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Haar Inequality in Hereditary Setting and Applications.

PRIMO BRANDI - CRISTINA MARCELLI(*)

ABSTRACT - We present a functional extension of Haar's lemma under Carathéodory assumptions. As a consequence, we derive uniqueness and continuous dependence criteria for the solutions of nonlinear hereditary Cauchy problems of the first order.

1. - Introduction.

A great deal of research has been devoted to differential-functional inequalities in order to study the behaviour of the solutions of partial differential functional problems, mainly for C^1 -solutions (see, e.g., [21], [12], [13], [22], [14], [19], [1], [15], [2], [16], [5], [4], [23]).

The aim of this paper is to discuss differential functional inequalities in the Carathéodory sense and then to derive uniqueness and continuous dependence criteria for generalized solutions of partial differential hereditary Cauchy problems.

More precisely, we deal with the following Cauchy problem

$$\begin{cases} D_t u_i(t, x) = f_i(t, x, u(t, x), u(\cdot), D_x u_i(t, x)), \\ (t, x) \in G, \quad t\text{-a.e.}, \quad i = 1, \dots, n, \\ u(t, x) = \phi(t, x), \quad (t, x) \in G_0, \end{cases}$$

where $\phi \in C^0(G_0, \mathbb{R}^n)$ is the initial data and $f: G \times \mathbb{R}^n \times C^0(G_0 \cup G) \times \mathbb{R}^{mn} \rightarrow \mathbb{R}^n$ is a Carathéodory function satisfying Volterra condition.

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We first give two comparison results (Theorems 4 and 5) which allow us to estimate functions of more variables by means of a one-variable function, which is the maximal solution of a suitable ordinary comparison problem.

These theorems, which can be regarded as a generalization of Haar's lemma to the Carathéodory and hereditary setting, extend analogous results obtained by A. Salvadori ([19]) and J. Turo ([23]).

As a consequence, uniqueness and continuous dependence criteria for generalized solutions of functional Cauchy problems are derived. These last results extend analogous criteria established by K. Zima ([24]), J. Szarski ([21]), A. Salvadori ([19]), and can be regarded as the Carathéodory version of the results given by Z. Kamont ([12]) and Z. Kamont-K. Przadka ([16]).

2. - Notations and statement of the problem.

Following J. Szarski ([20]), we will consider the partial orders in \mathbb{R}^n defined as follows: given two vectors $y = (y_1, \dots, y_n)$ and $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_n)$, we say that $y \leq \tilde{y}$ if $y_j \leq \tilde{y}_j$, $j = 1, 2, \dots, n$; we say that $y \leq^i \tilde{y}$ if $y \leq \tilde{y}$ and $y_i = \tilde{y}_i$. Moreover, for every $y \in \mathbb{R}^n$ we put $|y| = (|y_1|, \dots, |y_n|) \in \mathbb{R}^n$.

Given a set $A \subset \mathbb{R}^{m+1}$, we denote by

$$\Pi(A) = \{t \in \mathbb{R} : (t, x) \in A \text{ for some } x \in \mathbb{R}^m\},$$

i.e. the projection of A on the t -axis. Moreover, for every $t \in \Pi(A)$ we put

$$S_t(A) = \{x \in \mathbb{R}^m : (t, x) \in A\}, \quad A_t = \{(\tau, x) \in A : \tau \leq t\}.$$

We will say that a property \mathcal{P} holds in A , t -a.e., if it is satisfied for every $(t, x) \in A$ with the exception, at most, of a set of points whose projection on the t -axis has null measure.

We denote by $C(A)$ the space of the continuous functions defined in A and taking values on \mathbb{R}^n , endowed with the compact-open topology.

Let $a > 0$ be a fixed real number. We put $I = [0, a[$, and $I_0 =]-\infty, 0]$, $I_t =]-\infty, t]$, for every $t \in I$. Moreover, let $E = I \times \mathbb{R}^m$ and $E_0 = I_0 \times \mathbb{R}^m$.

Let $h \in L^1(I, \mathbb{R}^m)$ be a given nonnegative function and let $b \in \mathbb{R}^m$ be a given vector such that $\int_0^a h(t) dt \leq b$.

We put

$$\Delta = \left\{ (t, x) \in \mathbb{R}^{m+1} : t \in I, -b + \int_0^t h(\tau) d\tau \leq x \leq b - \int_0^t h(\tau) d\tau \right\}.$$

Note that Δ can be regarded as a generalization of Haar's pyramid. Furthermore, given a vector $d \geq b$, we put $\Delta_0 = I_0 \times [-d, d]$.

Finally, in what follows, (G_0, G) will denote either the pair (E_0, E) or the pair (Δ_0, Δ) indifferently.

DEFINITION 1. We will denote by $\mathcal{X}(G_0 \cup G)$ the class of the continuous functions $u: G_0 \cup G \rightarrow \mathbb{R}^n$ satisfying the conditions:

- (K₁) for a.e. $t_0 \in I$ and every $i \in \{1, \dots, n\}$ the function $u^i(t_0, \cdot)$ is differentiable; for every $\bar{x} \in S_{t_0}$ the derivative $D_{x_j} u^i(t_0, \dots, \bar{x}_j, \dots)$ is continuous for every $j = 1, \dots, m$, and there exists the derivative $D_t u^i(t_0, \bar{x})$;
- (K₂) for every compact set $K \subset \mathbb{R}^m$ a function $\theta \in L^1_{loc}(I)$ exists such that for every $(t, x), (t^*, x) \in G$, with $t \leq t^*$ and $x \in K$

$$|u(t, x) - u(t^*, x)| \leq \int_t^{t^*} \theta(\tau) d\tau$$

i.e. the function $u(\cdot, x)$ is absolutely continuous, locally uniformly with respect to x .

Moreover, we will denote by $\mathcal{X}_B(G_0 \cup G)$ the subset of the functions $u \in \mathcal{X}(G_0 \cup G)$ which are bounded and uniformly continuous in G .

REMARK 1. Of course, in the case $m = 1$ no continuity assumption on the derivative $D_x u^i(t_0, \cdot)$ is required.

Note that $C^1(G_0 \cup G) \subset \mathcal{X}(G_0 \cup G)$. In fact, condition (K₂) is satisfied if for every compact set $K \subset \mathbb{R}^m$ a function $\theta \in L^1_{loc}(I)$ exists such that $|u_t(\tau, x)| \leq \theta(\tau)$ for a.e. $\tau \in I$ and $x \in K$.

Moreover, $\mathcal{X}(G_0 \cup G)$ contains the class of solutions considered by M. Cinquini Cibrario-S. Cinquini in [8], Z. Kamont-J. Turo in [17], T. Czlapiński in [11], where the existence of generalized solutions of hyperbolic Cauchy problems is discussed.

In the following we briefly denote by

$$D_t u(t, x) = (D_t u^1(t, x), \dots, D_t u^n(t, x)) \in \mathbb{R}^n,$$

$$D_x u(t, x) = \begin{pmatrix} D_{x_1} u^1(t, x) & \dots & D_{x_1} u^n(t, x) \\ \vdots & & \vdots \\ D_{x_m} u^1(t, x) & \dots & D_{x_m} u^n(t, x) \end{pmatrix} \in \mathbb{R}^{mn}.$$

Let $\mathcal{V} \subset C(G_0 \cup G)$ be a fixed open set and let $\mathcal{O} = G \times \mathbb{R}^n \times \mathcal{V} \times \mathbb{R}^{mn}$.

DEFINITION 2. A function $f: \mathcal{O} \rightarrow \mathbb{R}^n$ is said to satisfy Volterra condition if

(V) for a.e. $t \in I$ and every pair $z, \tilde{z} \in \mathcal{V}$, such that

$$z(\tau, \xi) = \tilde{z}(\tau, \xi) \quad \text{for every } (\tau, \xi) \in (G_0 \cup G_t)$$

we have

$$f(t, x, y, z, q) = f(t, x, y, \tilde{z}, q) \quad \text{for every } x \in S_t(G), (y, q) \in \mathbb{R}^{n+mn}.$$

Given a Carathéodory function $f: \mathcal{O} \rightarrow \mathbb{R}^n$ satisfying condition (V) and a function $\phi \in C(G_0)$, we deal with the following functional Cauchy problem

$$P(f, \phi) \quad \begin{cases} D_t u(t, x) = f(t, x, u(t, x), u, D_x u(t, x)) & \text{a.e. in } G & \text{(p.1),} \\ u(t, x) = \phi(t, x) & \text{in } G_0 & \text{(p.2).} \end{cases}$$

Denoted by $q_{j,i}$ the entries of the matrix $q \in \mathbb{R}^{mn}$, we will assume that $f_i = f_i(t, x, y, z, (q_{1,i}, \dots, q_{m,i}))$, for every $i = 1, \dots, n$.

Under this assumption the previous system is hyperbolic of a special type since in each equation the first-order derivative of only one unknown function appears.

DEFINITION 3. We will say that a function $u: G_0 \cup G \rightarrow \mathbb{R}^n$ is a solution of problem $P(f, \phi)$ provided:

- i) $u \in \mathcal{X}(G_0 \cup G)$;
- ii) equation (p.1) is satisfied for every $(t, x) \in G$, t -a.e.;
- iii) equation (p.2) is satisfied for every $(t, x) \in G_0$.

The family of the solutions of problem $P(f, \phi)$ will be denoted by $SP(f, \phi)$.

The main result of this paper is a comparison theorem (Theorem 4) which allows us to estimate the difference between two solutions of problem $P(f, \phi)$ by means of the maximal solution of the following ordi-

nary comparison problem:

$$(CP) = CP(g, \eta) \quad \begin{cases} \theta'(t) = g(t, \theta(t), \theta) & \text{a.e. in } I, \\ \theta(t) = \eta(t) & \text{in } I_0, \end{cases}$$

where $\eta \in C(I_0)$ and $g: I \times \mathbb{R}^n \times C(I_0 \cup I) \rightarrow \mathbb{R}^n$ are given functions.

We assume that function g is non-negative and satisfies the following conditions:

- (C) (Carathéodory): $g(\cdot, x, \theta)$ is measurable for every (x, θ) and $g(t, \cdot, \cdot)$ is continuous for almost every t ; moreover, for every $(\bar{t}, \bar{x}, \bar{\theta}) \in I \times \mathbb{R}^n \times C(I_0 \cup I)$ a real number $r > 0$ and a function $m \in L^1(B(\bar{t}, r))$ exist such that

$$|g(t, x, \theta)| \leq m(t) \quad \text{for every } (t, x, \theta) \in B(\bar{t}, \bar{x}, \bar{\theta}), r, t\text{-a.e.}$$

- (V) (Volterra): for every $(t, x) \in I \times \mathbb{R}^n$, t -a.e., and every $\theta, \tilde{\theta} \in C(I_0 \cup I)$ such that $\theta = \tilde{\theta}$ in I_t , we have

$$g(t, x, \theta) = g(t, x, \tilde{\theta}).$$

- (W⁺) (quasi-monotonicity): for every $(t, x, \theta), (t, \tilde{x}, \theta) \in I \times \mathbb{R}^n \times C(I_0 \cup I)$, t -a.e., such that $x \leq^i \tilde{x}$, we have

$$g_i(t, x, \theta) \leq g_i(t, \tilde{x}, \theta).$$

- (M) (monotonicity in the functional argument): for every $(t, x) \in I \times \mathbb{R}^n$, t -a.e., and every pair $\theta, \tilde{\theta} \in C(I_0 \cup I)$ such that $\theta(t) \leq \tilde{\theta}(t)$ for every $t \in I_0 \cup I$, we have

$$g(t, x, \theta) \leq g(t, x, \tilde{\theta}).$$

3. - Extension of Haar lemma.

In ([6]) we discussed the local existence of the maximal solution of problem (CP) and we established the following comparison result which is the key to obtain an extension of Gronwall inequality in hereditary setting.

THEOREM 1. *Let $\Omega: I_T \rightarrow \mathbb{R}^n$ be the maximal solution of problem CP(g, η), with $T > 0$, and let $\gamma: I_T \rightarrow \mathbb{R}^n$ be an absolutely continuous function such that*

- i) $\gamma(t) \leq \eta(t), \quad t \in I_0;$
- ii) $\gamma'(t) \leq g(t, \gamma(t), \gamma) \quad \text{for a.e. } t \in I_T.$

Then we have

$$\gamma(t) \leq \Omega(t) \quad \text{in } I_T.$$

Moreover, in [7] we derived uniqueness and continuous dependence criteria for extremal solutions of problem (CP). We now state a particular case of Theorem 5 in [7], that we will use in what follows.

LEMMA 2. Assume that the maximal solution Ω of problem $CP(g, \eta)$ exists in I_T .

Then, an integer \bar{k} exists such that for every $k \geq \bar{k}$ the maximal solution Ω_k of problem $CP(g + 1/k, \eta + 1/k)$ exists in I_T .

Moreover, the sequence $(\Omega_k)_{k \geq \bar{k}}$ converges to Ω in $C(I_T)$.

Here we will deduce from Theorem 1 a functional extension of Haar lemma. Let us first prove the following result.

LEMMA 3. Given a function $u \in \mathcal{X}(\Delta_0 \cup \Delta)$, the function $M: I \rightarrow \mathbb{R}^n$ defined by

$$M^i(t) = \max_{(\tau, \xi) \in \Delta_t} |u^i(\tau, \xi)| \quad i = 1, \dots, n$$

is absolutely continuous in every interval $[0, t^*] \subset I$.

PROOF. Let us fix an index $i \in \{1, \dots, n\}$ and an interval $[0, t^*] \subset I$. Let $\varepsilon > 0$ be a given real number and let $\theta \in L^1([0, t^*])$ be the summable function in assumption (K_2) (see Definition 1). Let $\delta = \delta(\varepsilon)$ be a positive real number such that

$$(1) \quad \int_F \theta(t) dt < \varepsilon \quad \text{for every set } F \subset [0, t^*] \text{ with } \text{meas}(F) < \delta.$$

Let $\{[\alpha_s, \beta_s], s = 1, \dots, p\}$ be a finite collection of nonoverlapping intervals in $[0, t^*]$ such that

$$(2) \quad \sum_{s=1}^p (\beta_s - \alpha_s) < \delta.$$

By virtue of the monotonicity of function M , it is not restrictive to assume that $M^i(\alpha_s) < M^i(\beta_s)$, $s = 1, \dots, p$. Then, for every s a point $(t_s, x_s) \in \Delta$ exists such that $\alpha_s \leq t_s \leq \beta_s$ and

$$(3) \quad M^i(\beta_s) = |u^i(t_s, x_s)|.$$

Let us observe that $(\alpha_s, x_s) \in \Delta$, and we have $M^i(\alpha_s) \geq |u^i(\alpha_s, x_s)|$. Therefore, from (1), (2) and (3) we deduce

$$\begin{aligned} \sum_{s=1}^p [M^i(\beta_s) - M^i(\alpha_s)] &\leq \sum_{s=1}^p [|u^i(t_s, x_s)| - |u^i(\alpha_s, x_s)|] \leq \\ &\leq \sum_{s=1}^p |u^i(t_s, x_s) - u^i(\alpha_s, x_s)| \leq \sum_{s=1}^p \int_{\alpha_s}^{t_s} \theta(t) dt < \varepsilon. \end{aligned}$$

This concludes the proof.

The following theorem is the main result of this paper. It provides a comparison result which allows us to estimate functions of more variables by means of the maximal solution of comparison problem $CP(g, \eta)$.

THEOREM 4 (Extension of Haar lemma). *Let $u \in \mathcal{X}(\Delta_0 \cup \Delta)$ be a given function such that for every $(t, x) \in \Delta_T$, t -a.e., with $T > 0$, we have*

$$(4) \quad |D_t u^i(t, x)| \leq \sum_{j=1}^m h_j(t) |D_{x_j} u^i(t, x)| + g^i(t, |u(t, x)|, M), \quad i = 1, \dots, n$$

where $M: I \rightarrow \mathbb{R}^n$ is defined by $M^i(t) = \max_{(\tau, \xi) \in \Delta_t} |u^i(\tau, x)|$, $i = 1, \dots, n$. Moreover assume that

$$(5) \quad |u(t, x)| \leq \eta(t) \quad \text{for every } (t, x) \in \Delta_0.$$

Then we have

$$|u(t, x)| \leq \Omega(t) \quad \text{for every } (t, x) \in \Delta_0 \cup \Delta_T$$

where $\Omega: I_T \rightarrow \mathbb{R}^n$ is the maximal solution of comparison problem $CP(g, \eta)$.

PROOF. It is sufficient to prove that $M(t) \leq \Omega(t)$ for every $t \in [0, T]$. To this purpose, taking account of Lemma 3 and Theorem 1, it is sufficient to prove that for every index $i \in \{1, \dots, n\}$ we have

$$M^{i'}(t) \leq g^i(t, M(t), M) \quad \text{for a.e. } t \in [0, T].$$

Let $i \in \{1, \dots, n\}$ be fixed. Put $m^i(t) = \max_{x \in S_t(\Delta_0 \cup \Delta)} |u^i(t, x)|$, $t \in$

$\in I_0 \cup I$, note that for a.e. $t \in [0, T]$ with $M^{i'}(t) > 0$ we have

$$M^i(t) = m^i(t) \quad \text{and} \quad M^{i'}(t) = m^{i'}(t).$$

Therefore, since the function M^i is monotone and g^i is nonnegative, and taking property (W^+) into account, it is sufficient to prove that for every $\varepsilon > 0$ a measurable set $A \subset [0, T]$ exists, with $\text{meas}(A) > T - \varepsilon$, such that

$$(6) \quad m^{i'}(t) \leq g^i(t, m(t), M) \quad \text{for every } t \in A \text{ with } M^{i'}(t) > 0.$$

Note that the multifunction $\tilde{M}: I_0 \cup I \rightarrow 2^{\mathbb{R}^m}$ defined by $\tilde{M}(t) = \{x \in S_t(\Delta_0 \cup \Delta): |u^i(t, x)| = m^i(t)\}$ has a closed graph, then a measurable function $z: I_0 \cup I \rightarrow \mathbb{R}^m$ exists such that

$$|u^i(t, z_1(t), \dots, z_m(t))| = m^i(t), \quad t \in I_0 \cup I.$$

By Lusin theorem, a closed set $C_\varepsilon \subset [0, T]$, with $\text{meas}(C_\varepsilon) > T - \varepsilon/2$ exists such that the function $z|_{C_\varepsilon}$ is continuous.

For every $t \in C_\varepsilon$ we put

$$H(t) = \{r \in \mathbb{R}: t - r \in C_\varepsilon\}.$$

Of course, $H(t)$ is closed.

Let us now consider the function $\alpha: I_0 \cup I \rightarrow \mathbb{R}^m$ defined by

$$\alpha_j(t) = \begin{cases} b_j - \int_0^t h_j(\tau) d\tau, & t \geq 0, \\ d_j & t < 0, \end{cases} \quad j = 1, \dots, m,$$

and let

$$\Psi = \{(I, J, K): I \cup J \cup K = \{1, \dots, m\};$$

$$I, J, K \text{ are disjoint and } I \cup J \neq \emptyset\}.$$

For every triplet $(I, J, K) \in \Psi$ and every triplet of vectors $\alpha, \beta, z \in \mathbb{R}^m$ let $(\alpha, \beta, z)_{I, J, K}$ be the vector

$$(\alpha, \beta, z)_{I, J, K}^j = \begin{cases} \alpha_j & \text{if } j \in I, \\ \beta_j & \text{if } j \in J, \\ z_j & \text{if } j \in K, \end{cases} \quad j = 1, \dots, m,$$

and let $s_{(I, J, K)}: \{1, \dots, m\} \rightarrow \mathbb{R}$ be the function defined by

$$s_{(I, J, K)}(j) = s(j) = \begin{cases} -1 & \text{if } j \in I, \\ +1 & \text{if } j \in J, \\ 0 & \text{if } j \in K. \end{cases}$$

We put

$$u^i(t, [\alpha]_I, [\beta]_J, [z]_K) = u^i(t, (\alpha, \beta, z)_{I, J, K}).$$

Finally, let $R_{I, J, K}: C_\varepsilon \times [T - a, +\infty[\rightarrow \mathbb{R}$ be the Carathéodory function defined by

$$R_{I, J, K}(t, r) = \begin{cases} r^{-1} \{ u^i(t, [\alpha(t)]_I, [-\alpha(t)]_J, [z(t-r)]_K) + \\ - u^i(t, [\alpha(t-r)]_I, [-\alpha(t-r)]_J, [z(t-r)]_K) \} & \text{if } r \in H(t) \setminus \{0\}, \\ \sum_{j=1}^m s(j) h_j(t) D_{x_j} u^i(t, [\alpha(t)]_I, [-\alpha(t)]_J, [z(t)]_K) & \text{if } r = 0, \\ \text{linear} & \text{if } r \notin H(t). \end{cases}$$

Of course, $R_{I, J, K}(\cdot, r)$ is measurable. Moreover, put

$B_\varepsilon = \{t \in C_\varepsilon: C_\varepsilon$ has metric density 1 at t , function $u(t, \cdot)$ satisfies condition (K_1) and the derivatives $m^{i'}(t), \alpha_j'(t), j = 1, \dots, m$, exist finite},

let us now prove that for every fixed $t \in B_\varepsilon$ the linear part of the function $R_{I, J, K}$ can be chosen in such a way that $R_{I, J, K}(t, \cdot)$ is continuous in $[T - a, +\infty[$.

By the differentiability of $u^i(t, \cdot)$ and the continuity of $D_{x_j} u^i(t, \dots, \bar{x}_j, \dots)$ for every $j \in I \cup J$ we have

$$\begin{aligned} \lim_{r \rightarrow 0, r \in H(t)} R_{I, J, K}(t, r) &= \\ &= \lim_{r \rightarrow 0, r \in H(t)} r^{-1} [u^i(t, [\alpha(t)]_I, [-\alpha(t)]_J, [z(t-r)]_K) + \\ &\quad - u^i(t, [\alpha(t-r)]_I, [-\alpha(t-r)]_J, [z(t-r)]_K)] = \\ &= \lim_{r \rightarrow 0, r \in H(t)} \left\{ - \sum_{j=1}^m s(j) D_{x_j} u^i(t, [\alpha(t)]_I, [-\alpha(t)]_J, [z(t-r)]_K) \cdot \right. \\ &\quad \left. \cdot \frac{\alpha_j(t) - \alpha_j(t-r)}{r} + \frac{o(r)}{r} \right\} = \end{aligned}$$

$$\begin{aligned}
&= \sum_{j \in J} h_j(t) D_{x_j} u^i(t, [\alpha(t)]_I, [-\alpha(t)]_J, [z(t)]_K) - \\
&- \sum_{j \in I} h_j(t) D_{x_j} u^i(t, [\alpha(t)]_I, [-\alpha(t)]_J, [z(t)]_K).
\end{aligned}$$

Let N be the number of the triplets (I, J, K) in Ψ . By virtue of Scorza-Dragnoni property, there exists a closed set $G_{I, J, K} \subset B_\varepsilon$, with $\text{meas}(G_{I, J, K}) > \text{meas}(B_\varepsilon) - \varepsilon/2N$, such that the function $R_{I, J, K}$ is continuous in $G_{I, J, K} \times [T - a, +\infty[$.

Finally, let $A = \bigcap_{(I, J, K) \in \Psi} G_{I, J, K}$. Of course we have that

$$\text{meas}(A) \geq \text{meas}(B_\varepsilon) - \frac{\varepsilon}{2} > T - \varepsilon.$$

Let us now fix a point $t_0 \in A$, with $M^{i'}(t_0) > 0$, such that the set A has metric density 1 at t_0 , and let us prove that (6) holds.

Assume that $m^i(t_0) = u^i(t_0, z(t_0))$ (the proof is analogous in the case $m^i(t_0) = -u^i(t_0, z(t_0))$). Put

$$I = \{j: z_j(t_0) = \alpha_j(t_0)\}, \quad J = \{j: z_j(t_0) = -\alpha_j(t_0)\},$$

$$K = \{j: -\alpha_j(t_0) < z_j(t_0) < \alpha_j(t_0)\},$$

note that

$$(7) \quad D_{x_j} u^i(t_0, z(t_0)) \geq 0 \quad \text{if } j \in I, \quad D_{x_j} u^i(t_0, z(t_0)) \leq 0 \quad \text{if } j \in J,$$

$$(8) \quad D_{x_j} u^i(t_0, z(t_0)) = 0 \quad \text{if } j \in K.$$

Assume first that $I \cup J = \emptyset$. Since

$$\frac{m^i(t_0) - m^i(t_0 - r)}{r} \leq \frac{u^i(t_0, z(t_0)) - u^i(t_0 - r, z(t_0))}{r} \quad \text{for every } r > 0,$$

by virtue of assumption (W^+) we have

$$\begin{aligned}
m^{i'}(t_0) &\leq D_t u^i(t_0, z(t_0)) = \\
&= D_t u^i(t_0, z(t_0)) - \sum_{j=1}^m h_j(t_0) |D_{x_j} u^i(t_0, z(t_0))| \leq g^i(t_0, m(t_0), M)
\end{aligned}$$

and (6) is proved.

Let us now assume that $I \cup J \neq \emptyset$. Let $(r_n)_n$ be a sequence of positive

real numbers convergent to 0, such that for every $n \in \mathbb{N}$ we have

$$(9) \quad t_0 - r_n \in G_{I, J, K}.$$

By virtue of (7), (8), (9) we have

$$\begin{aligned} m^{i'}(t_0) &= \lim_{n \rightarrow +\infty} \frac{m^i(t_0) - m^i(t_0 - r_n)}{r_n} \leq \\ &\leq \lim_{n \rightarrow +\infty} \frac{u^i(t_0, z(t_0)) - u^i(t_0 - r_n, [\alpha(t_0 - r_n)]_I, [-\alpha(t_0 - r_n)]_J, [z(t_0)]_K)}{r_n} = \\ &= \lim_{n \rightarrow +\infty} \frac{u^i(t_0, z(t_0)) - u^i(t_0 - r_n, z(t_0))}{r_n} + \\ &+ \lim_{n \rightarrow +\infty} \frac{u^i(t_0 - r_n, z(t_0)) - u^i(t_0 - r_n, [\alpha(t_0 - r_n)]_I, [-\alpha(t_0 - r_n)]_J, [z(t_0)]_K)}{r_n} = \\ &= D_t u^i(t_0, z(t_0)) + \lim_{n \rightarrow +\infty} R_{I, J, K}(t_0 - r_n, -r_n) = \\ &= D_t u^i(t_0, z(t_0)) + R_{I, J, K}(t_0, 0) = \\ &= D_t u^i(t_0, z(t_0)) + \sum_{j=1}^m s(j) h_j(t_0) D_{x_j} u^i(t_0, z(t_0)) = \\ &= D_t u^i(t_0, z(t_0)) - \sum_{j=1}^m h_j(t_0) |D_{x_j} u^i(t_0, z(t_0))| \leq g^i(t_0, m(t_0), M) \end{aligned}$$

and this concludes the proof.

REMARK 2. Note that the proof of the previous theorem still holds if we weaken the hypotheses on function u as follows: the derivatives $D_t u(t, x)$, $D_x u(t, x)$ exist in the interior of pyramid Δ , t -a.e., and condition (K_1) holds only on the boundary of Δ .

For the sake of comparison of Theorem 4 with analogous results, observe that it extends Haar's classical lemma and its generalization established in [19] (Lemma 1) and in [23], since in (4) we consider a generic function g depending also on the functional argument, instead of a linear function in the second variable, without retarded argument. Moreover, Theorem 4 is the Carathéodory version of the analogous result given in [12] (Theorem 2), [16] (Lemma 1).

We now present a version of Theorem 4 for unbounded domains.

THEOREM 5. *Let $u \in \mathcal{X}_B(E_0 \cup E)$ be a given function and let $h \in L^1([0, T], \mathbb{R}^n)$ be a given nonnegative function, with $T > 0$.*

Assume that for every $(t, x) \in E_T$, t-a.e., we have

$$(4') \quad |D_t u^i(t, x)| \leq \sum_{j=1}^m h_j(t) |D_{x_j} u^i(t, x)| + g^i(t, |u(t, x)|, M),$$

$$i = 1, \dots, n$$

where $M: I \rightarrow \mathbb{R}^n$ is defined by $M^i(t) = \sup_{(\tau, \xi) \in E_t} |u^i(\tau, \xi)|$, $i = 1, \dots, n$.

Moreover, assume that

$$(5') \quad |u(t, x)| \leq \eta(t) \quad \text{for every } (t, x) \in E_0.$$

Finally, suppose that function $g(t, x, \cdot)$ is continuous uniformly with respect to the pair (t, x) .

Then we have

$$|u(t, x)| \leq \Omega(t) \quad \text{for every } (t, x) \in E_0 \cup E_T,$$

where $\Omega: I_T \rightarrow \mathbb{R}^n$ is the maximal solution of comparison problem $CP(g, \eta)$.

PROOF. Note that from the definition of class $\mathcal{X}_B(E_0 \cup E)$ it follows that function M is continuous.

Let us fix $\varepsilon > 0$. By virtue of Lemma 2, a real number $\delta = \delta(\varepsilon) > 0$ exists such that denoted by Ω_δ the maximal solution of comparison problem $CP(g + \delta, \eta)$, we have $\Omega_\delta(t) \leq \Omega(t) + \varepsilon$ in I_T .

Let \tilde{k} be an integer such that $\int_0^T h_j(\tau) d\tau \leq \tilde{k}$, $j = 1, \dots, m$, and for every integer $k \geq \tilde{k}$ we put

$$\Delta^k = \left\{ (t, x): t \in [0, T], -k + \int_0^t h_j(\tau) d\tau \leq x_j \leq k - \int_0^t h_j(\tau) d\tau, j = 1, \dots, m \right\},$$

$$M_k^i(t) = \max_{(\tau, \xi) \in (\Delta^k)_t} |u^i(\tau, \xi)|, \quad i = 1, \dots, n.$$

Observe that, by virtue of Lemma 3, functions M_k , $k \in \mathbb{N}$, are absolutely continuous. Moreover, the sequence $(M_k)_k$ uniformly converges to M in $[0, T]$.

Therefore, recalling the uniformity of continuity of $g(t, x, \cdot)$, we de-

duce that an integer $k^* \in \mathbb{N}$ exists such that

$$g(t, x, M) \leq g(t, x, M_k) + \delta \quad \text{for every } (t, x) \in E \text{ and every } k \geq k^*.$$

Thus, we can apply Theorem 4 to obtain that $M_k(t) \leq \Omega_\delta(t) \leq \Omega(t) + \varepsilon$ for every $t \in I_T$, $k \geq k^*$. Hence, taking the limit for $k \rightarrow +\infty$ we obtain

$$M(t) \leq \Omega(t) + \varepsilon \quad \text{for every } (t, x) \in E.$$

The assertion follows by the arbitrariness of ε .

As an application of Theorem 4 we now derive the following comparison result, which is the Carathéodory version of the analogous results given in [12] (Theorem 4), [16] (Theorem 1).

COROLLARY 6. *Let $f, \tilde{f}: \mathcal{D} \rightarrow \mathbb{R}^n$ and $\phi, \tilde{\phi} \in C(\Delta_0)$ be given functions. Assume that for a.e. $t \in I$, every $x \in S_t(\Delta)$, $(y, z), (\bar{y}, \bar{z}) \in \mathbb{R}^n \times \mathcal{X}(\Delta_0 \cup \Delta)$, $q, \tilde{q} \in \mathbb{R}^{mn}$, we have:*

$$(10) \quad |f^i(t, x, y, z, q) - f^i(t, x, y, z, \tilde{q})| \leq \sum_{j=1}^m h_j(t) |q_j^i - \tilde{q}_j^i|,$$

$i = 1, \dots, n;$

$$(11) \quad |f^i(t, x, y, z, q) - \tilde{f}^i(t, x, \bar{y}, \bar{z}, q)| \leq g^i(t, |y - \bar{y}|, M),$$

$i = 1, \dots, n$

where

$$M^i(t) = \max_{(\tau, x) \in \Delta_t} |z^i(\tau, x) - \bar{z}^i(\tau, x)|, \quad i = 1, \dots, n.$$

Finally, assume that

$$(12) \quad |\phi(t, x) - \tilde{\phi}(t, x)| \leq \eta(t) \quad \text{in } \Delta_0.$$

Then, for every $u, v \in \mathcal{X}(\Delta_0 \cup \Delta)$, with $u \in SP(f, \phi)$, $v \in SP(\tilde{f}, \tilde{\phi})$, we have

$$|u(t, x) - v(t, x)| \leq \Omega(t) \quad \text{for every } (t, x) \in \Delta_0 \cup \Delta_T, \quad i = 1, \dots, n$$

where $\Omega: I_T \rightarrow \mathbb{R}^n$ is the maximal solution of comparison problem $CP(g, \eta)$.

PROOF. Put $w(t, x) = u(t, x) - v(t, x)$. By assumption (12) we have

$$(13) \quad |w(t, x)| \leq \eta(t) \quad \text{in } \Delta_0.$$

Moreover

$$\begin{aligned} D_t w^i(t, x) &= f^i(t, x, u(t, x), u, D_x u(t, x)) - \bar{f}^i(t, x, v(t, x), v, D_x v(t, x)) = \\ &= f^i(t, x, u(t, x), u, D_x u(t, x)) - f^i(t, x, u(t, x), u, D_x v(t, x)) + \\ &+ [f^i(t, x, u(t, x), u, D_x v(t, x)) - \bar{f}^i(t, x, v(t, x), v, D_x v(t, x))]. \end{aligned}$$

Thus, from (10), (11) it follows that

$$|D_t w^i(t, x)| \leq \sum_{j=1}^m h_j(t) |D_{x_j} w^i(t, x)| + g^i(t, |w(t, x)|, M)$$

for a.e. $t \in [0, a[$ and every $x \in S_t(\Delta)$. Therefore the assertion follows from (13) and Theorem 4.

REMARK 3. In force of Theorem 5, the previous result still holds if we replace Δ with an unbounded set E and $\mathcal{K}(\Delta_0 \cup \Delta)$ with the class $\mathcal{K}_B(E_0 \cup E)$, provided we assume that $g(t, x, \cdot)$ is continuous uniformly with respect to the pair (t, x) .

4. - Uniqueness and continuous dependence.

In this section we derive uniqueness and continuous dependence criteria for solutions of functional Cauchy problems. These results extend analogous theorems given in [24], [21], [19] for C^1 -solutions; moreover they are the Carathéodory version of the uniqueness and continuous dependence criteria established in [12] (Theorems 5, 6), [16] (Theorem 2).

We recall that the comparison function g of problem (CP) was assumed to satisfy conditions (C), (V), (W^+) and (M). In order to obtain uniqueness and continuous dependence criteria, as usual, we assume the further assumption

- (*) the maximal solution Ω of ordinary comparison problem $P(g, \eta_0)$, with $\eta_0 \equiv 0$, is the null-function.

Of course, this is equivalent to the requirement that $g(t, 0, 0) \equiv 0$ and problem $P(g, \eta_0)$ admits a unique solution.

REMARK 4. Observe that condition (*) is satisfied by functions $g = (g_1, \dots, g_n)$ of the type

$$1) \ g^i(t, x, \theta) = |\theta^i|_t, \text{ where } |\theta^i|_t = \max_{0 \leq \tau \leq t} |\theta^i(\tau)|;$$

$$2) \ g^i(t, x, \theta) = \psi(t) \int_0^t |\theta^i(\tau)| d\tau, \text{ where } \psi \in L^1(I) \text{ is nonnegative.}$$

In fact, $g(t, 0, 0) \equiv 0$. Moreover, by virtue of the uniqueness result in [3], problem $P(g, \eta_0)$ admits a unique solution.

Other examples of functions satisfying condition (*) are provided by $\gamma(\|g(t, x, \theta)\|)$, where γ is a nonnegative Lipschitzian function with $\gamma(0) = 0$, and g is one of the previous functions.

For simplicity we consider the cases of solutions defined in a pyramid Δ or in unbounded set E , separately.

4.a. *Bounded domain.*

The following uniqueness criterium is an immediate application of Corollary 6.

THEOREM 7 (Uniqueness). *Let $f: \mathcal{D} \rightarrow \mathbb{R}^n$ be a given function.*

Assume that for every $(t, x) \in E$, t-a.e., and every $(y, z, q), (\bar{y}, \bar{z}, \bar{q}) \in \mathbb{R}^n \times \mathcal{V} \times \mathbb{R}^{mn}$ we have

$$(14) \quad |f^i(t, x, y, z, q) - f^i(t, x, \bar{y}, \bar{z}, \bar{q})| \leq \leq g^i(t, |y - \bar{y}|, M) + \sum_{j=1}^m h_j(t) |q_j^i - \bar{q}_j^i|$$

where $M^i(t) = \max_{(\tau, x) \in \Delta_t} |z^i(\tau, x) - \bar{z}^i(\tau, x)|$.

Then, for every function $\phi \in C(\Delta_0)$, Cauchy-problem $P(f, \phi)$ admits at most one solution in class $\mathcal{X}(\Delta_0 \cup \Delta)$.

THEOREM 8 (Continuous dependence). *Let $f, \tilde{f}: \mathcal{D} \rightarrow \mathbb{R}^n$, $\phi, \tilde{\phi} \in C(\Delta_0)$ be given functions, and let $T > 0$ be a fixed real number.*

Assume that all the assumptions of Theorem 7 hold and that Cauchy-problem $P(f, \phi)$ admits the (unique) solution $u \in \mathcal{X}(\Delta_0 \cup \Delta)$ in $\Delta_0 \cup \Delta_T$.

Then, for every $\varepsilon > 0$ a real number $\delta = \delta(\varepsilon) > 0$ exists such that if

$$|f^i(t, x, y, z, q) - \tilde{f}^i(t, x, y, z, q)| < \delta \quad i = 1, \dots, n,$$

for every $(t, x, y, z, q) \in \Delta_T \times \mathbb{R}^n \times \mathcal{V} \times \mathbb{R}^{mn}$, *t-a.e.*, and

$$|\phi^i(t, x) - \tilde{\phi}^i(t, x)| < \delta, \quad \text{for every } (t, x) \in \Delta_0,$$

we have

$$|u^i(t, x) - v^i(t, x)| < \varepsilon$$

for every $v \in SP(\tilde{f}, \tilde{\phi})$, and every $(t, x) \in \Delta_0 \cup \Delta_T$.

PROOF. By virtue of Lemma 2, for every $\varepsilon > 0$ a positive real number $\delta = \delta(\varepsilon)$ exists such that the maximal solution Ω_δ of Cauchy problem $P(g + \delta, \delta)$ exists in I_T and we have $\Omega_\delta(t) < \varepsilon$ for every $t \in I_T$.

Let us now observe that for every $(t, x) \in \Delta$, *t-a.e.*, and every $(y, z, q), (\bar{y}, \bar{z}, q) \in \mathbb{R}^n \times \mathcal{V} \times \mathbb{R}^{mn}$ we have

$$\begin{aligned} |f^i(t, x, y, z, q) - \tilde{f}^i(t, x, \bar{y}, \bar{z}, q)| &\leq \\ &\leq |f^i(t, x, y, z, q) - f^i(t, x, \bar{y}, \bar{z}, q)| + \\ &+ |f^i(t, x, \bar{y}, \bar{z}, q) - \tilde{f}^i(t, x, \bar{y}, \bar{z}, q)| \leq \\ &\leq g^i(t, |\bar{y} - y|, M) + \delta, \quad i = 1, \dots, n, \end{aligned}$$

hence, by applying Corollary 6 we have

$$|u(t, x) - v(t, x)| \leq \Omega_\delta(t) < \varepsilon.$$

4.b. Unbounded domain.

It is easy to prove that all the results of the previous section 4.a hold in class $\mathcal{X}_B(E_0 \cup E)$, provided $M^i(t) = \sup_{(\tau, x) \in E_t} |z^i(\tau, x) - \bar{z}^i(\tau, x)|$, $i = 1, \dots, n$ (see also Remark 1).

Furthermore, the following continuous dependence criterium in class $\mathcal{X}(E_0 \cup E)$ also holds as an immediate application of Theorem 8.

COROLLARY 9. Let $(\phi_s)_{s \geq 0}$ be a sequence of initial data and let $f_s: \mathcal{O} \rightarrow \mathbb{R}^n$, $s \geq 0$, be a sequence of Carathéodory functions satisfying Volterra condition.

Assume that for every compact set $K \subset \mathbb{R}^m$ the sequence $(\phi_s)_s$ uniformly converges to ϕ_0 in $I_0 \times K$, and that the sequence $(f_s)_s$ uniformly converges to f_0 in $I \times K \times \mathbb{R}^n \times \mathcal{V} \times \mathbb{R}^{mn}$.

Finally, assume that function f_0 satisfies (14).

Then, every sequence $(v_s)_s$ with $v_s \in SP(f_s, \phi_s)$ converges in $C(E_0 \cup E)$ to the unique solution of problem $P(f_0, \phi_0)$.

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