RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

JOHN C. LENNOX

On soluble groups in which centralizers are finitely generated

Rendiconti del Seminario Matematico della Università di Padova, tome 96 (1996), p. 131-135

<http://www.numdam.org/item?id=RSMUP_1996__96__131_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 96 (1996)

On Soluble Groups in which Centralizers Are Finitely Generated.

JOHN C. LENNOX (*)

The aim of this paper is to consider some evidence for an affirmative answer to the following question of the author (see [3]): is a soluble group polycyclic if all centralizers of its finitely generated subgroups are finitely generated?

Of course all subgroups of polycyclic groups are finitely generated and so the converse question has a positive answer.

We note that the hypothesis implies that the group in question is finitely generated.

In what follows we shall abbreviate finitely generated to f.g. and if G is a group we shall abbreviate the usual centralizer notation $c_G(X)$ to $c(X)$ where there is no ambiguity as to the identity of the group G with respect to which centralizers are being taken.

THEOREM A. *A soluble group of finite rank is polycyclic if all centralizers of f.g. subgroups are f.g.*

Now soluble groups of finite rank are nilpotent by abelian by finite and so Theorem A is an immediate consequence of the somewhat stronger

THEOREM B. *Suppose that G is a nilpotent by polycyclic group in which the centralizer of each polycyclic subgroup is f.g. Then G is polycyclic.*

(*) Indirizzo dell'A.: School of Mathematics, University of Wales, Cardiff, Wales, U.K.

The author wishes to thank the Alexander von Humboldt Stiftung for a Fellowship and Professor Hermann Heineken of the University of Wurzburg for hospitality during the completion of this work.

As a further corollary of Theorem B we have that a soluble linear group is polycyclic if the centralizer of each of its polycyclic subgroups is f.g.

We remark that the hypothesis of Theorem B only requires the centralizers of polycyclic subgroups to be f.g. If we further weaken this hypothesis to require only that centralizers of cyclic suybgroups are f.g., we have

THEOREM C. *Suppose that G is an abelian by nilpotent group in which the centralizer of each cyclic subgroup is f.g. Then G is polycyclic.*

This result is deduced in turn from

THEOREM D. *Suppose that a group G has an abelian normal subgroup A such that G/A is nilpotent and $c_G(a)$ is f.g. for all a in A. Then G is polycyclic.*

We do not know whether Theorems C and D hold if «nilpotent» is replaced by «polycyclic».

Proof.

PROOF OF THEOREM B. We first of all need.

LEMMA 1. *Suppose that G is a soluble group and that $c(X)$ is f.g. for every polycyclic subgroup X of G. Then this property passes to G/P, where P is a polycyclic normal subgroup of G.*

PROOF. Suppose that $L/P = c_{G/P}(H/P)$, where H/P is polycyclic. Then L normalizes H. Now $n_G(H)/c_G(H)$ is a soluble group of automorphisms of a polycyclic group and is therefore polycyclic by Mal'cev's Theorem. Since $c_G(H)$ is f.g. and contained in L, it now follows that L is f.g., as required.

We now proceed with the proof of Theorem B and we suppose that G is a f.g. nilpotent by polycyclic group in which $c_G(H)$ is f.g. for every polycyclic subgroup H of G. Let N be a nilpotent normal subgroup of G such that G/N is polycyclic. Since G is f.g. it is countable and so N is countable.

Suppose that $N = \{a_1, a_2, \dots, a_n, \dots\}$. Set $U_n = \langle a_1, a_2, \dots, a_n \rangle$. Then U_n is f.g. nilpotent and so polycyclic and hence $C_n = c(U_n)$ is f.g. by hypothesis.

(∗) Furthermore, $C_1 N \geq C_2 N \geq \ldots \geq C_n N \geq \ldots$ so that, since G/N is polycyclic, we must have that there exists a positive integer k with $|C_k N \colon C_{k+r} N|$ finite for all $r \geq 1$.

We set $K = C_k$ and consider $V = \mathrm{core}_G(KN)$. Since G/N is polycyclic, it follows by a result of Rhemtulla [4], that V is the intersection of finitely many conjugates of KN. Suppose that $g_1, \ldots, g_s$ will do as the conjugating elements and put $x = g_j$. Then $a_1^x, \ldots, a_k^x$ belong to N and so there exists a positive integer t such that U_k^x is contained in U_t. It then follows that $C_t \leq K^x$. Hence $C_t N \leq (KN)^x$ and using (∗) we have that the latter subgroup contains K^a for some positive integer a. Since s is finite we deduce that $K^b \leq V$ for some positive integer b.

Since K is f.g. soluble we have that K/K^b is finite, so that KN/V is finite (note that $N \leq V$). Now suppose that a is any element of N. Then certainly $K^c \leq c(a)N$ for some $c > 0$. We also have for all g in G that $(K^g)^b = (K^b)^g \leq V \leq KN$. Thus for any h in G

$$[a^h, K^{cb}] = [a, (K^{h^{-1}})^{cb}]^h \leq [a, K^c N]^h \leq N',$$

since $K^c \leq c(a)N$. So K^{cb} centralizes $a^G N'/N'$.

But G/N' is a f.g. abelian by polycyclic group so, by a theorem of P. Hall [2], N/N' is f.g. as a G-module. It follows at once from the above that $K^w = M$ centralizes N/N' for some $w > 0$. Moreover, N is nilpotent and we may apply a result of Robinson [5] to deduce that $[N, {}_q M] = 1$, for some $q > 0$.

We now set $W = MZ$, where Z is the centre of N. Then W is f.g. since it is of finite index in K and K is f.g. But $[N, {}_q M] = 1$ and so W is a f.g. hypercentral by polycyclic group. Hence $W/Z_q(W)$ is polycyclic. Hence W is polycyclic and so Z is a f.g. abelian group.

By Lemma 1 the hypotheses pass to G/Z and induction on the nilpotency class of N completes the proof.

PROOF OF THEOREM D. Here we shall need

LEMMA 2. *Suppose that G is a group and A an abelian normal subgroup of G such that G/A is polycyclic and $c_G(a)$ is f.g. for all a in A. Then if B is a normal subgroup of G contained in A, we have that $c_{G/B}(aB)$ is f.g. for all a in A.*

PROOF. Set $c_{G/B}(aB) = H/B$, so that $A \leq c_G(a) \leq H$. Moreover, H/A is polycyclic and $c_G(a)$ is f.g. by hypothesis, so that H is f.g. and the result follows.

We now proceed with the proof of Theorem D and suppose that G is a non-polycyclic group with an abelian normal subgroup A such that G/A is nilpotent and $c_G(a)$ is f.g. for all a in A. Thus $G = c_G(1)$ is f.g. and so by Hall's theorem A satisfies Max-G. This fact together with Lemma 2 allows to assume that if B is a non-trivial normal subgroup of G contained in A then G/B is polycyclic. From this it is not difficult to deduce that we may assume that G is just non-polycyclic (j.n.p.): for either G is j.n.p. or there is a non-trivial normal subgroup N of G with G/N not polycyclic. It follows that $N \cap A = 1$ and so N is polycyclic. In G/N we have $[gN, aN] = 1$ if and only if $[g, a] \in N \cap A = 1$. Thus G/N inherits the hypothesis in an obvious way. Since G satisfies Max-n we may factor out the maximal such N and so assume G is j.n.p. as stated.

By results of Groves [1] (or Robinson and Wilson [6]) there are two cases that arise:

Case 1. A is torsion free of finite rank.

Case 2. A has finite exponent a prime p.

In Case 1 (see [6]) G is a finite extension of a f.g. metabelian group. It not hard to see that we can come down to a subgroup of finite index and assume that G is metabelian. (If the new group is not j.n.p. we can repeat the argument and assume that it is). But then $c_G(a)$ is normal in G for all a in A. Hence a^G is contained in the centre of the f.g. metabelian group $c_G(a)$ and so, again by Hall's theorem, a^G is f.g. as an abelian group. But G is j.n.p. and so G/a^G is polycyclic. Hence G is polycyclic, a contradiction.

So we may assume that we case in Case 2. Put $C = c_G(A)$. Then G/C is nilpotent. Let Z/C the centre of G/C. By Hall's theorem A is the normal closure in G of finitely many elements $a_1, \dots, a_n$, say. Denote $c_G(a_i)$ by C_i. The C_i contains C and is normalized by Z, since $[Z, G] \leq C$.

Hence $[a_i^Z, C_i] = 1$. But C_i is f.g. abelian by nilpotent and so by the usual argument its centre and hence a_i^Z is a f.g. abelian group and therefore is finite since it is a p-group. It follows that the normal closure B of $\langle a_1, \dots, a_n \rangle$ under Z is finite. Furthermore, $C \leq c_Z(B) \leq Z$, so that $D = c_Z(B)$ is normal in G.

However, $1 = [B, D] = [B, D]^G = [A, D]$ so that $D = C$.

Thus $Z/C = Z/D$ embeds in $\operatorname{Aut} B$, which is a finite group. Therefore Z/C is finite. But G/C is a f.g. nilpotent group and hence is finite. Thus C is f.g. and abelian by nilpotent. By a final application of Hall's result, the centre of C is f.g. Hence A is f.g. and so G is polycyclic, a contradiction.

REFERENCES

[1] J. GROVES, *Soluble groups with every proper quotient polycyclic*, Illinois J. Math., **22** (1978), pp. 90-95.

[2] P. HALL, *Finiteness conditions for soluble groups*, Proc. London Math. Soc., (3) **4** (1954), pp. 419-436.

[3] J. C. LENNOX, *Centrality in soluble groups*, Proceedings «Infinite Groups 1994», de Gruyter, Berlin, 1996.

[4] A. H. RHEMTULLA, *A minimality property of polycyclic groups*, J. London Math. Soc., **42** (1967), pp. 456-462.

[5] D. J. S. ROBINSON, *A property of the lower central series of a group*, Math. Z., **107** (1968), pp. 225-231.

[6] D. J. S. ROBINSON - J. S. WILSON, *Ssoluble groups with many polycyclic quotients*, Proc. London Math. Soc., (3) **48** (1984), pp. 193-229.

Manoscritto pervenuto in redazione il 23 febbraio 1995.