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Regular time-optimal syntheses for smooth planar systems

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Regular Time-Optimal Syntheses for Smooth Planar Systems.

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ABSTRACT - The paper is concerned with the time optimal stabilizing problem for the control system

$$\dot{x} = F(x) + G(x)u, \qquad |u(t)| \leq 1 .$$

We show that under generic assumptions on F, G in $\mathcal{C}^3$, there exists a regular synthesis and all time-optimal trajectories are concatenations of a finite number of smooth arcs.

1. - Introduction.

This paper is concerned with the standard problem of reaching the origin in minimum time, for the control system

$$\dot{x} = F(x) + G(x)u, \qquad u(t) \in [-1, 1] \quad \text{a.e.}, \tag{1.1}$$

where F, G are $\mathcal{C}^3$ vector fields on the plane, with $F(0) = 0$. Calling $A(\tau)$ the set of points which can be steered to the origin within a fixed time τ, by a regular optimal feedback synthesis we mean a partition of $A(\tau)$ into finitely many embedded manifolds $\mathcal{M}_i$ and a feedback control law $u = u(x)$, whose restriction to each $\mathcal{M}_i$ is smooth, such that every Carathéodory trajectory of the (usually discontinuous) O.D.E.

$$\dot{x} = F(x) + G(x)u(x)$$

starting within $A(\tau)$ reaches the origin in minimum time. In general, the controllable set $A(\tau)$ will thus be divided into finitely many open re-

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gions where $u(x) \equiv \pm 1$, separated either by switching curves, or by singular arcs, or else by «overlap curves», consisting of points which can be steered to the origin optimally by two distinct control functions.

In the case of analytic vector fields, the existence of a regular optimal feedback was established in [11, 12]. Aim of this paper is to prove that the result remains valid for generic vector fields $F, G \in \mathcal{C}^3$. More precisely, using the Pontryagin Maximum Principle, we will single out a set of generic assumption on $F, G \in \mathcal{C}^3$, which imply that all optimal trajectories are a finite concatenations of integral curves of the three flows

$$(1.2) \quad \dot{x} = F(x) + G(x), \qquad \dot{x} = F(x) - G(x), \qquad \dot{x} = F(x) + G(x)\varphi_S(x).$$

Here the feedback function $u = \varphi_S(x)$ is defined at (3.18), in terms of F, G and of the Lie bracket $[F, G]$. A uniform bound on the number of arcs forming these trajectories is derived. Relying on this «finite dimensional reduction» of the time-optimal problem, we prove the existence of a time optimal synthesis, valid for generic vector fields $F, G \in \mathcal{C}^3$.

This work represents the first step of a research program aimed at the classification of generic planar time optimal feedbacks, under a topological equivalence relation, in analogy with the well established theory valid for smooth O.D.E. in the plane [7, 8]. The classification of local singularities will be done in the forthcoming paper.

2. – Basic definitions.

If $x \in \mathbb{R}^n$ and $r \geq 0$, by $B(x, r)$, $\overline{B}(x, r)$ we denote respectively the open and the closed ball centered at x with radius r. Given a set $C \subset \mathbb{R}^n$ we write $\mathrm{Int}(C)$, $\mathrm{Cl}(C)$ and $\mathrm{Fr}(C)$ for the interior, the closure and the topological frontier of C. If C is a submanifold with boundary, we will use the symbol ∂C to denote the boundary of C.

A *curve* in $\mathbb{R}^n$ is a continuous map $\gamma: I \mapsto \mathbb{R}^n$, where I is some real interval. Its domain is thus $\mathrm{Dom}(\gamma) = I$. If $x \in \{\gamma(t): t \in \mathrm{Dom}(\gamma)\}$, we simply write $x \in \gamma$. The symbol $\gamma \restriction J$, where $J \subset \mathrm{Dom}(\gamma)$ is an interval, denotes the restriction of γ to J. Similarly, if Σ is a control system defined in $\mathbb{R}^2$ and $U \subset \mathbb{R}^2$ is an open set then $\Sigma \restriction U$ is the restriction of Σ to U.

As usual, the space of vector fields $F = (F_1, F_2)$ on $\mathbb{R}^2$ whose partial derivatives of order $\leqslant 3$ are bounded on $\mathbb{R}^2$, will be endowed with the norm:

$$\|F\|_{\mathcal{C}^3} = \sup\{D^\alpha F_i(x) \colon x \in \mathbb{R}^2,\ 0 \leqslant \alpha \leqslant 3,\ i = 1, 2\}$$

where $D^\alpha = D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2}$ is a differential operator of order $|\alpha| = \alpha_1 + \alpha_2$.

A vector field F on $\mathbb{R}^2$ can be written in the form:

$$F = \alpha \partial_{x_1} + \beta \partial_{x_2} \tag{2.1}$$

where ∂_{x_1}, ∂_{x_2} are the constant vector fields with components $(1, 0)$, $(0, 1)$, respectively. By ∇F we denote the Jacobian matrix of first order partial derivatives of the vector field F in (2.1). We write $e^{tF}(\bar{x})$ for the value at time t of the solution of the Cauchy problem:

$$\dot{x} = F(x), \qquad x(0) = \bar{x}, \tag{2.2}$$

while $(e^{tF})_*$ will denote the Jacobian matrix of the map

$$x \mapsto e^{tF}(x). \tag{2.3}$$

We recall that the *Lie bracket* of two vector fields F, G is the vector field

$$[F, G] \doteq \nabla G \cdot F - \nabla F \cdot G \tag{2.4}$$

We use the symbol Ξ to denote the subspace of all pairs of vector fields on the plane $(F, G) \in \mathcal{C}^3 \times \mathcal{C}^3$ with $F(0) = 0 \in \mathbb{R}^2$. From now on, we fix a pair $(F, G) \in \Xi$ and consider the control system:

$$\dot{x} = F(x) + uG(x), \qquad |u| \leqslant 1. \tag{2.5}$$

We shall usually write $\Sigma = (F, G)$ to indicate the control system (2.5).

A *control* is a measurable function $u \colon [a, b] \mapsto [-1, 1]$ where $-\infty < a \leqslant b < +\infty$. A *trajectory of* Σ corresponding to u is an absolutely continuous curve $\gamma \colon \mathrm{Dom}(u) \mapsto \mathbb{R}^2$ which satisfies the equation:

$$\dot{\gamma}(t) = F(\gamma(t)) + u(t)\, G(\gamma(t)) \tag{2.6}$$

for almost every t in the domain of u. The set of all traiectories of Σ is denoted by $\mathrm{Traj}(\Sigma)$. If $\gamma \colon [a, b] \mapsto \mathbb{R}^2$ is a trajectory of Σ we define the initial and terminal points of γ as $\mathrm{In}(\gamma) \doteq \gamma(a)$ and $\mathrm{Term}(\gamma) \doteq \gamma(b)$. The time along γ is defined as

$$T(\gamma) = b - a. \tag{2.7}$$

A trajectory $\gamma \in \mathrm{Traj}(\Sigma)$ is *time optimal* if, for every trajectory $\gamma' \in \mathrm{Traj}(\Sigma)$ with $\mathrm{In}(\gamma') = \mathrm{In}(\gamma)$ and $\mathrm{Term}(\gamma') = \mathrm{Term}(\gamma)$, one has $T(\gamma') \geqslant T(\gamma)$.

If $u_1 : [a, b] \mapsto [-1, 1]$ and $u_2 : [b, c] \mapsto [-1, 1]$ are controls, their *concatenation* $u_2 * u_1$ is the control:

$$(u_2 * u_1)(t) \doteq \begin{cases} u_1(t) & \text{for } t \in [a, b], \\ u_2(t) & \text{for } t \in (b, c]. \end{cases}$$

If $\gamma_1 : [a, b] \mapsto \mathbb{R}^2$, $\gamma_2 : [b, c] \mapsto \mathbb{R}^2$ are trajectories of Σ for u_1 and u_2 such that $\gamma_1(b) = \gamma_2(b)$, then the *concatenation* $\gamma_2 * \gamma_1$ is the trajectory:

$$(\gamma_2 * \gamma_1)(t) = \begin{cases} \gamma_1(t) & \text{for } t \in [a, b], \\ \gamma_2(t) & \text{for } t \in [b, c]. \end{cases}$$

For convenience, we also define the vector fields

$$X = F - G, \qquad Y = F + G. \tag{2.8}$$

We use $\mathrm{Traj}(X)$ [$\mathrm{Traj}(Y)$] to denote the set of all trajectories of Σ which correspond to the constant control $u \equiv -1$ [$u \equiv 1$]. Elements of $\mathrm{Traj}(X)$, $\mathrm{Traj}(Y)$ will be called *X-trajectories* and *Y-trajectories*, respectively. A *bang-bang trajectory* is a trajectory obtained as a finite concatenation of X- and Y-trajectories. We write $\mathrm{Traj}(\xi_1 * \ldots * \xi_n)$, where $\xi_i = X$ or $\xi_i = Y$, to denote the set of all concatenations $\gamma = \gamma_1 * \ldots * \gamma_n$ where $\gamma_i \in \mathrm{Traj}(\xi_i)$ and is not trivial, i.e. its domain is not a single point. We also say that γ is of type $\xi_1 * \ldots * \xi_n$ (see [10] for complete description of this notation).

Instead of steering the system to the origin in minimum time, throughout the following we shall consider the entirely equivalent problem of reaching points in $\mathbb{R}^2$ in minimum time, starting from the origin. If $\tau \geqslant 0$, we denote by $R(\tau)$ the *reachable set* within time τ:

$$R(\tau) = \{x : \exists \gamma \in \mathrm{Traj}(\Sigma) \text{ s.t. } \gamma(0) = 0 \in \mathbb{R}^2,\ \gamma(t) = x, \text{ for some } t \leqslant \tau\}. \tag{2.9}$$

The *minimum time function*, T: $\mathbb{R}^2 \mapsto [0, +\infty]$ is defined by

$$T(x) \doteq \inf\{\tau : x \in R(\tau)\}. \tag{2.10}$$

Recalling (2.9) we have:

$$T(x) = \inf\{T(\gamma) : \gamma \in \mathrm{Traj}(\Sigma),\ \mathrm{In}(\gamma) = 0,\ \mathrm{Term}(\gamma) = x\}.$$

Clearly, a trajectory $\gamma \in \mathrm{Traj}(\Sigma)$, with $\mathrm{Dom}(\gamma) = [0, b]$, $\mathrm{In}(\gamma) = 0$, is optimal if and only if $T(\gamma(b)) = b$. In this case we write $\gamma \in \mathrm{Opt}(\Sigma)$.

The convexity of the set $\{F(x) + uG(x)\colon |u| \leqslant 1\}$ and the bound on the derivatives of F and G imply the following:

LEMMA 2.1. *If $0 \leqslant \tau < +\infty$ then the set $R(\tau)$ is compact. For any $x \in \mathrm{R}^2$, if $T(x) = \tau$ then there exists $\gamma \in \mathrm{Traj}(\Sigma)$ such that $\gamma(0) = 0$, $\gamma(\tau) = x$.*

For the proof see [5] Th. 20.1 p. 107.

The control system Σ is *locally controllable* if, for each $\tau > 0$, the set $R(\tau)$ contains a neighborhood of the origin. The following results are well known [6, p. 366]:

LEMMA 2.2. *If the system Σ is locally controllable then the minimum time function is continuous and, for every $\tau > \sigma > 0$, one has*

$$R(\sigma) \subseteq \mathrm{Int}(R(\tau)). \tag{2.11}$$

LEMMA 2.3. *If $F(0) = 0$ and the vector fields G, $[F, G]$ are linearly independent at the origin, then the system $\Sigma = (F, G)$ in (2.5) is locally controllable.*

A *synthesis* for the control system Σ at time τ is a family $\Gamma = \{\gamma_x\colon [0, b_x] \mapsto \mathrm{R}^2 \,|\, x \in R(\tau)\}$ of trajectories satisfying the following conditions:

a) For each $x \in R(\tau)$ one has $\gamma_x(0) = 0$, $\gamma_x(b_x) = x$.

b) If $y = \gamma_x(t)$ where $t \in \mathrm{Dom}(\gamma_x)$ then $\gamma_y = \gamma_x \restriction [0, t]$.

A synthesis for the system Σ is *time optimal* if, for each $x \in R(\tau)$, one has $\gamma_x(T(x)) = x$, where T is the minimum time function defined at (2.10).

3. - Pontryagin maximum principle and special curves.

An *admissible pair* for the system Σ is a couple (u, γ) such that u is a control and γ is a trajectory corresponding to u. We use the symbol $\mathrm{Adm}(\Sigma)$ to denote the set of admissible pairs and we say that $(u, \gamma) \in \mathrm{Adm}(\Sigma)$ is optimal if γ is optimal.

A *variational vector field along* $(u, \gamma) \in \mathrm{Adm}(\Sigma)$ is a vector-valued absolutely continuous function v: $\mathrm{Dom}(\gamma) \mapsto \mathbb{R}^2$ that satisfies the equation:

$$\dot{v}(t) = ((\nabla F)(\gamma(t)) + u(t)(\nabla G)(\gamma(t))) \cdot v(t) \tag{3.1}$$

for almost all $t \in \mathrm{Dom}(\gamma)$.

A *variational covector field along* $(u, \gamma) \in \mathrm{Adm}(\Sigma)$ is an absolutely continuous function λ: $\mathrm{Dom}(\gamma) \mapsto \mathbb{R}^2_*$ that satisfies the equation:

$$\dot{\lambda}(t) = -\lambda(t) \cdot ((\nabla F)(\gamma(t)) + u(t)(\nabla G)(\gamma(t))) \tag{3.2}$$

for almost all $t \in \mathrm{Dom}(\gamma)$. Here $\mathbb{R}^2_*$ denotes the space of row vectors.

The Hamiltonian $\mathcal{H}$: $\mathbb{R}^2_* \times \mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{R}$ is defined as

$$\mathcal{H}(\lambda, x, u) = \lambda \cdot (F(x) + uG(x)) \,. \tag{3.3}$$

If λ is a variational covector field along $(u, \gamma) \in \mathrm{Adm}(\Sigma)$, we say that λ is *maximizing* if:

$$\mathcal{H}(\lambda(t), \gamma(t), u(t)) = \max\{\mathcal{H}(\lambda(t), \gamma(t), w) \colon |w| \leq 1\} \tag{3.4}$$

for almost all $t \in \mathrm{Dom}(\gamma)$.

The *Pontryagin Maximum Principle* (PMP) states that, if $(u, \gamma) \in \mathrm{Adm}(\Sigma)$ is time optimal, then there exists:

(PMP1) *A non trivial maximizing variational covector field* λ *along* (u, γ),

(PMP2) *A constant* $\lambda_0 \leq 0$ *such that*: $\mathcal{H}(\lambda(t), \gamma(t), u(t)) + \lambda_0 = 0$ *for almost all* $t \in \mathrm{Dom}(\gamma)$.

In this case λ is called an *adjoint covector field along* (u, γ) or simply an *adjoint variable*, and we say that (γ, λ) satisfies the PMP, or that γ is an *extremal trajectory*.

If λ is an adjoint covector field along $(u, \gamma) \in \mathrm{Adm}(\Sigma)$, the corresponding *switching function* is defined as

$$\phi_\lambda(t) = \lambda(t) \cdot G(\gamma(t)) \,. \tag{3.5}$$

A time $t \in \mathrm{Dom}(\gamma)$ is called a *switching time* for γ if, for each $\varepsilon > 0$, $\gamma \restriction [t-\varepsilon, t+\varepsilon]$ is neither an X-trajectory nor a Y-trajectory. If t is a switching time for γ then we say that $\gamma(t)$ is a *switching point for* γ, or that γ has a switching at $\gamma(t)$.

LEMMA 3.1. *If* $(u, \gamma) \in \mathrm{Adm}(\Sigma)$ *is extremal and* λ *is an adjoint covector field along* (u, γ) *then* $\phi_\lambda(t) = 0$ *at every switching time* t.

Consider $(u, \gamma) \in \mathrm{Adm}(\Sigma)$, $t_0 \in \mathrm{Dom}(\gamma)$ and $v_0 \in \mathbb{R}^2$. We write $v(v_0, t_0; t)$ to denote the value at time t of the variational vector field along (u, γ) satisfying (3.1) together with the boundary condition $v(t_0) = v_0$. If $t_0, t_1 \in \mathrm{Dom}(\gamma)$ we say that t_0 and t_1 are *conjugate* along γ if the vectors $v(G(\gamma(t_1)), t_1; t_0)$ and $G(\gamma(t_0))$ are linearly dependent.

Let D, D' be two $\mathcal{C}^2$ connected one-dimensional embedded submanifolds of $\mathbb{R}^2$. We say that D' is a *conjugate curve* to D along the X-trajectories if there is a bijective function $\psi\colon D \mapsto D'$ with the following properties. If γ_x is the X-trajectory satisfying $\gamma_x(0) = x$, then $\psi(x) = \gamma_x(t(x))$ for some time t depending continuously on x, and the times 0, $t(x)$ are conjugate along γ_x. Conjugate curves along the Y-trajectories are defined similarly.

LEMMA 3.2. *If $\gamma \in \mathrm{Traj}(\Sigma)$ is extremal and t_0, t_1 are switching times for γ, then t_0 and t_1 are conjugate along γ.*

For the proof of this lemma see [10]. A straightforward computation yields:

LEMMA 3.3. *If $(u, \gamma) \in \mathrm{Adm}(\Sigma)$ is extremal and λ is an adjoint covector field along (u, γ), then the switching function ϕ_λ is $\mathcal{C}^1$ and its derivative is given by*:

$$\dot{\phi}_\lambda(t) = \lambda(t) \cdot [F, G](\gamma(t)) . \tag{3.6}$$

For each $x \in \mathbb{R}^2$, one can form the 2×2 matrices whose columns are the vectors F, G, or $[F, G]$. As in [10], we shall use the following scalar functions on $\mathbb{R}^2$:

$$\Delta_A(x) \doteq \det(F(x), G(x)) = F(x) \wedge G(x) , \tag{3.7}$$

$$\Delta_B(x) \doteq \det(G(x), [F, G](x)) = G(x) \wedge [F, G](x) , \tag{3.8}$$

where det stands for determinant and $\wedge$ denotes an exterior product. A point $x \in \mathbb{R}^2$ is called an *ordinary point* if

$$\Delta_A(x) \cdot \Delta_B(x) \neq 0 . \tag{3.9}$$

On the set of ordinary points we define the scalar functions f, g as the coefficients of the linear combination

$$[F, G](x) = f(x) F(x) + g(x) G(x) . \tag{3.10}$$

In [10, p. 447] it was shown that

$$f(x) = -\frac{\Delta_B(x)}{\Delta_A(x)}. \tag{3.11}$$

In the following, given two nonzero vectors $v, v' \in \mathbb{R}^2$, by $\arg(v, v') \in [-\pi, \pi]$ we denote the angle between them, oriented from v to v'. If v_0 is a constant vector and $v(t) \neq 0$, one has

$$\frac{d}{dt}\{\arg(v_0, v(t))\} = \frac{v(t) \wedge \dot{v}(t)}{\|v(t)\|^2}. \tag{3.12}$$

LEMMA 3.4. *Let $(u, \gamma) \in \mathrm{Adm}(\Sigma)$, $t_0 \in \mathrm{Dom}(\gamma)$. For every t such that $G(\gamma(t)) \neq 0$, define the angle*

$$\alpha(t) = \arg(G(\gamma(t_0)), v((G(\gamma(t)), t; t_0)), \tag{3.13}$$

Then, one has

$$\operatorname{sgn}(\dot{\alpha}(t)) = \operatorname{sgn}(\Delta_B(\gamma(t))). \tag{3.14}$$

Indeed, for any t at which $G(\gamma(t))$ does not vanish, one has

$$\begin{aligned} \frac{d}{dt} v(G(\gamma(t)), t; t_0) &= \\ &= \lim_{\varepsilon \to 0} \frac{v(G(\gamma(t+\varepsilon)), t+\varepsilon; t_0) - v(G(\gamma(t)), t; t_0)}{\varepsilon} = \\ &= M(t_0, t) \cdot \lim_{\varepsilon \to 0} \frac{v(G(\gamma(t+\varepsilon)), t+\varepsilon; t) - G(\gamma(t))}{\varepsilon} = \\ &= M(t_0, t) \cdot [F + u(t)G, G](\gamma(t)) = M(t_0, t) \cdot [F, G](\gamma(t)), \end{aligned} \tag{3.15}$$

where the matrix $M(t_0, t)$ is defined by

$$M(t_0, t)\, w = v(w, t; t_0). \tag{3.16}$$

Since $M(t_0, t)$ preserves orientation, by (3.12) we have

$$\begin{aligned} \operatorname{sgn}(\dot{\alpha}(t)) &= \operatorname{sgn}(M(t_0, t)\, G(\gamma(t)) \wedge M(t_0, t)[F, G](\gamma(t))) = \\ &= \operatorname{sgn}(G(\gamma(t)) \wedge [F, G](\gamma(t))), \end{aligned}$$

proving (3.14).

THEOREM 3.5. *Let $U \subset \mathbb{R}^2$ be an open set such that each $x \in U$ is an* *ordinary point. Then all optimal trajectories γ of $\Sigma \restriction U$ are bang-bang with at most one switching. Moreover if $f > 0$ throughout U then γ is an X-, Y- or $Y * X$-trajectory, if $f < 0$ throughout U then γ is an X-, Y- or $X * Y$-trajectory.*

For the proof see [10] Theorem 3.9 p. 443.

A point x at which $\Delta_A(x)\Delta_B(x) = 0$ is called a *nonordinary point.* A *nonordinary arc* is a $\mathcal{C}^2$ one-dimensional connected embedded submanifold S of $\mathbb{R}^2$, with the property that every $x \in S$ is a nonordinary point. A nonordinary arc will be said *isolated* if there exists a set U satisfying the following conditions:

(C1) U is an open connected subset of $\mathbb{R}^2$.

(C2) S is a relatively closed subset of U.

(C3) If $x \in U \setminus S$ then x is an ordinary point.

(C4) The set $U \setminus S$ has exactly two connected components.

An *open turnpike* is an isolated nonordinary arc that satisfies the following conditions:

(S1) For each $x \in S$ the vectors $X(x)$ and $Y(x)$ are not tangent to S and point to opposite sides of S.

(S2) For each $x \in S$ one has $\Delta_B(x) = 0$ and $\Delta_A(x) \neq 0$.

(S3) Let U be an open set which satisfies (C1-4) above. If U_X and U_Y are the connected components of $U \setminus S$ labelled in such a way that $X(x)$ points into U_X and $Y(x)$ points into U_Y, then the function f in (3.10) satisfies

$$f(x) > 0 \quad \text{on } U_Y \qquad f(x) < 0 \quad \text{on } U_X .$$

A $\mathcal{C}^2$ one-dimensional connected, embedded submanifold with boundary $S \subset \mathbb{R}^2$ is a *turnpike* if $S \setminus \partial S$ is an open turnpike. Next, consider a turnpike S and a point $x_0 \in S$. We wish to construct a trajectory $\gamma \in \mathrm{Traj}(\Sigma)$ such that $\gamma(t_0) = x_0$ and $\gamma(t) \in S$ for each $t \in \mathrm{Dom}(\gamma) \doteq [t_0, t_1]$. Clearly, one should have $\Delta_B(\gamma(t)) \equiv 0$ for all t. Since $\Delta_B(\gamma(t_0)) = 0$, it suffices to verify that:

$$\frac{d}{dt}\Delta_B(\gamma(t)) = (\nabla\Delta_B \cdot \dot{\gamma})(t) = 0 .$$

The above holds provided that

$$(\nabla\Delta_B \cdot uG)(\gamma(t)) + (\nabla\Delta_B \cdot F)(\gamma(t)) = 0 .$$

From (S1) we have that:

$$(\nabla\Delta_B \cdot G)(x) \neq 0 \qquad \forall x \in S, \tag{3.17}$$

the values of the control u are thus uniquely determined by

$$u = \varphi_S(x) \doteq -\frac{\nabla\Delta_B \cdot F(x)}{\nabla\Delta_B \cdot G(x)}. \tag{3.18}$$

The turnpike S is said to be *regular* if the function φ_S in (3.18) satisfies:

$$|\varphi_S(x)| < 1 \qquad \forall x \in S \setminus \partial S. \tag{3.19}$$

A curve $\gamma \in \mathrm{Traj}(\Sigma)$ is said to be a *Z-trajectory* if there exists a regular turnpike S such that $\{\gamma(t)\colon t \in \mathrm{Dom}(\gamma)\} \subset S$, in this case we write $\gamma \in \mathrm{Traj}(Z)$.

A an isolated nonordinary arc (or INOA) S is said to be of the *turnpike type* if it verifies (S1), (S3) and:

(S2′) Each of the function Δ_A, Δ_B is either identically zero on S or nowhere zero on S

then every turnpike is of the turnpike type but not viceversa.

A an isolated nonordinary arc (or INOA) is said to be of the *antiturnpike type* if verifies (S1), (S2′) and the following condition:

(S3′) Let U be an open set which satisfies (C1-4) above. If U_X and U_Y are the connected components of $U \setminus S$ labelled in such a way that $X(x)$ points into U_X and $Y(x)$ points into U_Y, then the function f in (3.10) satisfies

$$f(x) < 0 \quad \text{on } U_Y \qquad f(x) > 0 \quad \text{on } U_X.$$

We say that S is a *nondegenerate* INOA of the antiturnpike type if it verifies:

(SN) If $\Delta_B \equiv 0$ on S then either $X \cdot \nabla\Delta_B$ or $Y \cdot \nabla\Delta_B$ never vanish on S.

A point $x \in \mathbb{R}^2$ is a *near-ordinary point* if it is an ordinary point or belongs to an INOA that is either of the turnpike type or nondegenerate of the antiturnpike type. In [10, p. 459] it was proved the following:

THEOREM 3.6. *Let x be a near-ordinary point. Then there exists a neighborhood U of x such that every optimal trajectory γ of*

$\Sigma \restriction U$ is concatenation of at most five trajectories each of which is an X-, Y- or Z-trajectory.

It can happen that $R(\tau)$ contains curves whose points can be reached in minimum time using different optimal controls. An *overlap curve* is a $\mathcal{C}^2$ one-dimensional connected embedded submanifold K of $\mathbb{R}^2$, with the property that for each point of K there exist two distinct time optimal trajectories $\gamma_1, \gamma_2: [0, b] \mapsto \mathbb{R}^2$, and $\varepsilon > 0$ such that:

$$\gamma_1(0) = \gamma_2(0) = 0, \qquad \gamma_1(b) = \gamma_2(b) = x, \qquad T(x) = b,$$

and $\gamma_1 \restriction [b - \varepsilon, b]$ is an X-trajectory, while $\gamma_2 \restriction [b - \varepsilon, b]$ is a Y-trajectory.

4.– Bounds on the number of arcs.

The aim of this section is to prove, given $\tau > 0$, the existence of generic conditions on F, G ensuring that every time optimal trajectory in $R(\tau)$ is a finite concatenation of X-, Y- and Z-trajectories; more precisely for each Σ in a generic subset of Ξ there exists $N(\Sigma)$ that bounds the number of these trajectories.

Given a trajectory $\gamma \in \mathrm{Traj}(\Sigma)$ we denote by $n(\gamma)$ the smallest integer such that there exist $\gamma_i \in \mathrm{Traj}(X) \cup \mathrm{Traj}(Y) \cup \mathrm{Traj}(Z)$, $i = 1, \ldots, n(\gamma)$ verifying:

$$\gamma = \gamma_{n(\gamma)} * \cdots * \gamma_1 .$$

We call $n(\gamma)$ the number of arcs of γ.

Given $\tau > 0$ let define Π_τ to be the class of systems having an a priori bound on the number of arcs of optimal trajectories:

$$\Pi_\tau = \{\Sigma \in \Xi: \exists N(\Sigma) \text{ s.t. } \forall \gamma \in \mathrm{Opt}(\Sigma \restriction R(\tau)),\ n(\gamma) \leq N(\Sigma)\}.$$

A subset of Ξ is said to be *generic* if it contains an open and dense subset of Ξ. A *condition* for $\Sigma = (F, G) \in \Xi$ is a logic proposition involving the components of the vector fields (F, G), their derivatives or set and functions that can be defined using them. Given a condition P for $\Sigma \in \Xi$ we write $P(\Sigma)$ if the system satisfies the condition P. A condition P is said to be *generic* if $\{\Sigma \in \Xi: P(\Sigma)\}$ is generic. If $P_1, \ldots, P_n$ are generic conditions then it is easy to verify that $\{\Sigma \in \Xi: P_1(\Sigma), \ldots, P_n(\Sigma)\}$ is generic.

We now give a finite number of generic conditions $P_1, \ldots, P_n$ that assure the genericity of Π_τ, this means:

$$\{\Sigma \in \Xi \colon P_1(\Sigma), \ldots, P_n(\Sigma)\} \subset \Pi_\tau . \tag{4.1}$$

From now on we consider a fixed time $\tau > 0$ and a fixed system $\Sigma = (F, G) \in \Xi$ and we describe the conditions of (4.1). The first condition is:

(P_1) $F(0)$ and $[F, G](0)$ are linearly independent.

From (P_1) and Lemma 2.3 it follows that the system Σ is locally controllable. The second condition is:

(P_2) Zero is a regular value for the functions Δ_A and Δ_B

this means:

$$\forall x \in \mathbb{R}^2 \qquad \Delta_A(x) = 0 \Rightarrow \nabla \Delta_A(x) \neq 0 \tag{4.2}$$

and the similar condition with Δ_B rather than Δ_A in (4.2). From (P_2) we have that the sets $\Delta_A^{\leftarrow}(0) = \{x \in R(\tau) \colon \Delta_A(x) = 0\}$ and $\Delta_B^{\leftarrow}(0) = \{x \in R(\tau) \colon \Delta_B(x) = 0\}$ are $\mathcal{C}^2$ one-dimensional compact embedded submanifold of $\mathbb{R}^2$. So we can give the following generic condition:

(P_3) The set $\Delta_A^{\leftarrow}(0) \cap \Delta_B^{\leftarrow}(0)$ is finite.

Let Tan_A be the set of points $x \in \Delta_A^{\leftarrow}(0)$ such that $X(x)$ or $Y(x)$ is tangent to $\Delta_A^{\leftarrow}(0)$. Define Tan_B in the same way using Δ_B rather than Δ_A.

(P_4) Tan_A and Tan_B are finite sets.

We will call *bad points* the elements of the set:

$$\mathrm{Bad}(\tau) = (\Delta_A^{\leftarrow}(0) \cap \Delta_B^{\leftarrow}(0)) \cup (\mathrm{Tan}_A \cup \mathrm{Tan}_B). \tag{4.3}$$

It is easy to verify that if $x \in R(\tau) \setminus Bad(\tau)$ then x is a near-ordinary point. From (P_3), (P_4) we obtain:

(P_5) $\mathrm{Bad}(\tau)$ is finite.

From Theorems 3.5 and 3.6 we know the structure of time opimal trajectories in a neighborhood of a near ordinary point; it remains to consider the case of bad points.

LEMMA 4.1. *If* $x \in \mathrm{Bad}(\tau)$, $G(x) \neq 0$ *then* $x \in (\Delta_A^{\leftarrow}(0) \cap \Delta_B^{\leftarrow}(0))$ *if and only if* $x \in \mathrm{Tan}_A$.

PROOF. Being $G(x) \neq 0$ we can choose a local system of coordinates such that $G = (1, 0)$, then we have $\alpha F(x) = G(x)$ $(\alpha \in \mathbb{R})$ and:

$$\nabla(\Delta_A)(x) = \nabla(F_1 G_2 - G_1 F_2)(x) = -\nabla F_2(x),$$

$$[F, G](x) = -\nabla F \cdot G.$$

Assume, first, that $x \in \Delta_A^{\leftarrow}(0) \cap \Delta_B^{\leftarrow}(0)$. From $\Delta_B(x) = 0$ we have $[F, G](x)$, $G(x)$ are collinear and then:

$$0 = (\partial_1 F_2)(x) G_1(x) + (\partial_2 F_2)(x) G_2(x) = (\partial_1 F_2)(x)$$

finally:

$$\nabla(\Delta_A)(x) \cdot G(x) = 0.$$

We conclude that $x \in \mathrm{Tan}_A$.

In the same way if $x \in \mathrm{Tan}_A$ then $\Delta_B(x) = 0$.

We can now prove the following:

THEOREM 4.2. *Under generic conditions on $F, G \in \mathcal{C}^3$, for every $x \in \mathrm{Bad}(\tau)$ there exist U_x, neighborhood of x, and $N_x \in \mathbb{N}$ such that if $\gamma \in \mathrm{Opt}(\Sigma)$ and $\{\gamma(t)\colon t \in [b_0, b_1]\} \subset U_x$ then:*

$$n(\gamma \restriction [b_0, b_1]) \leq N_x.$$

PROOF. Consider x and γ satisfying the assumptions above. For sake of simplicity in the proof we write γ instead of $\gamma \restriction [b_0, b_1]$. We have three cases:

(1) $G(x) = 0$,

(2) $G(x) \neq 0$, $x \in (\Delta_A^{\leftarrow}(0) \cap \Delta_B^{\leftarrow}(0)) \cap \mathrm{Tan}_A$,

(3) $x \in \mathrm{Tan}_B$.

Case (1). We suppose that:

(P_6) $F(x) \cdot \nabla(\Delta_A)(x) \neq 0$, $F(x) \cdot \nabla(\Delta_B)(x) \neq 0$.

Take U_x open connected such that x is the only bad point in U_x and $U_x - (\Delta_A^{\leftarrow}(0) \cap \Delta_B^{\leftarrow}(0))$ has four connected components $U_1, \ldots, U_4$. Assume that, say, $F(x)$ points into U_1 and $-F(x)$ into U_2. See Fig. 1. Then it is clear that for U_x sufficiently small the same happens for $X(y)$, $Y(y)$ $y \in U_x$. Following γ, we can move from U_2 into any other component, while from U_3 or U_4 we can move into U_1; there are no other possibilities. From Theorem 3.6 we have that for each $\mathrm{Cl}(U_i)$ $(i = 1, \ldots, 4)$, $n(\gamma \restriction \{t\colon \gamma(t) \in \mathrm{Cl}(U_i)\}) \leq 5$ therefore the conclusion holds, with $N_x = 15$.

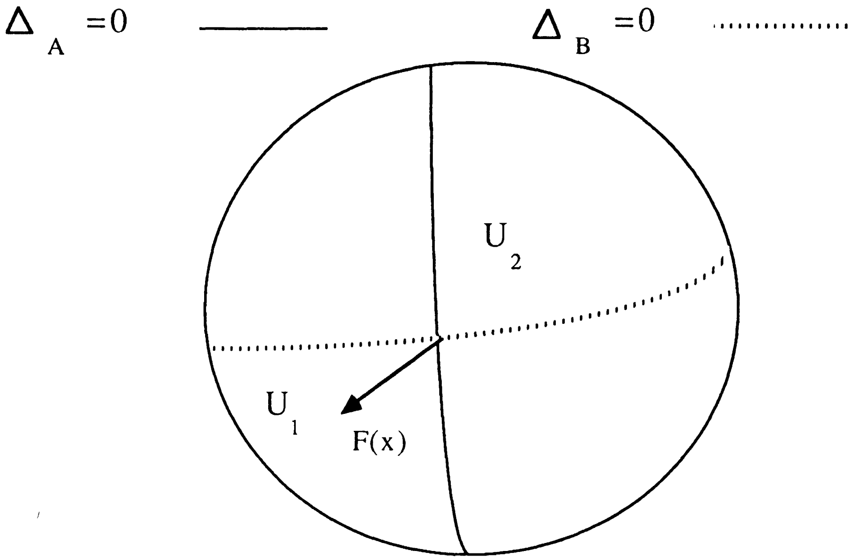

Figure 1.

Case (2). Let y range on $\Delta_A^{\leftarrow}(0)$ and assume that:

$$(\mathrm{P}_7) \quad \frac{\partial}{\partial y}(X \cdot \nabla \Delta_A)\big|_{y=x} \neq 0, \qquad X(x) \neq 0, \ \ Y(x) \neq 0 .$$

If X and Y have the same orientation, then we can proceed as in case (1). In the following, we thus study the case where they have opposite orientations.

Take U_x open connected such that x is the only bad point in U_x and $U_x \setminus (\Delta_A^{\leftarrow}(0) \cap \Delta_B^{\leftarrow}(0))$ has four connected components $U_1, \ldots, U_4$. Let A_1 be the connected component of $(\Delta_A^{\leftarrow}(0) \cap U_x) \setminus \{x\}$ that comes before x in the orientation of $X(x)$ and A_2 the other component. We label $U_1, \ldots, U_4$ in such a way that, see Fig. 2:

$$X(y) \text{ points into } U_1, \ Y(y) \text{ points into } U_2 \text{ for } y \in A_1,$$
$$X(y) \text{ points into } U_3, \ Y(y) \text{ points into } U_4 \text{ for } y \in A_2.$$

Choosing U_x smaller if necessary, we can assume that $X \neq 0 \neq Y$ on U_x. Moreover, we can assume that $X(y)$ points into U_3 for each $y \in \mathrm{Fr}(U_3) \cap U_x$, $Y(y)$ points into U_2 for each $y \in \mathrm{Fr}(U_2) \cap U_x$ and $X(y)$, resp. $Y(y)$, points into $U_3 \cup U_4$, resp. $U_1 \cup U_2$, for every $y \in \Delta_B^{\leftarrow}(0) \cap U_x$.

Suppose first that $f < 0$ (see (3.11)) on U_1 then $f > 0$ on U_2. From

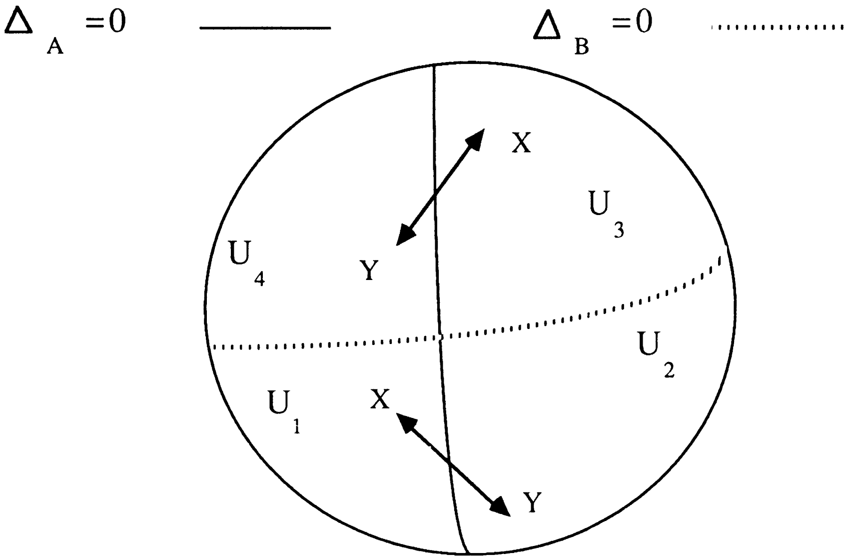

Figure 2.

Theorem 3.5 it follows that if γ enters U_3 then it remains in U_3 and the same happens for U_2.

The set $B_1 \doteq \Delta_B^{\leftarrow}(0) \cap \mathrm{Fr}(U_1)$ is an INOA of the antiturnpike type and there exists k, depending on U_x, such that:

$$(4.4) \quad |(X\Delta_B)(y)| \geqslant k > 0\,, \qquad |(Y\Delta_B)(y)| \geqslant k > 0 \qquad \forall y \in B_1\,.$$

Following the proof of Theorem 6.4 in [10, pp. 459-465] we obtain that, for U_x sufficiently small, every time optimal trajectory in $U_1 \cup U_4 \cup B_1$ does not contain any trajectory of the type $X * Y * X$ or $Y * X * Y$. The proof relies essentially on the construction of an envelope for such trajectories. For the theory of envelopes we refer to [9] and [13]. The uniform bound in (4.4) assure the admissibility of these envelopes.

More precisely let γ' be an $Y * X * Y$ trajectory. Let t_1 be the first switching time of γ', $\varepsilon > 0$, and define the extremal trajectories γ_s which correspond to control $+1$ up to time $s \in [t_1 - \varepsilon, t_1 + \varepsilon]$ and then switch to control -1. The second switching points of γ_s form a curve C. Repeating the reasoning of Theorem 6.4 of [10], we obtain that there exists a feedback control $u_C(y) \in]-1, 1[$, $y \in C$, such that C is run by a trajectory γ_C corresponding to u_C. Moreover, if σ is the second switching time of γ_s then $\widetilde{\gamma} = \gamma_C * \gamma_s \restriction [0, \sigma]$ takes the same time as γ' to steer its initial point to its terminal point. Therefore if γ' is optimal then $\widetilde{\gamma}$ is optimal too. But $\widetilde{\gamma}$ does not satisfy the PMP, hence γ' cannot be optimal.

Then it is clear that $N(\gamma)$ is finite unless γ has a switching point on $\Delta_A^{\leftarrow}(0)$. In this case γ switches precisely at the points where it meets $\Delta_A^{\leftarrow}(0)$, indeed two switching points have to be conjugate. Assume that X, Y points toward $\Delta_A^{\leftarrow}(0)$ otherwise consider γ with reversed time. Choose a local coordinate system such that $x=(0,0)$, $\Delta_A^{\leftarrow}(0) = \{(p,0)\colon p\in \mathbb{R}\}$ and $\Delta_B^{\leftarrow}(0)=\{(0,q)\colon q\in\mathbb{R}\}$. Suppose that $\gamma(t_0) = (0,q_0)$, $\gamma(t_1)=(0,q_1)$ and that γ switches at $(p_0,0)=\gamma(s_0)$, $s_0\in]t_0,t_1[$. We have that:

$$q_0 = ap_0^2 + bp_0^3 + o(p_0^3), \qquad q_1 = ap_0^2 + cp_0^3 + o(p_0^3), \tag{4.5}$$

$$X(0) = (p_X, 0), \qquad t_1 - t_0 \simeq 2\frac{p_0}{p_X} \simeq 2\frac{\sqrt{q_0}}{p_X}, \tag{4.6}$$

in fact X, Y are parallel along $\Delta_A^{\leftarrow}(0)$. From (4.5), (4.6) we have that:

$$q_1 = q_0 + (c-b)p_0^3 + o(p_0^3) = q_0 + (c-b)q_0^{3/2} + O(q_0^2)$$

and if we call t_2 the next time in which γ touches the ordinate axis we have:

$$t_2 - t_1 = 2\frac{\sqrt{q_1}}{p_X} = \frac{2}{p_X}\sqrt{q_0 + (c-b)q_0^{3/2} + O(q_0^2)} =$$
$$= (t_1 - t_0)\sqrt{1 + \frac{p_X(c-b)}{2}(t_1-t_0) + o(t_1-t_0)}\,.$$

Then calling t_n the time of the n-th crossing of the ordinate axis we obtain:

$$\sum_n t_n - t_{n-1} = +\infty\,. \tag{4.7}$$

If $\bar{t}$ is the time between the first two switchings t_0, t_1 on $\Delta_A^{\leftarrow}(0)\cap U_x$ then we have a lower bound on $\bar{t}$. Indeed, if moving backwards along γ we intersect $\Delta_A^{\leftarrow}(0)\cap U_x$ again, then t_0 is not the first switching. Hence, given $\tau>0$, from this lower bound and from (4.7) we have a bound on the number of arcs for γ.

It remains to consider the case in which $f>0$ on U_1 and then $f<0$ on U_2. Let γ_X, γ_Y be, respectively, the X-, Y-trajectory with $\mathrm{In}(\gamma_X)=\mathrm{In}(\gamma_Y)=x$; let V_1 be the connected component of $U_x\setminus(\gamma_X\cup\gamma_Y)$ containing U_1, U_4, and let V_2 be the other connected component.

If $\Delta_A < 0$ on U_2 and γ moves from V_1 to V_2, then it cannot return back to V_1 remaining in U_x the opposite happens if $\Delta_A > 0$ on U_2.

If γ is contained in V_2, we can use (4.4) to obtain, as above, a bound on $n(\gamma)$.

Suppose that γ is contained in V_1. If γ either has a switching in $U_1 \cup \cup U_4$, or switches at t_0 on $\Delta_B^{\leftarrow}(0)$, then it cannot switch again. Indeed from Theorem 3.5 and the geometry of U_x, we have that γ does not cross $\Delta_B^{\leftarrow}(0)$ before a second switching. Moreover, from Lemma 3.4, we have that $\alpha(t_0) = 0$, α defined in (3.13), and α is monotone since the sign of his derivative is equal to the sign of Δ_B. Hence, for U_x sufficiently small γ cannot have another switching in U_x.

If γ switches on $\Delta_A^{\leftarrow}(0)$ we proceed as above. We obtain a bound on the number of arcs unless γ has a sequence of switchings on $V \doteq (V_1 \cap \cap (U_2 \cup U_3))$. In this case by (4.4) and (P_7) we can use the same construction of Theorem 6.4 in [10, pp. 459-465], i.e. we can construct an admissible envelope. This concludes the proof of the second case.

Case (3). We assume that:

(P_8) $\Delta_A(x) \neq 0$.

Suppose, for example, that $X(x) \cdot \nabla\Delta_B(x) = 0$. Take U_x open connected such that x is the only bad point in U_x and $\Delta_A \neq 0$ on U_x. Let γ_Y be the maximal Y-trajectory passing through x and let U_1, U_2 be the connected components of $U_x \setminus \gamma_Y$. See Fig. 3. For U_x sufficiently small $\gamma_Y \cap \Delta_B^{\leftarrow}(0) \cap U_x = \{x\}$, and $X(y)$ points to the same side of γ_Y for every $y \in \gamma_Y \cap U_x$. If $X(x)$ points into U_1 then γ cannot cross from U_1 into U_2, and viceversa if $X(x)$ points into U_2. Since in U_1, U_2 we have an a priori bound on the number of arcs of γ, as in the preceding cases, the proof is completed.

By the previous analysis, under the generic assumptions (P_6), (P_7), (P_8) the conclusion of the theorem holds.

Using Theorem 4.2 for each $x \in R(\tau)$ we can select an neighborhood U_x such that every optimal trajectory remaining in U_x is the concatenation of $\leq N_x$ regular arcs. Choose $\varepsilon_x > 0$ such that $B(x, 2\varepsilon_x) \subset U_x$. Since $R(\tau) \subset \bigcup_{x \in R(\tau)} B(x, \varepsilon_x)$, by compactness we can extract a finite subcover $B(x_i, \varepsilon_i)$, $i = 1, \ldots, n$, $\varepsilon_i \doteq \varepsilon_{x_i}$. Consider an extremal pair (u, γ), $\gamma\colon [0, \tau] \mapsto \mathbb{R}^2$, $\mathrm{In}(\gamma) = 0$. Define:

$$\varepsilon \doteq \min_{i=1,\ldots,n} \varepsilon_i, \qquad N = \max_{i=1,\ldots,n} N_{x_i}. \tag{4.8}$$

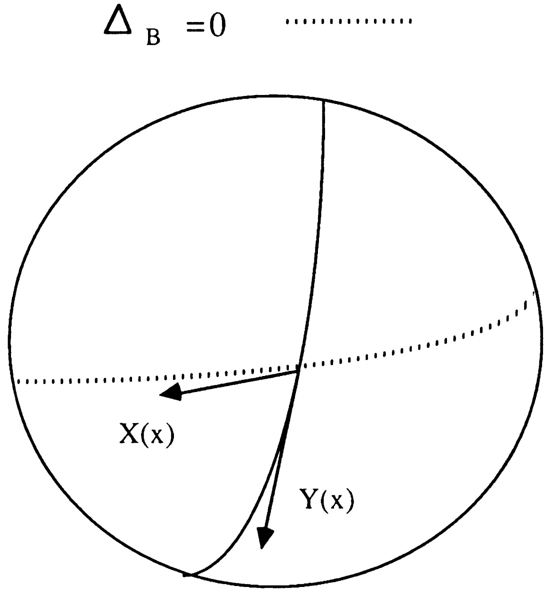

Figure 3.

Choose i_1 such that $0 \in B(x_{i_1}, \varepsilon)$. Let t_1 be either the first time such that:

$$\gamma(t_1) \notin B(x_{i_1}, 2\varepsilon)$$

or $t_1 = \tau$ if γ remains in $B(x_{i_1}, 2\varepsilon)$. Then there exists $i_2 \neq i_1$ such that $\gamma(t_1) \in B(x_{i_2}, \varepsilon)$. Let t_2 be either the first time for which $\gamma(t_2) \notin$ $\notin B(x_{i_2}, 2\varepsilon)$ or $t_2 = \tau$ if γ remains in $B(x_{i_2}, 2\varepsilon)$. We proceed in the same way defining a set of increasing times $\{t_0 \doteq 0, t_1, \dots, t_\nu = \tau\}$. If $M \doteq \max\{|F(x)| + |G(x)| : x \in R(\tau)\}$ denotes the maximum speed of trajectories inside $R(\tau)$, it's clear that $t_j - t_{j-1} \geqslant \varepsilon / M$. Therefore:

$$\nu \leqslant \frac{M\tau}{\varepsilon}. \tag{4.9}$$

By definition, for each t_j, $j = 1, \dots, \nu$, we have that $\{\gamma(t) : t \in [t_{j-1}, t_j[\}$ is contained in $B(x_{i_j}, 2\varepsilon)$. Using Theorems 3.5, 3.6, 4.2 toghether with (4.9) we obtain:

$$n(\gamma) \leqslant N(\Sigma) \doteq N \frac{M\tau}{\varepsilon}.$$

We have thus proved the following:

COROLLARY 4.3. *For every $\tau > 0$ the set Π_τ is a generic subset of Ξ.*

5. – Existence of a regular synthesis.

Given a system $\Sigma \in \Pi_\tau$ we can construct a synthesis Γ for Σ. We can follow the classical idea of constructing extremal trajectories and deleting those trajectories which are not globally optimal. At the end we obtain a set of trajectories from which we can extract a synthesis. This synthesis is optimal by construction. For synthesis theory see [1-3] and [14].

We can describe an algorithm $\mathcal{A}$ by induction. At step N, we construct precisely those trajectories $x(\cdot)$ which are concatenation of N bang- or singular arcs and satisfy the Pontryagin maximum principle. The endpoints of the arcs forming these trajectories, corresponding to the switching times of the control, are determined by certain nonlinear equations. Under generic conditions such equations can be solved by the implicit function theorem, thus determining a smooth switching locus. Eventually the algorithm will partition the reachable set $R(\tau)$ into finitely many open regions (where the optimal feedback control is either $u = 1$ or $u = -1$), separated by boundary curves and points, here called *frame curves* and *frame points*, respectively.

At each step, it may happen that distinct extremal trajectories reach the same point x_0, at different times. It is therefore necessary to delete from the synthesis those trajectories which are not globally optimal. This procedure will usually produce new «overlap curves», consisting of points reached in minimum time by two distinct trajectories, one ending with the control value $u = 1$, the other with $u = -1$.

If at step N the algorithm $\mathcal{A}$ does not construct any new trajectory then we say that $\mathcal{A}$ *stops at step N (for Σ at time τ)*. From Corollary 4.3 it is clear that under generic assumptions, there exists $N(\Sigma)$ such that $\mathcal{A}$ stops before step $N(\Sigma)$ and, by construction, we have that $\{\gamma: \gamma$ is constructed by $\mathcal{A}\} = \mathrm{Opt}(\Sigma)$. In this case we define $R_{\mathcal{A}}(\tau)$ to be the set of points reached by the trajectories constructed by $\mathcal{A}$; notice that $R_{\mathcal{A}}(\tau) = R(\tau)$. We let $\mathrm{Fr}(R_{\mathcal{A}}(\tau))$ be a frame curve and let its intersections with other frame curves be frame points.

If $\mathcal{A}$ stops then for each $x \in R(\tau)$ there exists a set of constructed trajectories that reach x. Define $\Gamma_x \doteq \{\gamma: \gamma$ is constructed by $\mathcal{A}$, $\mathrm{Term}(\gamma) = x\}$.

We want to select, for each $x \in R(\tau)$, a trajectory from Γ_x to form a synthesis. Define K_k to be the set of points $x \in R(\tau)$ reached by at least one constructed trajectory γ satisfying $n(\gamma) \leqslant k$. Notice that K_k is compact for each k and $K_{N(\Sigma)} = R(\tau)$. We proceed by induction on k. Given $x \in K_k \setminus K_{k-1}$, we consider the optimal trajectories $\gamma \in \Gamma_x$ formed by k arcs, for which the following holds. If $y = \gamma(t)$ is the initial point of the last arc of γ then $\gamma \restriction [0, t]$ has been selected from Γ_y by induction. Finally, if there are more than one such trajectories then we select one, say according to the preference order X, Y, Z on the type of the last arc.

In this way, at step $N(\Sigma)$, we have constructed a synthesis for Σ at time τ. We use the symbol $\Gamma_{\mathcal{A}}(\Sigma, \tau)$ to denote this synthesis and we call it *the synthesis generated by the algorithm* $\mathcal{A}$. We have the following:

THEOREM 5.1. *Consider $\Sigma \in \Xi$ and $\tau > 0$. If $\mathcal{A}$ stops for Σ at time τ then $\Gamma_{\mathcal{A}}(\Sigma, \tau)$ is an optimal synthesis.*

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