RENDICONTI del SEMINARIO MATEMATICO della UNIVERSITÀ DI PADOVA

KRYSTYNA TWARDOWSKA

An approximation theorem of Wong-Zakai type for stochastic Navier-Stokes equations

Rendiconti del Seminario Matematico della Università di Padova, tome 96 (1996), p. 15-36

<http://www.numdam.org/item?id=RSMUP_1996__96__15_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 96 (1996)

An Approximation Theorem of Wong-Zakai Type for Stochastic Navier-Stokes Equations.

KRYSTYNA TWARDOWSKA (*)

SUMMARY - We present an extension of the Wong-Zakai approximation theorem for stochastic Navier-Stokes equations defined in abstract spaces and with some Hilbert space valued disturbances given by the Wiener process.By approximating these disturbances we obtain in the limit equation the Itô correction term for the infinite dimensional case. Such form of the correction term was proved in the author's papers [24] and [25], where the approximation theorem for semilinear stochastic evolution equations in Hilbert spaces was studied. A theorem of this type for nonlinear stochastic partial differential equations, more exactly for the model considered by Pardoux ([19]) one can find in [23].

0. – Introduction.

We consider a generalization of the Wong-Zakai approximation theorem ([27]) for stochastic Navier-Stokes differential equations. Similar equations were already studied e.g. by Bensoussan ([2]), Bensoussan and Temam ([3]), Brzeźniak and others ([4]), Capiński ([6]), Capiński and Cutland ([7]), Fujita-Yashima ([10]). In the above papers mainly the existence and uniqueness theorems were given.

We are dealing with the approximation theorems of Wong-Zakai type. A thorough discussion of generalizations of this theorem in finite and infinite dimensions can be found in [25]. We only mention that for a linear equation in the infinite-dimensional case some generalizations

(*) Indirizzo dell'A.: Institute of Mathematics, Warsaw University of Technology, Plac Politechniki 1, 00-661 Warsaw, Poland.

The research of the author was partially supported by KBN grant No. 2 P301 052 03.

are known, where the Wiener process is one-dimensional and the state space is infinite-dimensional (see Aquistapace and Terreni ([1]), Brzeźniak and others ([5]), Gyöngy ([12])). In [24] and [25] some extensions of the Wong-Zakai theorem to nonlinear functional stochastic differential equations as well as to stochastic semilinear evolution equations in a Hilbert space with a Hilbert space valued Wiener process are given. For the latter equations the unbounded operator is the infinitesimal generator of a semigroup of contraction type and the other operators are nonlinear and bounded. The Wong-Zakai approximation theorem for stochastic nonlinear partial differential equations with unbounded monotone and coercive operators defined in Gelfand triples is given in [23]. The infinite-dimensional Itô correction term derived here coming from a Hilbert space valued Wiener process is exactly the same as in [9] and [23]-[25].

The author is grateful to the referee for helpful remarks which enabled the author to improve this paper.

1. – Definitions and notation.

Let $(\Omega, F, (F_t)_{t \in [0, T]}, P)$ be a filtered probability space on which an increasing and right-continuous family $(F_t)_{t \in [0, T]}$ of sub-σ-algebras of F is defined such that F_0 contains all P-null sets in F.

Let $L(X, Y)$ denote the vector space of continuous linear operators from X to Y, with the operator norm $\| \cdot \|_{L(X, Y)}$, where X and Y are arbitrary Banach spaces (we put $L(X) = L(X, X)$); $L^p(\Omega; X)$, $\infty \geq p \geq 1$, denotes the usual Banach space of equivalence classes of random variables with values in X which are p-integrable (essentially bounded for $p = \infty$) with the norm $\| \cdot \|_{L^p(\Omega; X)}$. We put $L^p(\Omega) = L^p(\Omega; \mathbb{R})$.

Moreover, $\mathcal{L}^1(X, Y)$ is the Banach space of nuclear operators from X to Y with the trace norm $\| \cdot \|_{\mathcal{L}^1(X, Y)}$ and $\mathcal{L}^2(X, Y)$ is the Hilbert space of Hilbert-Schmidt operators with the norm $\| \cdot \|_{HS}$, where X and Y are arbitrary separable Hilbert spaces. $\mathcal{L}^1(X, Y)$ and $\mathcal{L}^2(X, Y)$ are some subspaces of $L(X, Y)$.

Let H and K be real separable Hilbert spaces with scalar products $(\cdot, \cdot)_H$, $(\cdot, \cdot)_K$ and with orthonormal bases $\{l_n\}_{n=1}^{\infty}$, $\{k_n\}_{n=1}^{\infty}$ of H and K, respectively. We also consider a real separable Hilbert spaces V and W which are continuously and densely embedded in the Hilbert space H. Moreover, the inclusion $V \to H$ is compact. Then, identifying H with its dual space H^* (by the scalar product in H) we have, denoting by V^* and W^* the dual spaces to V and W, respectively,

$$W \subset V \subset H = H^* \subset V^* \subset W^* .$$

The embeddings are continuous with dense ranges. The above spaces are endowed with the norms $\|\cdot\|_W$, $\|\cdot\|_V$, $\|\cdot\|_H$ and $\|\cdot\|_{V^*}$, $\|\cdot\|_{W^*}$, respectively. The pairing between V and V^* (as well as between W and W^*) is denoted by $\langle\cdot, \cdot\rangle$.We consider a K-valued Wiener process $(w(t))_{t\in[0,T]}$, adapted to the family F_t, with nuclear covariance operator J. It is known ([8], Chapter 5) that there are real-valued independent Wiener processes $\{\overset{j}{w}(t)\}_{j=1}^{\infty}$ on $[0, T]$ such that

$$w(t) = \sum_{j=1}^{\infty} \overset{j}{w}(t)\, k_j$$

almost everywhere in $(t, \omega) \in [0, T] \times \Omega$, where $\{k_j\}_{j=1}^{\infty}$ is an orthonormal basis of eigenvectors of J corresponding to eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$, $\sum_{j=1}^{\infty} \lambda_j < \infty$, with

$$(*) \qquad E[\Delta \overset{i}{w}\, \Delta \overset{j}{w}] = (t-s)\,\lambda_i \delta_{ij}$$

for $\Delta \overset{j}{w} = \overset{j}{w}(t) - \overset{j}{w}(s)$ and $s < t$ (δ_{ij} is the Kronecker delta). We put

$$(1.1) \qquad w^m(t) = \sum_{j=1}^{m} \overset{j}{w}(t)\, k_j = \sum_{j=1}^{m} (w(t), k_j)_K k_j .$$

Now we define the n-th polygonal approximations of the processes $(w(t))_{t\in[0,T]}$ and $(w^m(t))_{t\in[0,T]}$, respectively, by

$$(1.2) \qquad w_{(n)}(t) = \sum_{j=1}^{\infty} w_{j,n}(t)\, k_j ,$$

$$(1.3) \qquad w_{(n)}^{m}(t) = \sum_{j=1}^{m} w_{j,n}(t)\, k_j ,$$

where for $t_{i-1}^n < t \leq t_i^n$ with $0 = t_0^n < \dots < t_n^n = T$,

$$(1.4) \qquad w_{j,n}(t) = \frac{t - t_{i-1}^n}{t_i^n - t_{i-1}^n}\, \overset{j}{w}(t_i^n) + \frac{t_i^n - t}{t_i^n - t_{i-1}^n}\, \overset{j}{w}(t_{i-1}^n).$$

DEFINITION 1.1. Denote by $\mathcal{P}$ the σ-algebra of sets on $[0, T] \times \Omega$ generated by all F_t-adapted and left continuous X-valued stochastic processes.

An X-valued stochastic process $(X_t)_{t\in[0,T]}$ is called predictable if the mapping $(t, \omega) \to X_t(\omega)$ is $\mathcal{P}$-measurable.

2. - Description of the abstract model.

We consider the stochastic nonlinear differential equation

$$
(2.1)\qquad \begin{cases} du(t) + A(t)u(t)\,dt + G(u(t))\,dt + B(t, u(t))\,dw(t) = f(t)\,dt\,, \\ u(0) = u_0\,, \end{cases}
$$

where $(u(t))_{t \in [0, T]}$ is an H-valued stochastic process and

(A1) u_0 is an H-valued square integrable F_0-measurable random variable, that is, $u_0 \in L^2(\Omega, F_0, P; H)$.

For every $n \in \mathbb{N}$ we consider the approximation equation

$$
(2.2_n)\qquad \begin{cases} du_{(n)}(t) + A(t)u_{(n)}(t)\,dt + G(u_{(n)}(t))\,dt + \\ \qquad\qquad + B(t, u_{(n)}(t))\,dw_{(n)}(t) = f(t)\,dt\,, \\ u_{(n)}(0) = u_0\,, \end{cases}
$$

where $w_{(n)}(t)$ is given by (1.2). Moreover, we consider the equation

$$
(2.3)\qquad \begin{cases} d\widehat{u}(t) + A(t)\widehat{u}(t)\,dt + G(\widehat{u}(t))\,dt + B(t, \widehat{u}(t))\,dw(t) + \\ \qquad\qquad + \frac{1}{2}\,\widetilde{\mathrm{tr}}\,(JDB(t, \widehat{u}(t))B(t, \widehat{u}(t)))\,dt = f(t)\,dt\,, \\ \widehat{u}(0) = u_0\,, \end{cases}
$$

with $\widetilde{\mathrm{tr}}\,(JDB(t, \widehat{u}(t))B(t, \widehat{u}(t)))$ to be described later in this section.We assume that the family of operators $A(t) \in L(V, V^*)$ defined for almost every (a.e.) $t \in (0, T)$ has the following properties:

(A2) growth restriction: there exists a constant β such that

$$\|A(t)u\|_{V^*} \leq \beta\|u\|_V \text{ for every } u \in V \text{ and for a.e. } t,$$

(A3) coercivity: there exist constants $\alpha > 0$ and λ, $\widetilde{\nu}$ such that

$$2\langle A(t)u, u\rangle + \lambda\|u\|_H^2 + \widetilde{\nu} \geq \alpha\|u\|_V^2 + \|B(t, u)\|_{HS}^2$$

for every $u \in V$ and for a.e. t,

(A4) measurability: the mapping

$$(0, T) \ni t \to \langle A(t)u, v\rangle \in \mathbb{R}$$

is Lebesgue measurable for every $u, v \in V$.

The family of operators $B(t, \cdot)\colon V \to \mathcal{L}^2(K, H)$ defined for a.e. $t \in (0, T)$ satisfies the following assumptions:

(A5) boundedness of $B(t, \cdot)$: there exists a constant $\tilde{L}$ such that

$$\|B(t, u)\|_{HS} \leqslant \tilde{L} \quad \text{for all } u \in V,$$

(A6) the operator $B(t, \cdot) \in C_b^1$, i.e., is of class C^1 with bounded derivative (in the Hilbert-Schmidt topology) and this derivative is assumed to be globally Lipschitzean,

(A7) the boundedness of $DB(t, \cdot)$ is meant on V in the sense of the norm in H: there exists a constant $\tilde{\tilde{L}}$ such that

$$\|DB(t, u)\,h\|_{HS} \leqslant \tilde{\tilde{L}}\,\|h\|_H \quad \text{for all } u \in V,\ h \in H,$$

(A8) measurability: for every $u \in V$ the mapping

$$(0, T) \ni t \to B(t, u) \in \mathcal{L}^2(K, H)$$

is Lebesgue measurable.

The bilinear continuous mapping $G\colon V \times V \to W^*$ satisfies the following assumptions:

(A9) $\langle G(u, v), v \rangle = 0$ for every $u \in V$ and $v \in W$,

(A10) boundedness: there exists a constant $\tilde{C}$ such that

$$\|G(u, v)\|_{W^*} \leqslant \tilde{C}\|u\|_H^{1/2}\|v\|_H^{1/2}\|u\|_V^{1/2}\|v\|_V^{1/2}$$

for all $u, v \in V$.

Finally, we assume

(A11) $f \in L^2((0, T) \times \Omega;\ V^*)$ and f is nonanticipating.

Put

$$G(u) = G(u, u)$$

for every $u \in V$.

REMARK 2.1. Assumption (A7) ensures the correctness of the definition of $DB(t, h_1) \circ B(t, h_1) \in L(K, L(K, H))$ for $h_1 \in H$ because $DB(t, h_1)\colon H \supset V \to L(K, H)$ is bounded on V (in the Hilbert-Schmidt topology) in the norm of H.

REMARK 2.2. We shall also use a weaker assumption than (A6) for the operator B, that is, the following Lipschitz condition: for all $h \in H$, $k \in K$ and $N \in \mathbb{R}_+$ there exists a constant $L = L(h, k, N)$ such that

$$\text{(A12)} \qquad |(h, B(t, u)k)_H - (h, B(t, v)k)_H| \leqslant L\|u - v\|_V$$

for all $u, v \in V$ with $\|u\|_V, \|v\|_V \leqslant N$.

By assumption (A7) we may replace (A12) by

$$\text{(A12')} \qquad |(h, B(t, u)k)_H - (h, B(t, v)k)_H| \leqslant L\|u - v\|_H .$$

Now we observe ([9], [23]-[25]) that the Fréchet derivative $DB(t, h_1) \in L(V, L(K, H))$ for $h_1 \in V$ and a.e. t.

Consider the composition $DB(t, h_1) \circ B(t, h_1) \in L(K, L(K, H))$, where the Fréchet derivative is computed for $h_1 \in H$ due to the extension made in (A7). Let $\Psi \in L(K, L(K, H))$ and define

$$B_{\tilde{h}_1}(h, h') := (\Psi(h)(h'), \tilde{h}_1)_H \in \mathbb{R}$$

for $h, h' \in K$. By the Riesz theorem, for every $\tilde{h}_1 \in H$ there exists a unique operator $\tilde{\Psi}(\tilde{h}_1) \in L(K)$ such that for all $h, h' \in K$,

$$(\tilde{\Psi}(\tilde{h}_1)(h), h')_K = (\Psi(h)(h'), \tilde{h}_1)_H .$$

Now, the covariance operator J has finite trace and therefore the mapping

$$\tilde{\xi}\colon H \ni \tilde{h}_1 \to \operatorname{tr}(J\tilde{\Psi}(\tilde{h}_1)) \in \mathbb{R}$$

is a linear bounded functional on H. Therefore, using the Riesz theorem we find a unique $\tilde{\tilde{h}}_1 \in H$ such that $\tilde{\xi}(\tilde{h}_1) = (\tilde{\tilde{h}}_1, \tilde{h}_1)_H$. Denote

$$\tilde{\tilde{h}}_1 = \widetilde{\operatorname{tr}}(J\Psi) .$$

We observe that $(\tilde{\tilde{h}}_1, \tilde{h}_1)_H$ is the trace of the operator $J\tilde{\Psi}(\tilde{h}_1) \in L(K)$ but $\widetilde{\operatorname{tr}}(J\Psi)$ is merely a symbol for $\tilde{\tilde{h}}_1$.

DEFINITION 2.1. Suppose we are given an H-valued initial random variable u_0 and a K-valued Wiener process $(w(t))_{t \in [0, T]}$. Suppose further that an H-valued stochastic process $(u(t))_{t \in [0, T]}$ has the following properties:

(i) $(u(t))_{t \in [0, T]}$ is predictable,

(ii) $u(t) \in L^2((0, T) \times \Omega; V) \cap L^2(\Omega; L^\infty(0, T; H))$,

(iii) there exists a set $\Omega' \subset \Omega$ such that $P(\Omega') = 1$ and for all $(t, \omega) \in [0, T] \times \Omega'$ and $y \in Y \subset H$ (Y is an everywhere dense, in the strong topology, subset of H) equation (2.1) is satisfied in the following sense:

$$(2.4)\qquad (y, u(t, \omega))_H = (y, u_0(\omega))_H - \int_0^t (\langle A(s)u(s, \omega), y\rangle + $$

$$+\langle G(u(s, \omega)), y\rangle)\, ds - \left(y, \int_0^t B(s, u(s, \omega))\, dw(s)\right)_H + \langle f(t, \omega), y\rangle .$$

An equivalent formulation of (2.4) is understood in W^* (see e.g. [7]).

It is known ([14], [19]) that the above integrals are well defined. Moreover (see [14]),

$$\left(y, \int_0^t B(s, u(s, \omega))\, dw(s)\right)_H = \int_0^t \hat{y}B(s, u(s, \omega))\, dw(s),$$

where $\hat{y} \in L(H, \mathbb{R})$ is the operator given by the formula $\hat{y}v = (y, v)_H$, $v \in H$.

Then $(u(t))_{t \in [0, T]}$ is called a solution to (2.1) with initial condition u_0.

DEFINITION 2.2. Let $n \in \mathbb{N}$. We say that a mapping $u_{(n)}: [0, T] \to H$ is a solution to equation (2.2_n) if $u_{(n)} \in L^2(0, T; V) \cap L^\infty(0, T; H)$, $\dot{u}_{(n)} \in L^2(0, T; W^*)$ and if equation (2.2_n) is satisfied for all $0 \leqslant t \leqslant T$.

3. – Applications for Navier-Stokes equations.

Here we denote by $\mathcal{O}$ an open bounded set of $\mathbb{R}^2$ with a regular boundary $\partial\mathcal{O}$. Let $H^s(\mathcal{O})$ be the Sobolev space of functions y which are in $L^2(\mathcal{O})$ together with all their derivatives of order $\leqslant s$; $s > 2/2 + 1$. Further, $H_0^1(\mathcal{O})$ is the Hilbert subspace of $H^1(\mathcal{O})$, made up of functions vanishing on $\partial\mathcal{O}$. We also introduce the product Hilbert spaces $(L^2(\mathcal{O}))^2$, $(H_0^1(\mathcal{O}))^2$, $(H^s(\mathcal{O}))^2$.

We consider the set $\mathcal{V}(\mathcal{O})$ of functions from $\mathcal{C}^\infty$ with a compact support in $\mathcal{O}$. Put

$$\mathcal{E} = \left\{ y = (y_1, y_2):\ y_i \in \mathcal{V}(\mathcal{O}),\ \operatorname{div} y = \sum_{i=1}^{2} \frac{\partial y_i}{\partial x_i} = 0 \right\}$$

and

$$H = \text{the closure of } \mathcal{E} \text{ in } (L^2(\mathcal{O}))^2,$$

$$V = \text{the closure of } \mathcal{E} \text{ in } (H_0^1(\mathcal{O}))^2,$$

$$W = \text{the closure of } \mathcal{E} \text{ in } (H^s(\mathcal{O}))^2.$$

For $y, z \in H$ we put

$$((y, z))_H = \sum_{i=1}^{2} \int_{\mathcal{O}} y_i(x) z_i(x)\, dx.$$

For $y, z \in V$ we put

$$((y, z))_V = \sum_{i,j=1}^{2} \int_{\mathcal{O}} \frac{\partial y_i(x)}{\partial x_j} \frac{\partial z_i(x)}{\partial x_j}\, dx$$

to obtain

$$\langle -\nu \Delta y, z \rangle = \nu((y, z))_V.$$

These spaces have the structure of the Hilbert spaces induced by $(L^2(\mathcal{O}))^2$, $(H_0^1(\mathcal{O}))^2$, $(H^s(\mathcal{O}))^2$, that is

$$((y, z))_H = \sum_{i=1}^{2} (y_i, z_i)_{L^2(\mathcal{O})},$$

$$((y, z))_V = \sum_{i=1}^{2} (y_i, z_i)_{H_0^1(\mathcal{O})},$$

$$((y, z))_W = \sum_{i=1}^{2} (y_i, z_i)_{H^s(\mathcal{O})}.$$

It is obvious that W, V and H have all properties from § 1.

Let $\nu > 0$ be fixed. We define the family of operators $A(t) \in L(V, V^*)$ by

$$\langle A(t) y, z \rangle = \nu((y, z))_V$$

for all $y, z \in V$. Therefore, assumptions (A2), (A3) and (A4) (for $\alpha = \nu$, $\lambda = 0$, $\tilde{\nu} = 0$) are satisfied. Further we consider a trilinear form

$$b(y, z, w) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} y_i(x) \frac{\partial z_j(x)}{\partial x_i} w_j(x)\, dx$$

defined and continuous on $V \times V \times W$. We recall ([15], p. 67 and p. 71) that for a positive constant C_1 we have

$$(3.1) \qquad |b(y, y, w)| \leqslant C_1 \|y\|^2_{(L^4(\mathcal{O}))^2} \|w\|_{(H^1(\mathcal{O}))^2}$$

for all $y, w \in V$.

It is also proved in [15], Lemma 6.2, p. 70, that there exists a positive constant C_2 such that

$$(3.2) \qquad \|v\|^2_{(L^4(\mathcal{O}))^2} \leqslant C_2 \|v\|_{(H_0^1(\mathcal{O}))^2} \|v\|_{(L^2(\mathcal{O}))^2}$$

for all $v \in (H_0^1(\mathcal{O}))^2$.

Now we define a bilinear continuous operator $G\colon V \times V \to W^*$ by

$$\langle G(y, z), w \rangle = b(y, z, w).$$

for all $y, z \in V$ and $w \in W$.

It is easily to check that assumptions (A9) and (A10) are satisfied.

We consider the following stochastic Navier-Stokes equation

$$(3.3) \qquad \begin{cases} du - \nu \Delta u\, dt + (u \cdot \nabla) u\, dt + \nabla p\, dt + B(u)\, dw(t) = f(t)\, dt, \\ u = 0 \quad \text{on } \Sigma = [0, T] \times \partial\mathcal{O}, \\ u(0) = u_0 \quad \text{in } \mathcal{O}, \\ \operatorname{div} u = 0 \quad \text{in } [0, T] \times \mathcal{O}, \end{cases}$$

where $u = u(t, x)$ is the velocity field of a fluid and $p = p(t, x)$ is the pressure.

The reduction to the abstract form (2.1) is completely calssical (see [21]) and we omit it. Further, we shall understand equation (2.1) as the above Navier-Stokes equation for $\mathcal{O} \subset \mathbb{R}^2$.

The uniqueness of solution is understood in the sense of trajectories.

The existence and uniqueness of solution to (2.3) under assumptions (A1)-(A4), (A8)-(A12) follows from the following modification of Theorem 6.3 in [7]. Namely, we omit the assumption on the periodic boundary condition that we only need to prove the uniqueness of the solution. The uniqueness we obtain from [6].

For each $n \in \mathbb{N}$ the existence and uniqueness of solution to (2.2_n) under assumptions (A1)-(A4), (A8)-(A12) follows e.g. from a slight modification of the existence and uniqueness theorems in [15].

4. - Auxiliary lemmas.

Let us denote by $W_m = V_m = H_m = V_m^* = W_m^*$ the vector space spanned by the vectors $l_1, \ldots, l_m$ and let $P_m \in L(H, H_m)$ be the orthogonal projection. We recall that $\{l_n\}_{n=1}^{\infty}$ is an orthonormal base of H. We assume that $l_n \in W$ for every $n \in \mathbb{N}$. Otherwise the equalities

$$W_m = V_m = H_m = V_m^* = W_m^*$$

would not be satisfied. We introduce in H_m the norm

$$\|u\| = \left(\sum_{j=1}^{m} |u_j|^2\right)^{1/2}$$

for $u = (u_1, \ldots, u_m)$, and the usual scalar product $(\cdot, \cdot)$. We extend P_m to an operator $V^* \to V_m^*$ by

$$\tilde{P}_m u = \sum_{j=1}^{m} \langle u, l_j \rangle l_j \qquad \text{for } u \in V^* .$$

Analogously we define $\tilde{\tilde{P}}_m u$ for $u \in W^*$.

We denote by K_m the vector space spanned by the vectors $k_1, \ldots, k_m$. Let $\Pi_m \in L(K, K_m)$ be the orthogonal projection.

Now, we define the families of operators $A^m(t)\colon V_m \to V_m^*$ by

$$A^m(t)\,u := \tilde{P}_m A(t)\,u \qquad \text{for } u \in V_m , \tag{4.1}$$

and $G^m\colon V_m \times V_m \to W_m^*$ by

$$G^m(u, v) := \tilde{\tilde{P}}_m G(u, v) \qquad \text{for } u, v \in V_m , \tag{4.2}$$

as well as $B^m(t, \cdot)\colon H_m \to \mathcal{L}^2(K_m, H_m)$ by

$$B^m(t, u) := P_m B(t, u) \qquad \text{for } u \in H_m . \tag{4.3}$$

Let $w^m(t)$ be the Wiener process with values in K_m defined by

$$w^m(t) = \Pi_m w(t) .$$

Clearly, it can be represented by formula (1.1). Moreover, we put

$$f^m = P_m f \in L^2(\Omega \times (0, T);\ V_m^*)$$

and

$$u_0^m = P_m u_0 \in L^2(\Omega;\ H_m) .$$

Now, we consider the following stochastic differential equation of Itô type in the space $\mathbb{R}^m$ for the i-th coordinate of a process $v^m(t) = (v_1^m(t), \dots, v_m^m(t)) \in H_m$:

$$
\begin{aligned}
(4.4)\quad & dv_i^m(t) + (A^m(t)v^m(t))_i\,dt + G_i^m(v^m(t))\,dt + \\
&\quad + B_i^m(t, v^m(t))\,dw^m(t) + \\
&\quad + \frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m}\lambda_j \frac{\partial B_{ij}^m(t, v^m(t))}{\partial v_l^m} B_{lj}^m(t, v^m(t))\,dt = f_i^m(t)\,dt, \\
&\quad v^m(0) = u_0^m,
\end{aligned}
$$

where $(B_{ij}^m(t, v^m(t)))_{i,j=1,\dots,m}$ is the matrix representation of elements of $B^m(t, v^m(t))$, and $(A^m(t)v^m(t))_i$ is the i-th coefficient of the vector

$$A^m(t)v^m(t) = \tilde{P}_m A(t)u = \sum_{i=1}^{m}\langle A(t)v^m(t), l_i\rangle l_i .$$

Equation (4.4) is a finite dimensional stochastic differential equation. Therefore, from the slight modification of the existence and uniqueness theorems, e.g. in [2], [3], we observe that under our assumptions equation (4.4) has exactly one solution $v^m(t) \in H_m$ for any $m = 1, 2, \dots$ such that

$$v^m \in L^2((0, T) \times \Omega;\, V_m) \cap L^2(\Omega;\, L^\infty(0, T;\, H_m)).$$

For every $n \in \mathbb{N}$, we also consider the approximation equation

$$
(4.5_n^m)\quad \begin{cases} dv_{(n)}^m(t) + A^m(t)v_{(n)}^m(t)\,dt + G^m(v_{(n)}^m(t))\,dt + \\ \qquad\qquad + B^m(t, v_{(n)}^m(t))\,dw_{(n)}^m(t) = f^m(t)\,dt, \\ v_{(n)}^m(0) = u_0^m, \end{cases}
$$

where $w_{(n)}^m(t)$ is given by (1.3). We observe that $dw_{(n)}^m(t) = \dot{w}_{(n)}^m\,dt$ on every interval $(t_{i-1}^n, t_i^n]$ so equations (4.5_n^m) are of deterministic nature for almost every $\omega \in \Omega$.

Let us start from

LEMMA 4.1. *Let $v^m(t)$, $\hat{u}(t)$ be the solutions to equations* (4.4) *and* (2.3), *respectively, under assumptions* (A1)-(A4), (A8)-(A12). *Then for each $t \in [0, T]$, $0 < T < \infty$, we have*

$$(4.6)\qquad \lim_{m\to\infty} E[\|v^m(t) - \hat{u}(t)\|_H^2] = 0 .$$

PROOF. From the proof of the uniqueness of the solution to equation of the type (2.3), see Theorem 3.2 in [6], taking $u_1(t) = v^m(t)$ and $u_2(t) = \widehat{u}(t)$ we immediately deduce that $v^m(t) \to \widehat{u}(t)$ in the sense of (4.6).

Further, we have

LEMMA 4.2. *The correction term*

$$(**) \qquad \left(\frac{1}{2} \sum_{j=1}^{m} \sum_{l=1}^{m} \lambda_j \frac{\partial B_{ij}^m(t, v^m(t))}{\partial v_l^m} B_{lj}^m(t, v^m(t)) \right)_{i=1,\dots,m}$$

in equation (4.4) *is the result of applying the projection operators* P_m *and* Π_m *to operator* $B(t, \cdot)$ *and to the Wiener process* $(w(t))_{t\in[0,T]}$ *in the construction of the term* $(1/2)\,\widetilde{\operatorname{tr}}\,(JDB(t, \widehat{u}(t))\,B(t, \widehat{u}(t)))$, *that is,*

$$(4.7) \qquad \sum_{j=1}^{m} \sum_{l=1}^{m} \lambda_j \frac{\partial B_{\cdot j}^m(t, v^m(t))}{\partial v_l^m} B_{lj}^m(t, v^m(t))$$

converges to $\widetilde{\operatorname{tr}}\,(JDB(t, \widehat{u}(t))\,B(t, \widehat{u}(t)))$ *weakly in* H.

PROOF. We have $A^m(t)u = \widetilde{P}_m A(t)u = \sum_{i=1}^{m} \langle A(t)u, l_i\rangle l_i \in V_m^*$ for $u \in V_m$ and $B^m(t, u) = B(t, u)|_{K_m} = P_m \circ B(t, u)|_{K_m} \in \mathcal{L}^2(K_m, H_m)$ for $u \in H_m$ because $B(t, \cdot)|_{K_m}\colon H_m \to \mathcal{L}^2(K_m, H)$ and $B(t, u)|_{K_m} \in \mathcal{L}^2(K_m, H)$. The restriction mapping we understand in the following sense

$$(H_m \ni u \to B(t, u)|_{K_m})\colon H_m \to \mathcal{L}^2(K_m, H).$$

Now we consider the Fréchet derivative of $B^m(t, u)$ for $u \in H_m$, that is, $DB^m(t, u) \in L(H_m, \mathcal{L}^2(K_m, H_m))$ and $DB^m(t, u)(x) \in \mathcal{L}^2(K_m, H_m)$ for $x \in H_m$. Now we consider the composition $DB^m(t, u) \circ B^m(t, u) \in L(K_m, \mathcal{L}^2(K_m, H_m))$.

Then $DB^m(t, u)(x) \in \mathcal{L}^2(K_m, H_m)$ is given by the matrix

$$\widehat{A}(t, u)(x) = \begin{bmatrix} DB_{11}^m(t, u)(x) & \dots & DB_{1m}^m(t, u)(x) \\ \dots & \dots & \dots \\ DB_{m1}^m(t, u)(x) & \dots & DB_{mm}^m(t, u)(x) \end{bmatrix}.$$

We put

$$X=\begin{pmatrix}\xi_1\\ \vdots\\ \xi_m\end{pmatrix}\in K_m\,,\qquad Y=\begin{pmatrix}\eta_1\\ \vdots\\ \eta_m\end{pmatrix}\in K_m\,.$$

Let $DB^m(t, u)\circ B^m(t, u)X=\Psi(X)$. Then for $i=1, \dots, m$,

$$(\Psi(X)\,Y)_i=\sum_{j=1}^{m}\left(\sum_{l=1}^{m}\frac{\partial B_{ij}^m(t, u)}{\partial u_l}\left(\sum_{k=1}^{m}B_{lk}^m(t, u)\,\xi_k\right)\right)\eta_j$$

and for $p\in H_m$,

$$(\Psi(X)\,Y, p)_{H_m}=\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{l=1}^{m}\sum_{k=1}^{m}\frac{\partial B_{ij}^m(t, u)}{\partial u_l}B_{lk}^m(t, u)\,\xi_k\eta_j p_i\,.$$

Omitting X and Y in the last sum we obtain the matrix

$$\left(\sum_{i=1}^{m}\sum_{l=1}^{m}\frac{\partial B_{ij}^m(t, u)}{\partial u_l}B_{lk}^m(t, u)\,p_i\right)_{jk}=(\Psi_{jk})_{j, k=1, \dots, m}=\tilde{\Psi}(p)\,.$$

Consider the trace

$$\operatorname{tr}(J_m\tilde{\Psi}(p))=\sum_{j=1}^{m}\sum_{i=1}^{m}\sum_{l=1}^{m}\lambda_j\frac{\partial B_{ij}^m(t, u)}{\partial u_l}B_{lj}^m(t, u)\,p_i\,,$$

where $(J_m)_{jk}$, $j, k=1, \dots, m$, is the restriction of the covariance operator J to $\mathbb{R}^m$ $(=H_m)$. We rewrite it in the form of the inner product of two vectors in $\mathbb{R}^m$

$$\operatorname{tr}(J_m\tilde{\Psi}(p))=\sum_{i=1}^{m}\left(\sum_{j=1}^{m}\sum_{l=1}^{m}\lambda_j\frac{\partial B_{ij}^m(t, u)}{\partial u_l}B_{lj}^m(t, u)\right)p_i\,.$$

Taking into account the result of Lemma 4.1, we observe that the first vector is exactly the correction term $\tilde{\tilde{h}}_1$ obtained in Section 2. Replacing B by B^m and w by w^m we repeated step by step the construction of $\tilde{\tilde{h}}_1$ from Section 2. Therefore, we obtained the finite-dimensional form of the correction term that converges by the above construction to $\tilde{\operatorname{tr}}\,(JDB(t, \hat{u}(t))\,B(t, \hat{u}(t)))$ as $m\to\infty$. Thus we have proved (4.7).

Remark 4.1. Note that the definition of the correction term in the finite-dimensional case depends on the restriction to $\mathbb{R}^m$ of the operator J, i.e. it depends on $\lambda_1, \dots, \lambda_m$. This is because of our definition of $w(t)$

(see (*) in § 1). Moreover, since

$$\operatorname{tr}[J\tilde{\Psi}(\tilde{h}_1)] = \sum_{j=1}^{\infty} (J\tilde{\Psi}(\tilde{h}_1)k_j, k_j)_K = \sum_{j=1}^{\infty} (\tilde{\Psi}(\tilde{h}_1)k_j, J^* k_j)_K =$$

$$= \sum_{j=1}^{\infty} (\tilde{\Psi}(\tilde{h}_1)k_j, Jk_j)_K = \sum_{j=1}^{\infty} (\Psi(k_j)(Jk_j), \tilde{h}_1)_H =$$

$$= \sum_{j=1}^{\infty} (\Psi(k_j)(\lambda_j k_j), \tilde{h}_1)_H ,$$

taking in particular $\Psi = DB(t, h_1)B(t, h_1)$ we get

$$(***) \qquad \tilde{\tilde{h}} = \tilde{\operatorname{tr}}[J\Psi] = \sum_{j=1}^{\infty} \Psi(k_j)(Jk_j) = \sum_{j=1}^{\infty} [DB(t, h_1)B(t, h_1)(k_j)](\lambda_j k_j).$$

Thus in the infinite-dimensional case we have the same situation with the term $\tilde{\operatorname{tr}}\,(JDB(t, h_1)B(t, h_1))$ which depends on the covariance operator J, i.e. it depends on $\{\lambda_j\}_{j=1}^{\infty}$ (see (***)).

We also have

LEMMA 4.3. *Let $A^m(t, \cdot)$, $G^m(\cdot)$ and $B^m(t, \cdot)$ be given by* (4.1)-(4.3), *respectively, under assumptions* (A1)-(A4), (A8)-(A12). *Let $w^m_{(n)}(t)$ be given by* (1.3). *Assume that $v^m_{(n)}(t)$ and $u_{(n)}(t)$ are solutions to equations* (4.5^m_n) *and* (2.2_n), *respectively. Then, for every $t \in [0, T]$, $0 < T < \infty$, we have*

$$(4.8) \qquad \lim_{m\to\infty} E[\|v^m_{(n(m))}(t) - u_{(n(m))}(t)\|^2_H] = 0 ,$$

where $\{n(m)\}$ is an arbitrary increasing sequence depending on m.

PROOF. For every $m > 0$ we choose an arbitrary increasing sequence $\{n(m)\}$ depending on m, that is, $n = n(m)$ is a function of m. Then $\{u_{(n(m))}\}$ and $\{v^m_{(n(m))}\}$ are arbitrary subsequences of $\{u_{(n)}\}$ and $\{v^m_{(n)}\}$, respectively. Let us take $Y = \bigcup_{m=1}^{\infty} H_m$. It is obvious that for every $y \in Y$ there exists $m_0(y)$ such that for every $m \geq m_0(y)$ we have $y \in H_m$. Moreover,

$$\langle A(t)u, y\rangle = \langle A^m(t)u, y\rangle$$

if for $m \geq m_0(y)$ we take $y \in H_m$. We have the same equalities for B^m and G^m.

Now we set

$$h_m(t) = v^m_{(n(m))}(t) - u_{(n(m))}(t) = \sum_{j=1}^{\infty} g_{jm}(t)\, l_j = \sum_{j=1}^{\infty} (h_m(t), l_j)\, l_j .$$

We compute on $(t^n_{i-1}, t^n_i]$, for almost every $\omega \in \Omega$,

$$(h'_m(t), l_j) + \langle A(t)\, h_m, l_j\rangle = -\langle G(v^m_{(n(m))}(t)) - G(u_{(n(m))}(t)), l_j\rangle -$$

$$- (B(t, v^m_{(n(m))}(t))\, \alpha^{m,i}_{(n(m))} - B(t, u_{(n(m))}(t))\, \alpha^{i}_{(n(m))}, l_j) +$$

$$+ \langle f^m(t) - f(t), l_j\rangle,$$

where $\alpha^{m,i}_{(n(m))}$ and $\alpha^{i}_{(n(m))}$ are some constant derivatives on $(t^n_{i-1}, t^n_i]$ of $w^m_{(n(m))}(t)$ and $w_{(n(m))}(t)$, respectively.

We multiply the above equatlity by $g_{jm}(t)$ and take the sum over $j = 1, 2, \ldots$. We recall ([15], p. 71) that after the simple computations we obtain

$$b(v^m_{(n(m))}(t), v^m_{(n(m))}(t), h_m(t)) - b(u_{(n(m))}(t), u_{(n(m))}(t), h_m(t)) =$$

$$= b(h_m(t), v^m_{(n(m))}(t), h_m(t)) + b(v^m_{(n(m))}(t), h_m(t), h_m(t)) -$$

$$- b(h_m(t), h_m(t), h_m(t))$$

and

$$b(v^m_{(n(m))}(t), h_m(t), h_m(t)) = 0 , \qquad b(h_m(t), h_m(t), h_m(t)) = 0 .$$

We obtain

$$(h'_m(t), h_m(t)) + \langle A(t)\, h_m(t), h_m(t)\rangle = - b(h_m(t), v^m_{(n(m))}(t), h_m(t)) -$$

$$- (B(t, v^m_{(n(m))}(t))\, \alpha^{m,i}_{(n(m))} - B(t, u_{(n(m))}(t))\, \alpha^{m,i}_{(n(m))}, h_m(t)) -$$

$$- (B(t, u_{(n(m))}(t))\, \alpha^{m,i}_{(n(m))} - B(t, u_{(n(m))}(t))\, \alpha^{i}_{(n(m))}, h_m(t)) +$$

$$+ \langle f^m(t) - f(t), h_m(t)\rangle .$$

We replace $(h_m'(t), h_m(t))$ by $(1/2)(d/dt)\|h_m(t)\|^2$ and get

$$\|h_m(t)\|^2 + 2\int_{t_{i-1}^n}^{t} \langle A(s)h_m(s), h_m(s)\rangle ds =$$

$$= -2\int_{t_{i-1}^n}^{t} b(h_m(s), v_{(n(m))}^m(s), h_m(s))ds +$$

$$+ \|h_m(t_{i-1}^n)\|^2 - 2\int_{t_{i-1}^n}^{t} (B(s, v_{(n(m))}^m(s))\,a_{(n(m))}^{m,i} -$$

$$- B(s, u_{(n(m))}(s))\,a_{(n(m))}^{m,i}, h_m(s))ds - 2\int_{t_{i-1}^n}^{t} (B(s, u_{(n(m))}(s))\,a_{(n(m))}^{m,i} -$$

$$- B(s, u_{(n(m))}(s))\,a_{(n(m))}^{i}, h_m(s))ds + 2\int_{t_{i-1}^n}^{t} \langle f^m(s) - f(s), h_m(s)\rangle ds\,.$$

Let $C_1, \ldots, C_7$ be some constants. From (3.1), (3.2) and the inequality

$$a^{\gamma}b^{1-\gamma} \leqslant \gamma a + (1-\gamma)b \quad \text{for } a, b > 0,\ 0 \leqslant \gamma \leqslant 1,$$

which we will use in its equivalent form $ab \leqslant \nu a^2 + (4\nu)^{-1}b^2$, we have

$$(4.9)\qquad \left|\int_{t_{i-1}^n}^{t} b(h_m(s), v_{(n(m))}^m(s), h_m(s))ds\right| \leqslant$$

$$\leqslant C_1\int_{t_{i-1}^n}^{t} \|h_m(s)\|^2_{(L^4(\mathcal{O}))^2}\|v_{(n(m))}^m(s)\|_{(H_0^1(\mathcal{O}))^2}ds \leqslant$$

$$\leqslant C_2\int_{t_{i-1}^n}^{t} \|h_m(s)\|_{(H_0^1(\mathcal{O}))^2}\|h_m(s)\|_{(L^2(\mathcal{O}))^2}\|v_{(n(m))}^m(s)\|_{(H_0^1(\mathcal{O}))^2}ds \leqslant$$

$$\leqslant \nu\int_{t_{i-1}^n}^{t} \|h_m(s)\|^2_{(H_0^1(\mathcal{O}))^2}ds + C_3\int_{t_{i-1}^n}^{t} \|h_m(s)\|^2_{(L^2(\mathcal{O}))^2}\|v_{(n(m))}^m(s)\|^2_{(H_0^1(\mathcal{O}))^2}ds\,.$$

From (A5), (A3), (4.9) and (A12′) we get (further we shall use again the notation $\|\cdot\| = \|\cdot\|_{H_m} = \|\cdot\|_{(L^2(\mathcal{O}))^2}$)

$$\|h_m(t)\|^2 + \nu \int_{t_{i-1}^n}^t \|h_m(s)\|^2_{(H_0^1(\mathcal{O}))^2} ds \leqslant$$

$$\leqslant \|h_m(t_{i-1}^n)\|^2 + \nu \int_{t_{i-1}^n}^t \|h_m(s)\|^2_{(H_0^1(\mathcal{O}))^2} ds +$$

$$+ C_3 \int_{t_{i-1}^n}^t \|h_m(s)\|^2 \|v^m_{(n(m))}(s)\|^2_{(H_0^1(\mathcal{O}))^2} ds +$$

$$+ 2L \int_{t_{i-1}^n}^t \|\Pi_m a^i_{(n(m))}\|_K \|h_m(s)\|^2 ds +$$

$$+2 \int_{t_{i-1}^n}^t \|B(s, u_{(n(m))}(s))\|_{HS} \|\Pi_m a^i_{(n(m))} - a^i_{(n(m))}\|_K \|h_m(s)\| ds +$$

$$+2 \int_{t_{i-1}^n}^t \|f^m(s) - f(s)\|_{V^*} \|h_m(s)\| ds.$$

From (A5) we obtain

$$\|h_m(t)\|^2 \leqslant C_m + \tag{4.10}$$

$$+ C_4 \int_{t_{i-1}^n}^t (1 + \|v^m_{(n(m))}(s)\|^2_{(H_0^1(\mathcal{O}))^2} + \|\Pi_m a^i_{(n(m))}\|_K) \|h_m(s)\|^2 ds +$$

$$+ C_5 \int_{t_{i-1}^n}^t \|\Pi_m a^i_{(n(m))} - a^i_{(n(m))}\|_K \|h_m(s)\| ds$$

for a constant C_m, where $\|\Pi_m a^i_{(n(m))}\|_K$ can be estimated by its expected value.

Now taking the mathematical expectation, applying the above procedure to all intervals $(t_{i-1}^n, t_i^n]$ and using the Gronwall lemma conclude the proof. More exactly, we compute on the whole interval

$$E\left[\int_0^t \|\Pi_m \alpha^i_{(n(m))}(s) - \alpha^i_{(n(m))}(s)\|_K ds\right]^2 =$$

$$= E\left[\sum_{i=1}^n (t_i^{n(m)} - t_{i-1}^{n(m)})\sqrt{\left(\Pi_m \frac{w(t_i^{n(m)}) - w(t_{i-1}^{n(m)})}{t_i^{n(m)} - t_{i-1}^{n(m)}} - \frac{w(t_i^{n(m)}) - w(t_{i-1}^{n(m)})}{t_i^{n(m)} - t_{i-1}^{n(m)}}, \Pi_m \frac{w(t_i^{n(m)}) - w(t_{i-1}^{n(m)})}{t_i^{n(m)} - t_{i-1}^{n(m)}} - \frac{w(t_i^{n(m)}) - w(t_{i-1}^{n(m)})}{t_i^{n(m)} - t_{i-1}^{n(m)}}\right)}\right]^2 =$$

$$= \sum_{i=1}^n (t_i^{n(m)} - t_{i-1}^{n(m)})^2 E\left[\sqrt{\left(\Pi_m \frac{w(t_i^{n(m)}) - w(t_{i-1}^{n(m)})}{t_i^{n(m)} - t_{i-1}^{n(m)}} - \frac{w(t_i^{n(m)}) - w(t_{i-1}^{n(m)})}{t_i^{n(m)} - t_{i-1}^{n(m)}}, \Pi_m \frac{w(t_i^{n(m)}) - w(t_{i-1}^{n(m)})}{t_i^{n(m)} - t_{i-1}^{n(m)}} - \frac{w(t_i^{n(m)}) - w(t_{i-1}^{n(m)})}{t_i^{n(m)} - t_{i-1}^{n(m)}}\right)}\right]^2 =$$

$$= \sum_{i=1}^n E[\Pi_m w(t_i^{n(m)}) - \Pi_m w(t_{i-1}^{n(m)}) - w(t_i^{n(m)}) - w(t_{i-1}^{n(m)}),$$

$$\Pi_m w(t_i^{n(m)}) - \Pi_m w(t_{i-1}^{n(m)}) - w(t_i^{n(m)}) - w(t_{i-1}^{n(m)})] =$$

$$= \sum_{i=1}^n (a_m t_i^{n(m)} - 4a_m t_{i-1}^{n(m)} - a_m t_i^{n(m)} + 4a_m t_{i-1}^{n(m)} - at_{i-1}^{n(m)} +$$

$$+ at_{i-1}^{n(m)} - a_m t_i^{n(m)} + a_m t_{i-1}^{n(m)} + at_i^{n(m)} - at_{i-1}^{n(m)}) =$$

$$= \sum_{i=1}^n (-a_m t_i^{n(m)} + a_m t_{i-1}^{n(m)} + at_i^{n(m)} - at_{i-1}^{n(m)}) =$$

$$= \sum_{i=1}^n (a - a_m)(t_i^{n(m)} - at_{i-1}^{n(m)}) = (a - a_m)\cdot T \to 0,$$

as $m \to \infty$, where

$$\alpha^i_{(n(m))}(s) = \frac{w(t_i^{n(m)}) - w(t_{i-1}^{n(m)})}{t_i^{n(m)} - t_{i-1}^{n(m)}} \quad \text{for } s \in (t_{i-1}^{n(m)}, t_i^{n(m)}],$$

$$E[(w(t), w(s))_K] = \sum_{j=1}^\infty k_j^2 \min\{t, s\},$$

$$E[(\Pi_m w(t), w(s))_K] = \sum_{j=1}^m k_j^2 \min\{t, s\},$$

$$E[(\Pi_m w(t), \Pi_m w(s))_K] = \sum_{j=1}^m k_j^2 \min\{t, s\}$$

and

$$a = \sum_{j=1}^{\infty} k_j^2, \qquad a_m = \sum_{j=1}^{m} k_j^2.$$

Further, we have on $[0, T]$

$$E\left[\int_0^t \|\Pi_m a^i_{(n(m))}(s) - a^i_{(n(m))}(s)\|_K \|h_m(s)\| ds\right] \leq$$

$$\leq E\left[\sup_s \|h_m(s)\| \cdot \int_0^t \|\Pi_m a^i_{(n(m))}(s) - a^i_{(n(m))}(s)\|_K ds\right] \leq$$

$$\leq E\left[\sup_s \|h_m(s)\|^2\right]^{1/2} \left(E\left[\int_0^t \|\Pi_m a^i_{(n(m))}(s) - \right.\right.$$

$$\left.\left. - a^i_{(n(m))}(s)\|_K ds\right]^2\right)^{1/2} \leq \sqrt{t(a - a_m)}\sqrt{E\left[\sup_s \|h_m(s)\|^2\right]}.$$

Finally, we get on the whole interval $[0, T]$

$$E[\|h_m(t)\|^2] \leq C_6 \sqrt{t(a - a_m)}\sqrt{E\left[\sup_s \|h_m(s)\|^2\right]} + C'_m +$$

$$+ C_7 E\left[\int_0^t (1 + \|v^m_{(n(m))}(s)\|^2_{(H_0^1(\mathcal{O}))^2})\|h_m(s)\|^2 ds\right],$$

where $C'_m \to 0$ as $m \to \infty$.

We estimate $E[\sup_s \|h_m(s)\|^2]$ and $E[\sup_s \|v^m_{(n(m))}(s)\|^2]$ by a constant similarly as in [19], p. 112. Using the Gronwall lemma we obtain (4.8).

5. - Approximation theorem.

Let us start from a modification of the Wong-Zakai approximation theorem for equations in $\mathbb{R}^d$.

REMARK 5.1. We observe that we can modify the finite-dimensional version of the Wong-Zakai theorem in [13], Chapter VI, § 7, Theorem 7.2. Namely, we first change the assumption about the r-dimensional

Wiener process appearing as the disturbance in this theorem. We now assume that the Wiener process has different variances in different directions, that is, it satisfies relatoin (*) of the present paper. From this it follows that the expression c_{ij} defined by (7.6) in [13], Chapter VI, § 7, has now the form:

$$c_{ij} = s_{ij} + \frac{1}{2}\delta_{ij}\lambda_i, \qquad i, j = 1, \ldots, r.$$

Therefore, the correction term is now of the form (**) of the present paper. With $w(t)$, c_{ij} and the correction term defined in this way, we repeat the proof of the Wong-Zakai approximation theorem in [13].

We shall prove the following

THEOREM 5.1. *Let* $\widehat{u}(t)$ *and* $u_{(n)}(t)$ *be solutions to equations* (2.3) *and* (2.2_n), *respectively. Assume that assumptions* (A1)-(A11) *are satisfied. Take approximations* $w_{(n)}(t)$ *of the Wiener process* $w(t)$ *given by* (1.2). *Then, for each* $t \in [0, T]$, $0 < T < \infty$,

$$\lim_{n \to \infty} E[\|u_{(n)}(t) - \widehat{u}(t)\|_H^2] = 0. \tag{5.1}$$

PROOF. We have

$$(u_{(n)}(t) - \widehat{u}(t), y) = (u_{(n)}(t) - v_{(n)}^m(t), y) + \tag{5.2}$$
$$+ (v_{(n)}^m(t) - v^m(t), y) + (v^m(t) - \widehat{u}(t), y).$$

Observe that

$$\lim_{n \to \infty} E\left[\sup_{0 \le t \le T} |v_{(n)}^m(t) - v^m(t)|_{\mathrm{R}^m}^2\right] = 0 \tag{5.3}$$

because by the definitions of A^m, B^m, G^m and by Lemma 4.2 we can use the modified finite-dimensional version of the Wong-Zakai approximation theorem (see Remark 5.1) to obtain (5.3). Indeed, our equation is now in R^m and assumptions in [13], namely that the diffusion term σ is in C_b^2 and the drift term b is in C_b^1, can be used in the proof in [13] in a weaker form, just as our weaker assumptions, that is, the Lipschitz conditions on b and σ' instead of higher classes of continuity of b and σ'.

More exactly, for every $\varepsilon > 0$ and every $m > 0$ we choose $n(m)$ such that for every $n \ge n(m)$ we have from (5.3) the convergence of $v_{(n(m))}^m(t)$ to $v^m(t)$.

Now we choose m_0 such that for every $m \ge m_0$ we have by Lemma 4.1 the convergence of $v^m(t)$ to $\widehat{u}(t)$ and we put an appropriate $n(m)$ to get the previous convergence. Now from Lemma 4.3 we obtain for every $m \geqslant m_0$ the convergence of $v^m_{(n(m))}(t)$ to $u_{(n(m))}(t)$, which completes the proof.

REFERENCES

[1] P. AQUISTAPACE - B. TERRENI, *An approach to Itô linear equations in Hilbert spaces by approximation of white noise with coloured noise*, Stochastic Anal. Appl., **2** (1984), pp. 131-186.

[2] A. BENSOUSSAN, *A model of stochastic differential equation in Hilbert space applicable to Navier-Stokes equation in dimension* 2, in: *Stochastic Analysis, Liber Amicorum for Moshe Zakai*, Eds. E. Mayer-Wolf, E. Merzbach and A. Schwartz, Academic Press (1991), pp. 51-73.

[3] A. BENSOUSSAN - R. TEMAM, *Equations stochastiques du type Navier-Stokes*, J. Functional Analysis, **13** (1973), pp. 195-222.

[4] Z. BRZEŹNIAK - M. CAPIŃSKI - F. FLANDOLI, *Stochastic Navier-Stokes equations with multiplicative noise*, Stochastic Anal. Appl., **10**, 5 (1992), pp. 523-532.

[5] Z. BRZEŹNIAK - M. CAPIŃSKI - F. FLANDOLI, *A convergence result for stochastic partial differential equations*, Stochastics, **24** (1988), pp. 423-445.

[6] M. CAPIŃSKI, *A note on uniqueness of stochastic Navier-Stokes equations*, Universitatis Iagellonicae Acta Math., **30** (1993), pp. 219-228.

[7] M. CAPIŃSKI - N. CUTLAND, *Stochastic Navier-Stokes equations*, Acta Applicandae Math., **25** (1991), pp. 59-85.

[8] R. F. CURTAIN - A. J. PRITCHARD, *Infinite Dimensional Linear System Theory*, Springer, Berlin (1978).

[9] H. DOSS, *Liens entre équations différentielles stochastiques et ordinaires*, Ann. Inst. H. Poincaré, **13**, 2 (1977), pp. 99-125.

[10] H. FUJITA-YASHIMA, *Equations de Navier-Stokes stochastiques non homogénes et applications*, Scuola Normale Superiore, Pisa (1992).

[11] I. GYÖNGY, *On stochastic equations with respect to semimartingales III*, Stochastics, **7** (1982), pp. 231-254.

[12] I. GYÖNGY, *The stability of stochastic partial differential equations and applications. Theorems on supports*, Lecture Notes in Math., **1390**, Springer, Berlin (1989), pp. 91-118.

[13] N. IKEDA - S. WATANABE, *Stochastic Differential Equations and Diffusion Processes*, North-Holland, Amsterdam (1981).

[14] N. U. KRYLOV - B. L. ROZOVSKII, *On stochastic evolution equations*, Itogi Nauki i Techniki, Teor. Verojatn. Moscow, **14** (1979). pp. 71-146 (in Russian).

[15] J. L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris (1969).

[16] J. L. LIONS - E. MAGENES, *Non-Homogeneous Boundary Value Problems and Applications*, Springer, Berlin (1972).

[17] W. MACKEVIČIUS, *On the support of a solution of stochastic differential equation*, Lietuvos Matematikos Rinkinys, **26**, 1 (1986), pp. 91-98.

[18] S. NAKAO - Y. YAMATO, *Approximation theorem of stochastic differential equations*, Proc. Internat. Sympos. SDE Kyoto 1976, Tokyo (1978), pp. 283-296.

[19] E. PARDOUX, *Equations aux dérivées partielles stochastiques non linéaires monotones. Etude de solutions fortes de type Itô*, Thèse Doct. Sci. Math. Univ. Paris Sud (1975).

[20] B. SCHMALFUSS, *Measure attractors of the stochastic Navier-Stokes equations*, University Bremen, Report Nr. 258, Bremen (1991).

[21] R. TEMAM, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer, Berlin (1988).

[22] R. TEMAM, *Navier-Stokes Equations*, North-Holland, Amsterdam (1977).

[23] K. TWARDOWSKA, *An approximation theorem of Wong-Zakai type for nonlinear stochastic partial differential equations*, Stochastic Anal. Appl., **13**, 5 (1995), pp. 601-626.

[24] K. TWARDOWSKA, *An extension of the Wong-Zakai theorem for stochastic evolution equations in Hilbert spaces*, Stochastic Anal. Appl., **10**, 4 (1992), pp. 471-500.

[25] K. TWARDOWSKA, *Approximation theorems of Wong-Zakai type for stochastic differential equations in infinite dimensions*, Dissertationes Math., **325** (1993), pp. 1-54.

[26] M. J. VISHIK - A. V. FURSIKOV, *Mathematical Problems of Statistical Hydromechanics*, Kluwer, Dordrecht (1988).

[27] E. WONG - M. ZAKAI, *On the convergence of ordinary integrals to stochastic integrals*, Ann. Math. Statist., **36** (1965), pp. 1560-1564.

Manoscritto pervenuto in redazione il 9 marzo 1994
e, in forma revisionata, il 15 giugno 1995.