu(\zeta)| \leq \exp H_K(\zeta) \quad \forall \zeta \in V,$$

but of arbitrary high degree, so that we cannot find constants $c > 0$ and $M \in \mathbb{N}$ such that

$$|p_\nu(\zeta)| \leq c(1 + |\zeta|)^M \quad \forall \zeta \in V$$

for all $\nu \in \mathbb{N}$.

We have thus found functions which violate the Phragmén-Lindelöf principle.

Let us now assume that the points P_1, \dots, P_s can be approximated on E_M for some $M > 0$, and let us prove that the Phragmén-Lindelöf principle holds.

Indeed, for every polynomial p satisfying

$$(4.4) \quad |p(\zeta)| \leq (1 + |\zeta|)^M \exp H_K(\zeta) \quad \forall \zeta \in V,$$

since the points at infinity can be approximated by points where $H_K(\zeta) \leq M$, after passing to the uniformizing coordinate, we realize

that the order of pole of p in P_j is bounded because of (4.4), for $j = 1, \dots, s$.

Therefore we can find constants $c > 0$ and $m \in \mathbb{N}$ such that

$$|p(\zeta)| \leq c(1 + |\zeta|)^m \quad \forall \zeta \in V.$$

We have thus proved the thesis. ■

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