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The *S*-transform and its dual with applications to Prüfer extensions

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The S-Transform and its Dual with Applications to Prüfer Extensions.

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1. - Introduction.

We shall develop themes begun by J. Hays in [9] where the S-transform was introduced. Whereas the setting for [9], and for subsequent work in [1], was the quotient field of an integral domain, we shall deal with an arbitrary commutative ring U with 1 and a subring R containing 1. For an ideal I of R, $S(I)$ ($= S_R(I)$) consists of all $a \in U$ such that, for each $y \in I$, $y^n a \in R$ for some integer $n \geq 1$. The standard *ideal transform*, defined by $T(I) = \bigcup_n I^{-n}$ [15], satisfies $T(I) \subseteq S(I)$ with equality if I is finitely generated (see [9, Theorem 1.3]).

Hays called I a *maximal S-ideal* if $J \supset I$ implies $S(J) \neq S(I)$. The maximal S-ideals determine a transform S^{-1}, introduced in § 2, which takes subsets of U to ideals of R and has properties mirroring those of the S-transform. We show in particular that, for $A \subseteq U$, $S^{-1}(A)$ is the intersection of all prime ideals P of R such that $R_{[P]} \not\supseteq A$. Paragraph 3 concerns finiteness conditions. Extending results of J. Brewer [3] and P. Schenzel [19], we provide sufficient conditions on I for $S(I)$ to be of the form $T(J)$. Paragraph 4 concerns the effect of changing the base ring R, mainly by localisation.

In the remaining sections, we assume that R is Prüfer in U(see [18]). Some connections between the transforms $R_{[P]}$, $S(P)$, $T(P)$, P^{-1} and $P:P$ of a prime ideal P are obtained in § 5. Hays characterised prime ideals which are not maximal S-ideals in a valuation domain, and in § 6 we determine the extent to which his characterisation

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extends to Prüfer rings. This involves connections between S^{-1}-transforms and the property (#) studied in [6].

TERMINOLOGY. Unless otherwise stated, $S(I)$ means $S_R(I)$; *ideal* means R-ideal (ideal of R); *submodule* means R-submodule of U; *subalgebra* means R-subalgebra of U) containing R; *finitely generated* means finitely generated as a module over R. For $H \subseteq U$, $R[H]$ denotes the smallest subalgebra containing H. We write $H:K$ for $\{a \in U \mid aK \subseteq H\}$, $H:_A K$ for $(H:K) \cap A$, H^{-1} for $R:H$, and H^{-n} for $(H^n)^{-1}$ when $n \geqslant 2$. For a submodule K and a multiplicatively closed subset D of U, we take $K_{[D]} = \{a \in U \mid da \in K \text{ for some } d \in D\}$ but, for a prime ideal P, $K_{[R\setminus P]}$ is usually denoted by $K_{[P]}$ (cf. [8]). (Note: it can happen that $K_{[D]} \neq K \cdot R_{[D]}$.) For $I \subseteq R$, denote by $\mathbb{D}(I)$ the set of all prime ideals not containing I.

Recall that a *G-ideal* is a prime ideal which is not the intersection of the prime ideals strictly containing it. We call a prime ideal Q *branched* if it is neither minimal in R nor the union of a chain of prime ideals strictly within Q; otherwise call Q *unbranched*. This is consistent with Gilmer's use of the term «branched» in a Prüfer domain [5, Theorem 23.3(*e*)].

Given a subalgebra A, an ideal is called *A-regular* if it contains a finite intersection of submodules of the form $R:a$ where $a \in A$ (see [18]). In general «A-regular» differs from «regular» in its usual sense as in [11]; extreme examples are given by the cases $A = R$ and $A = R[X]$ with X an indeterminate. If A is the total quotient ring of R the two terms are equivalent. By [18, Lemma 1.1(1)], a prime ideal P is A-regular if and only if $R_{[P]} \not\supseteq A$ (a condition which will appear frequently).

2. – The transforms S and S^{-1}.

For $I \subseteq R$, we define the *S-transform of I* to be the subalgebra $S(I)$ $(= S_R(I)) = \{a \in U \mid \mathrm{rad}(R:_R a) \supseteq I\}$. As in [1, Prop. 2.2], $S\left(\sum_\alpha I_\alpha\right) = \bigcap_\alpha S(I_\alpha)$ for ideals I_α.

For $A \subseteq U$, we define the *S^{-1}-transform of A* to be the radical ideal $S^{-1}(A)$ $(= S_R^{-1}(A)) = \{x \in R \mid S(x) \supseteq A\}$. For submodules A_α, one verifies that $S^{-1}\left(\sum_\alpha A_\alpha\right) = \bigcap_\alpha S^{-1}(A_\alpha)$. Note that $S^{-1}(a) = \mathrm{rad}(R:_R a)$ for all $a \in U$. Hence $S(I) = \{a \in U \mid S^{-1}(a) \supseteq I\}$, which shows the dual nature of S and S^{-1}.

Our definition of S^{-1}-transforms is modelled on Hays' construction [9, Theorem 2.3] of maximal S-ideals (see § 1). Hays shows, in effect, that maximal S-ideals are those of the form $S^{-1}S(I)$, and that

$SS^{-1}S(I) = S(I)$. We add that, for $K \subseteq R$,

$$(*) \qquad \textit{if } S(K) = S(I) \quad \textit{then } K \subseteq S^{-1}S(I)$$

since $S(K) = \bigcap_{x \in K} S(x)$ and so $S(x) \supseteq S(I)$ for all $x \in K$. Using the dual of $(*)$ we obtain $SS^{-1}(A) \supseteq A$. It follows that $S^{-1}(A) \supseteq S^{-1}SS^{-1}(A)$. Since, by $(*)$, $I \subseteq S^{-1}S(I)$ where $I = S^{-1}(A)$, we obtain $S^{-1}SS^{-1}(A) = S^{-1}(A)$. Thus it is only the restrictions of the maps S and S^{-1}, each acting on the set of images of the other, which are really mutually inverse. One verifies that an ideal I is a maximal S-ideal if and only if it is an S^{-1}-transform. By duality, a subalgebra A is an S-transform if and only if it is maximal amongst all subalgebras B such that $S^{-1}(B) = S^{-1}(A)$.

(2.1) Proposition. For $I \subseteq R$, $S(I) = \bigcap\{R_{[P]} \mid P \in \mathrm{D}(I)\}$.

Proof. First, let $P \in \mathrm{D}(I)$. As in [9], choosing $x \in (I \backslash P)$ gives $S(I) \subseteq S(x) \subseteq R_{[P]}$. Secondly, let $a \in R_{[P]}$ for all $P \in \mathrm{D}(I)$. Then † (see Remark (2.2)) $(R :_R a) \not\subseteq P$ whenever $P \in \mathrm{D}(I)$. Hence $\mathrm{rad}(R :_R a) \supseteq I$ which means that $a \in S(I)$. ■

The second part of the proof of (2.1) is related to [1, Theorem 2.7] and, particularly, [10, Prop. 4.3]. Hays proved (2.1) for a domain in its quotient field. His proof cannot be used here since it depends on two results, one invalid for an arbitrary commutative ring (see [3, Footnote p. 301]) and the other [3, Cor. 1.2] requiring $xS(x) = S(x)$ (i.e., x is a unit in U) for all x in a system of generators for I.

(2.1^d) Theorem. For $A \subseteq U$, $S^{-1}(A) = \bigcap\{P \mid P \text{ is a prime ideal and } R_{[P]} \not\supseteq A\}$.

Proof. Since $S^{-1}(a) = \mathrm{rad}(R :_R a)$, for all a in U we have

$$S^{-1}(a) =^{\dagger} \bigcap\{P \mid P \supseteq (R :_R a)\} = \bigcap\{P \mid a \notin R_{[P]}\}.$$

Hence

$$S^{-1}(A) = \bigcap_{a \in A} S^{-1}(a) = \bigcap_{a \in A} (\bigcap \{P \mid a \notin R_{[P]}\}) =$$

$$= \bigcap\{P \mid a \notin R_{[P]} \text{ for some } a \in A\}. \quad ■$$

(2.2) Remark. Although there is an evident duality between statements (2.1^d) and (2.1) (regarding P and $R_{[P]}$ as dual entities), in neither case is the natural dual proof available. At † in the proof of (2.1) the dual fails since the implication $x \in P \Rightarrow (R:x) \not\subseteq R_{[P]}$ may be false if x is not a unit (e.g. take $x = 0$, $P = (0)$ in a domain in its quotient field). In the proof of (2.1^d), the dual of equation † (i.e. that $S(x) = \bigcap\{R_{[P]} \mid R_{[P]} \supseteq (R:x)\}$) is true, but its validity seems to depend on the case $I = Rx$ of (2.1). ■

We shall give particular attention to conditions for prime ideals to be S^{-1}-transforms. For R a Prüfer domain and U its quotient field, the next result is effectively contained in [9].

(2.3) PROPOSITION. For a prime ideal P, consider the conditions (i) $R_{[P]} \not\supseteq S(P)$, (ii) $P = S^{-1}(a)$ for some $a \in U$, (iii) $P = S^{-1}(A)$ for some $A \subseteq U$. Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). When P is a G-ideal, all three conditions are equivalent.

PROOF. (iii) implies $S(P) = SS^{-1}(A) \supseteq A$ and, if P is a G-ideal, (2.1^d) gives $R_{[P]} \not\supseteq A$ whence (i). For (i) $\Leftrightarrow$ (ii), note that $R_{[P]} \not\supseteq \bigcap_{Q \in \mathbb{D}(P)} R_{[Q]}$ is equivalent to the existence of an element a in U such that, for all $Q \in \mathbb{D}(P)$, $(R :_R a) \not\subseteq Q$ but $(R :_R a) \subseteq P$. By (2.1), $R_{[P]} \not\supseteq S(P)$ if and only if $P = \mathrm{rad}(R :_R a)$ for some a. ■

(2.3^d) REMARKS. Consider the dual conditions (i^d) $P \not\supseteq S^{-1}(R_{[P]})$, ($ii^d$) $R_{[P]} = S(x)$ for some $x \in R$, (iii^d) $R_{[P]} = S(I)$ for some $I \subseteq R$. It may be verified that (i^d) $\Rightarrow$ (ii^d) $\Rightarrow$ (iii^d). Also, (iii^d)$\Rightarrow$(i^d) when $R_{[P]}$ is not an intersection of rings of form $R_{[Q]}$ strictly containing it, with Q a prime ideal. However (ii^d) $\not\Rightarrow$ (i^d), e.g., take $(x) = P = (0)$ to be an intersection of S^{-1}-transforms in a domain in its quotient field. Also (iii^d) $\not\Rightarrow$ (ii^d), e.g., if P is both an intersection and a union of prime ideals distinct from P in a valuation domain, then $S(P) = R_{[P]}$ but, for all x, $R_{[P]} \neq S(x)$; this example also shows that (iii) $\not\Rightarrow$ (ii) in (2.3). ■

(2.4) REMARKS. Let $\mathbb{P}$ be a set of prime ideals with the property that if $P \in \mathbb{P}$ and Q is a prime ideal containing P then $Q \in \mathbb{P}$. Noting that every prime ideal is an intersection of G-ideals [13, § 1.3], it follows from (2.3) that all the ideals in $\mathbb{P}$ are S^{-1}-transforms if and only if each ideal in $\mathbb{P}$ is an intersection of prime ideals each of which is regular relative to its S-transform.

When R is a Hilbert ring, we deduce that each radical ideal (being an intersection of maximal ideals) is an S^{-1}-transform if and only if $R_{[M]} \not\supseteq S(M)$ for all maximal ideals M. ■

Recall from [13] that, for R a domain and U its quotient field, 0 is a G-ideal if and only if $U = S(x)$ for some $x \in (U\backslash 0)$, i.e., if and only if 0 is not an S^{-1}-transform. The next result adds to the «only if» part of this, but the case $U = R[X]$ with R a domain shows that the non-minimality assumption in (2.5) cannot, in general, be omitted. Concerning the converse of (2.5) in a Prüfer extension, see (6.3).

(2.5) COROLLARY. Suppose that $R :_R a$ is finitely generated for all $a \in U$. An unbranched G-ideal, which is not a minimal prime ideal in R, is not an S^{-1}-transform.

PROOF. Suppose that P is a G-ideal and an S^{-1}-transform. By (2.3)(iii) $\Rightarrow$ (ii), $P = \operatorname{rad}(R :_R a)$ for some a. If P were non-minimal unbranched, then $(R :_R a) \subseteq Q \subset P$ for some prime ideal Q; hence P must be branched. ∎

3. – Finiteness conditions.

As a possible dual for the T-transform we introduce the T^d-*transform* defined by $T^d(A) = \operatorname{rad}(A^{-1} \cap R)$ for all $A \subseteq U$. Then $T^d(a) = S^{-1}(a)$ for all $a \in U$. Whereas $R[A]$ and A have the same S^{-1}-transform, their T^d-transforms can differ (e.g. for R a d.v.r. with quotient field U, maximal ideal generated by x, and $A = Rx^{-1}$). Further duals of properties of T and S are given in the next result.

(3.1) PROPOSITION. (i) $T^d(A) \subseteq S^{-1}(A)$. (ii) If A is a finitely generated submodule then $T^d(A) = S^{-1}(A)$.

PROOF. (i) is clear. (ii) Let $A = \sum_{k=1}^{n} Ra_k$. Then

$$S^{-1}(A) \supseteq T^d(A) = \operatorname{rad}\left(\bigcap_k (R :_R a_k)\right) = \bigcap_k \operatorname{rad}(R :_R a_k) \supseteq S^{-1}(A). \qquad ∎$$

Next we exploit an argument used at the end of [19] where, in a Noetherian context, Schenzel effectively showed that if $IA = A \subseteq T(I)$ then a subalgebra A is finitely generated. In his domain case, Brewer [3] showed that $IA = A \subseteq T(I)$ implies $A = T(I) = S(I)$.

(3.2) THEOREM. Let I be an ideal and A a subalgebra. (i) If $IA = A \subseteq S(I)$ then $S(I)$ is a finitely generated subalgebra, $A = S(I) = S(J) = T(J) = S(B^{-1}) = T(B^{-1})$ and $S^{-1}S(I) = T^d(B)$ for finitely generated submodules J, B such that $J \subseteq I$ and $B \subseteq A$. (ii) If $IA = A \subseteq T(I)$ then $T(I) = S(I)$.

PROOF. (i) Let 1 be the finite sum $\sum x_k a_k$, where $x_k \in I$ and $a_k \in A$ for each k. Put $\sum Rx_k = J$ and $R[\{a_k\}] = C$. For each $b \in S(I)$, there exists $n \geq 1$ such that $J^n b \subseteq R$. Then $b = b(\sum x_k a_k)^n \in bJ^n C \subseteq C \subseteq A \subseteq$ $\subseteq S(I)$. Hence $C = A = S(I)$. But $JC = C = S(I) \subseteq S(J)$, and replacing I by J gives $C = S(J)$ which is $T(J)$ by the finite generation of J. Setting $R + \sum Ra_k = B$, we have $S^{-1}(C) = S^{-1}(B) = T^d(B)$ by (3.1). Also $I \subseteq$ $\subseteq S^{-1}S(I) = S^{-1}(C) = T^d(B) = \mathrm{rad}(K)$, where K denotes B^{-1}, and so $S(K) \subseteq S(I)$. If $a \in S(I)$ then, for some $n \geq 1$, $a \in B^n \subseteq K^{-n} \subseteq T(K)$, and so $S(I) \subseteq T(K)$. Since $T(K) \subseteq S(K)$, we obtain $S(I) = T(K) = S(K)$. (ii) Since $T(I) \subseteq S(I)$, (i) gives $T(I) = S(I)$. ■

(3.3) REMARK. It is clear that the sufficient conditions for $S(I) =$ $= T(I)$ given in [9, Lemma 1.11] do extend to our context. An argument similar to that of Hays shows that $S(I) = T(I)$ is also implied by the existence of a finitely generated ideal J such that $J \supseteq I$ and $S(J) =$ $= S(I)$. ■

For a submodule I, denote $(I^{-1})^{-1}$ by I_v.

(3.4) THEOREM. Let P be a prime ideal. (i) If $PS(P) = S(P)$ then $S^{-1}S(P) = P$. (ii) If $PT(P) = T(P)$ and P^{-1} is finitely generated then $P_v = P$.

PROOF. (i) By [18, Lemmas 1.1 and 1.2] we obtain $R_{[P]} \not\supseteq S(P)$. Hence, by (2.3), $P = S^{-1}(a)$ for some a, and so $S^{-1}S(P) = S^{-1}(a) = P$. (ii) By (3.2), $T(P) = S(P)$ and $T(P) = R[B]$ where B is a finitely generated submodule. Since $T(P) \supseteq P^{-1}$ we may assume that $B \supseteq P^{-1}$. Then $S^{-1}T(P) = S^{-1}(B) = T^d(B)$ by (3.1). Choose $k \geq 1$ such that $B \subseteq P^{-k}$ whence $T^d(B) \supseteq T^d(P^{-k})$. It may be verified that $(P_v)^k \subseteq (P^k)_v$ and so $T^d(P^{-k}) = T^d(P^{-1})$. It follows that $P = S^{-1}T(P) = T^d(B) \supseteq$ $\supseteq T^d(P^{-1}) = \mathrm{rad}(P_v)$ and so $P = P_v$. ■

4. – Change of the base ring.

So far we have taken R to be fixed in considering the transform $S_R(I)$. The following result is a companion to [16, Lemma 2.5]. In particular, we see that fixing I and taking a second S-transform based on the ring $S_R(I)$ gives nothing new.

(4.1) PROPOSITION. For a subalgebra B contained in $S_R(I)$, we have $S_B(I) = S_R(I)$.

PROOF. Let $a \in S_B(I)$. For all $x \in I$ there exists $n \geq 1$ such that $ax^n \in B \subseteq S_R(I)$, whence $ax^n x^m \in R$ for some $m \geq 1$. Thus $a \in S_R(I)$. Trivially $S_B(I) \subseteq S_R(I)$, whence equality. ■

Although the next result is not dual to the preceding one, there is some duality between the proofs.

(4.2) PROPOSITION. Let A, B be subalgebras such that $A \supseteq B$. Then $S_R^{-1}(A) = S_R^{-1}(B) \cap S_B^{-1}(A)$.

PROOF. If $x \in S_B^{-1}(A)$ then, for all $a \in A$, there exists $n \geqslant 1$ such that $x^n a \in B$. If also $x \in S_R^{-1}(B)$, there exists $m \geqslant 1$ such that $x^m x^n a \in R$. Thus $x \in S_R^{-1}(A)$. Since trivially $S_R^{-1}(A) \subseteq S_R^{-1}(B) \cap S_B^{-1}(A)$, the required equality follows. ■

From now on, D will denote a non-empty multiplicatively closed subset of R not containing 0. Whereas $S(I)$ may continue to denote $S_R(I)$, we shall denote $S_{R_D}(I_D)$ and $S_{R_{[D]}}(I_{[D]})$ by $S_D(I_D)$ and $S_{[D]}(I_{[D]})$, respectively. For a prime ideal Q, S_Q will denote $S_{(R \setminus Q)}$. Similar conventions will be used for S^{-1}. It may be shown that, for an ideal I, $S_D(I_D) \supseteq (S_R(I))_D$ with equality if $S_R(I) = S_R(K)$ for some finitely generated ideal $K \subseteq I$. Dually for submodule A, $S_D^{-1}(A_D) \supseteq (S_R^{-1}(A))$ with equality if $S_R^{-1}(A) = S_R^{-1}(C)$ for some finitely generated submodule $C \subseteq A$. Analogous statements hold with $[D]$ in place of D.

Consequences include that the ideals $H = (S_R^{-1} S_R(I))_D$ and $L = S_D^{-1} S_D(I_D)$ satisfy (i) $H \subseteq L$ if I is finitely generated, and (ii) $H \supseteq L$ if $S_R(I)$ is a finitely generated subalgebra; dual statements also hold. From (ii) we see that if I is a maximal S-ideal (i.e., $I = S_R^{-1} S_R(I)$) and $S(I)$ is a finitely generated subalgebra, then I_D is a maximal S_D-ideal.

The next result sharpens (2.1). Also, even in the domain case it effectively improves Hay's result [9, Prop. 1.9] that $S_R(I) = \bigcap S_M(I \cdot R_M)$ where the intersection is taken over all maximal ideals M of a domain R in its quotient field. For if M is a maximal ideal containing a prime ideal P, then $S_{[P]}(I_{[P]}) = S_{[P]}(I) \supseteq S_{[M]}(I) = S_{[M]}(I_{[M]})$. ■

(4.3) PROPOSITION. For an ideal I, $S_R(I) = \bigcap_P S_{[P]}(I_{[P]})$ where the intersection is taken over all $P \in \mathrm{D}(I)$. Dually, for a submodule A, $S_R^{-1}(A) = \bigcap_P S_{[P]}^{-1}(A_{[P]})$ where the intersection is taken over all prime ideals P such that $R_{[P]} \not\supseteq A$.

PROOF. Using (2.1) we have

$$S_R(I) = \bigcap_{x \in I} S_R(x) = \bigcap_{x \in I} \bigcap_{P \in \mathrm{D}(x)} R_{[P]} .$$

Since $S_R(x) \subseteq R_{[P]}$ if $x \notin P$, we obtain

$$S_R(I) = \bigcap_{x \in I} \bigcap_{P \in \mathrm{D}(x)} (S_R(x))_{[R \setminus P]} = \bigcap_{x \in I} \bigcap_{P \in \mathrm{D}(x)} S_{[P]}(x) = \bigcap_{x \in I} \bigcap_{P \in \mathrm{D}(I)} S_{[P]}(x) =$$

$$= \bigcap_{P \in \mathrm{D}(I)} \bigcap_{x \in I} S_{[P]}(x) = \bigcap_{P \in \mathrm{D}(I)} S_{[P]}(I) = \bigcap_{P \in \mathrm{D}(I)} S_{[P]}(I_{[P]}).$$

The dual is proved similarly. ■

The following correspondences will be useful in the Prüfer context. We shall use $\overline{G}$ to denote the natural image in $U_D/(R_D : U_D)$ of a subset G of U_D.

(4.4) Lemma. Consider the sets of ideals:

$\mathbb{A}_0 = \{I \mid I$ is a U-regular R-ideal and $I = I_{[D]} \cap R\}$,
$\mathbb{A} = \{H \mid H$ is a U-regular $R_{[D]}$-ideal such that $H = H_{[D]}\}$,
$\mathbb{B} = \{E \mid E$ is a U_D-regular R_D-ideal$\}$,
$\mathbb{C} = \{F \mid F$ is a $\overline{U_D}$-regular $\overline{R_D}$-ideal$\}$.

There are natural one-one correspondences between each two of these four sets, determined by the rule that, for a U-regular R-ideal I, the ideals $I_{[D]} \cap R$ in $\mathbb{A}_0$, $I_{[D]}$ in $\mathbb{A}$, I_D in $\mathbb{B}$, $\overline{I_D}$ in $\mathbb{C}$ correspond. In these correspondences regular primary ideals correspond, a U-regular primary R-ideal being in $\mathbb{A}_0$ precisely if it does not meet D.

Proof. The correspondence $I = I_{[D]} \cap R \leftrightarrow I_D$ between the set of all ideals of form $I_{[D]} \cap R$ in R and the ideals of R_D is standard. Replacing R by $R_{[D]}$ gives the correspondence between ideals of form $H = H_{[D]}$ in $R_{[D]}$ and the ideals of R_D, since $(R_{[D]})_D = R_D$. We note that it is implicit and easily verified that

$$(*) \qquad (K \cap R)_{[D]} = K_{[D]} \quad \text{for each } R_{[D]}\text{-ideal } K.$$

Now we turn to the effect of imposing the regularity conditions. First, take $I = I_{[D]} \cap R \in \mathbb{A}_0$ and suppose that $I \supseteq \bigcap_k (R : a_k)$ for some finite subset $\{a_k\}$ of U. Then $I_D \supseteq \bigcap (R : a_k)_D = \bigcap (R_D : a_k) \supseteq (R_D : U_D)$, hence $\overline{I_D} \supseteq \bigcap (\overline{R_D} : \overline{a_k})$ and $\overline{I_D} \in \mathbb{C}$. Next, let $\overline{I_D} \in \mathbb{C}$ where I is an R-ideal. Then $\overline{I_D} \supseteq \bigcap_k (\overline{R_D} : \overline{f_k})$ for some finite subset $\{f_k\}$ of U_D, and it follows easily that $I_D \supseteq \bigcap_k (R_D : f_k)$, whence $I_D \in \mathbb{B}$. Given $I_D \in \mathbb{B}$, we have $I_D \supseteq \bigcap_k (R_D : f_k)$ for a finite set of elements $f_k = a_k d_k^{-1}$ where $a_k \in U$, $d_k \in D$. Again, it follows that $I_{[D]} \supseteq \bigcap_k (R_{[D]} : a_k)$ and so $I_{[D]} \in \mathbb{A}$. Finally, let $H \in \mathbb{A}$. By $(*)$,

$H = I_{[D]}$ for some R-ideal I. For some finite set $\{a_k\}$ we have $I_{[D]} \supseteq \bigcap_k (R_{[D]} : a_k)$, and intersecting with $R(= R:1)$ gives that $I_{[D]} \cap R$ is U-regular. The rest is straightforward. ■

5. - Prüfer extensions.

From now on we shall assume that R is Prüfer in U. See [18] for a study of this condition. We shall obtain connections between $S(P)$ and other transforms of a prime ideal P. Various preliminaries are needed. We say that an ideal I is *U-invertible* if $IJ = R$ for some $J \subseteq U$. Our Prüfer assumption means that a U-regular R-ideal is U-invertible if and only if it is finitely generated. In particular, $R :_R a = R : (R + Ra)$ is U-invertible for each $a \in U$. By [18, Lemma 1.2 and Theorem 2.1], an ideal I is U-regular if and only if $IU = U$. The method of proof of $(2) \Rightarrow (1)$ in [18, Theorem 2.1] shows that $(R_{[P]}, P_{[P]})$ is a Manis valuation (MV) subalgebra of U for all U-regular prime ideals P. Then the conductor $R_{[P]} : U$ is contained in $P_{[P]}$ and is a prime ideal of both $R_{[P]}$ and U (see e.g. [17]).

(5.1) PROPOSITION. Let M be a U-regular prime ideal. (i) For an ideal I such that $I = I_{[M]} \cap R$, if I is U-regular then $I \supset (R_{[M]} :_R U)$, and if I is not U-regular then $I \subseteq (R_{[M]} :_R U)$. (ii) If J and K are ideals and J is U-regular then $J_{[M]}$ and $K_{[M]}$ are comparable. The U-regular prime ideals contained in M form a chain whose intersection is $R \cap N$, where $N = S_{[M]}^{-1}(U)$. (iii) A minimal prime ideal of R is not U-regular.

PROOF. Since $(R_{[M]}, M_{[M]})$ is (MV) and $(R_{[M]})_{M_{[M]}} = R_M$, [17, Theorem 2.5] gives that $\overline{R_M}$ is a valuation domain of the field $\overline{U_M}$. Taking $D = (R \backslash M)$ in (4.4), $\mathbb{C}$ is the chain of non-zero $\overline{R_M}$-ideals and $\mathbb{B}$ is the set of all $\overline{R_M}$-ideals which strictly contain the prime ideal $R_M : U_M$. For (i), suppose first that I is not U-regular. Then, by (4.4), I_M is not U_M-regular. By [17, Theorem 2.5], (R_M, M_M) is local (MV) in U_M. Hence, by [18, Lemma 1.3 (1,4)], I_M is contained in all U_M-regular ideals of R_M and so $I_M \subseteq (R_M : U_M)$. Taking inverse images in U we find that $I_{[M]} \subseteq (R_{[M]} : U)$, whence $I \subseteq (R_{[M]} :_R U)$. Now let I be U-regular. Since $I \in \mathbb{A}_0$ we have $I_M \in \mathbb{B}$ and $I_M \supset (R_M : U_M)$. Taking inverse images in R gives $I = I_{[M]} \cap R \supseteq (R_{[M]} :_R U)$. Hence $I \supset (R_{[M]} :_R U)$, since $R_{[M]} \neq U$ so $U(R_{[M]} :_R U) \neq U$ whence the $R_{[M]}$-ideal $R_{[M]} :_R U$ is not U-regular. (ii) If $K_{[M]}$ is U-regular, both $J_{[M]}$ and $K_{[M]}$ are in the set $\mathbb{A}$ of (4.4) and hence they are comparable since $\mathbb{C}$ is a chain. If $K_{[M]}$ is not U-regular then $J_{[M]} \cap R \supset (R_{[M]} :_R U) \supseteq \dot{K}_{[M]} \cap R$, and so $J_{[M]} \supset K_{[M]}$ by $(*)$ in (4.4). The U-regular prime ideals within M correspond to the U-regular prime

$R_{[M]}$-ideals. The latter form a chain and, by (2.1[d]), their intersection is N. (iii) If P is U-regular then $P \supset (R_{[P]} :_R U)$ and $R_{[P]} :_R U$ is prime. ■

(5.2) PROPOSITION. Let P be a U-regular prime ideal. Then P^n is P-primary for all $n \geqslant 1$.

PROOF. First, let M be a maximal ideal such that $M \supseteq P$. By [5, Theorem 17.3(*b*)], $(\overline{P_M})^n$ is a $\overline{P_M}$-primary ideal of the valuation domain $\overline{R_M}$. Since $PU = U$ we deduce that P^n and, hence, $(P^n)_{[M]} \cap R$ are U-regular. Now $((P^n)_{[M]} \cap R)_M = P_M^n$ so that, by the proof of (5.1), $P_M^n \supset (R_M : U_M)$. Thus the inverse image in R of $(\overline{P_M})^n$ is $(P^n)_{[M]} \cap R$, and this is P-primary. Secondly, if M is a maximal ideal such that $M \not\supseteq P$ then $(P^n)_{[M]} = R_{[M]}$. Hence, by [8, Prop. 9], P^n is an intersection of P-primary ideals and so is itself P-primary. ■

(5.3) LEMMA. Let J and Q be U-regular ideals such that Q is P-primary, $\operatorname{rad}(J) \supseteq P$ and $J \not\subseteq Q$. Then $J \supset Q$.

PROOF. Let M be a maximal ideal. If $M \not\supseteq P$ we have $M \not\supseteq \operatorname{rad}(J)$ and so $J_{[M]} = R_{[M]} = Q_{[M]}$. Suppose now that $M \supseteq P$. Then $Q_{[M]} = Q$ and, by (5.1), $J_{[M]}$ and $Q_{[M]}$ are comparable. Hence $J_{[M]} \supseteq Q_{[M]}$. Since $J_{[M]} \supseteq Q_{[M]}$ for all maximal M, it follows by [8, Prop. 9] that $J \supseteq Q$, so $J \supset Q$. ■

For a prime ideal P, we shall write $R_{[P]} \cap S(P) = Y(P)$. The proof of our next result involves adapting the proof of [6, Lemma 3] which, in the domain case, concerned the ring $R' = R_{[P]} \cap \bigcap_{\alpha} R_{[M_\alpha]}$, where $\{M_\alpha\}$ is the set of all maximal ideals not containing P. First, we verify that $R' = Y(P)$ by using (2.1). That $R' \supseteq Y(P)$ is clear. Let Q be a prime ideal such that $Q \not\supseteq P$. If Q is not U-regular then $R_{[Q]} = U \supseteq R'$. Suppose Q is U-regular and let M be a maximal ideal containing Q. If $M \supseteq P$ then $P \supset Q$ by (5.1), since P not U-regular would imply $P \subseteq (R_{[M]} :_R U) \subset Q$; hence $R_{[Q]} \supseteq R_{[P]} \supseteq R'$. If $M \not\supseteq P$ then $R_{[Q]} \supseteq R_{[M]} \supseteq R'$. It follows that $R' \subseteq Y(P)$.

(5.4) PROPOSITION. (i) For a prime ideal P, we have $Y(P) = (P:P)$. (ii) Either $Y(P) = S(P)$, or $Y(P)$ is the unique maximum proper subalgebra of $S(P)$.

PROOF. Using [18, Lemma 1.1(2)] and an argument in [7, Prop. 10], one verifies that each subalgebra is an intersection of rings of form $R_{[Q]}$ with Q a prime ideal. So if A is a subalgebra such that $A \subset S(P)$, then

$A = S(P) \cap \bigcap_{Q \in \mathbf{Q}} R_{[Q]}$ where $\mathbf{Q}$ is a set of prime ideals Q such that $Q \supseteq P$ and so $R_{[Q]} \subseteq R_{[P]}$. Thus $A \subseteq Y(P)$, which gives (ii). For (i), we obtain $(P:P) \supseteq Y(P)$ by adapting [6, Proof of Lemma 3]. Replacing D' and R_P in [6] by $Y(P)$ and $R_{[P]}$, the adaptation is straightforward apart from the need to verify that if M is a maximal ideal containing P then $PR_{[P]} \subseteq R_{[M]}$. For this, note that $R_{[M]} \subseteq R_{[P]}$ and that the R-ideal P is not $R_{[P]}$-regular. Replacing U by $R_{[P]}$ in (5.1) yields $P \subseteq (R_{[M]} : R_{[P]})$. Now suppose $(P:P) \neq Y(P)$. Since $S(P) \supseteq P^{-1} \supseteq (P:P) \supset Y(P)$, (ii) gives $S(P) = (P:P)$. Since $Y(P) \neq S(P)$, we have $R_{[P]} \not\supseteq S(P)$ and so $PS(P) = S(P)$, which contradicts $(P:P) = S(P)$. ■

The domain case of (5.5)(i ⇒ ii) was given in [9, Prop. 1.15].

(5.5) THEOREM. The following conditions are equivalent for a prime ideal P:

(i) $T(P) \neq S(P)$.

(ii) $T(P) = Y(P)$ and $R_{[P]} \not\supseteq S(P)$.

(iii) $T(P) = P^{-1}$ and $R_{[P]} \not\supseteq S(P)$.

(iv) $P = P^2$ and $P = S^{-1}(a)$ for some a.

(v) $P = \mathrm{rad}(I)$ for some U-invertible ideal I such that, for all $n \geq 1$, $P^n \not\subseteq I$.

PROOF. Since $S(P) \supseteq T(P) \supseteq P^{-1} \supseteq Y(P) = R_{[P]} \cap S(P)$ and, by (5.4), there is no ring strictly between $S(P)$ and $Y(P)$, we obtain (i) ⇒ (ii). (ii) ⇒ (iii) is clear. (iii) ⇒ (iv) By (2.3), $P = S^{-1}(a) = \mathrm{rad}(R :_R a)$ for some a. Supposing $P \neq P^2$, choose $x \in (P \backslash P^2)$ and put $Rx + (R :_R a) = J$. Then J is U-invertible, $\mathrm{rad}(J) = P$ and $J \not\subseteq P^2$. By (iii) P is U-regular, and so P^2 is P-primary by (5.2). Therefore $J \supset P^2$ by (5.3), and so $J^{-1} \subseteq P^{-2}$. Now $T(P) = P^{-1} = P^{-2}$. Hence $J^{-1} \subseteq P^{-1}$ and so $P = JJ^{-1}P \subseteq J$, whence $P = J$. Thus P is U-invertible and so $P^{-2} \neq P^{-1}$, a contradiction. We conclude that $P = P^2$. Now assume (iv). Then $P = \mathrm{rad}(I)$ where $I = (R :_R a)$ is U-invertible. Since $P \subseteq I$ would imply $P \neq P^2$, (v) holds. Finally, assuming (v), take $I^{-1} = \sum_{k=1}^{n} Ra_k$. Then, for each k, $P = \mathrm{rad}(I) \subseteq \mathrm{rad}(R :_R a_k)$ and so $a_k \in S(P)$. But if $a_k \in T(P)$ for each k, we would have $I^{-1}P^n \subseteq R$ for some n, and so $P^n \subseteq I$. Thus $a_k \notin T(P)$ for some k, whence (i) holds. ■

(5.6) COROLLARY. For each prime ideal P there is an ideal I such that $S(P) = T(I)$.

PROOF. Suppose $S(P) \neq T(P)$. Then $R_{[P]} \not\supseteq S(P)$ by (5.5), so $PS(P) = S(P)$ and, by (3.2), $S(P) = T(I)$ for some I. ■

(5.7) PROPOSITION. For a prime ideal P, the following conditions are equivalent: (i) $P^{-1} \neq S(P)$, (ii) $R_{[P]} \not\supseteq S(P)$, (iii) P is the radical of a U-invertible ideal.

PROOF. (i) implies $(P:P) \neq S(P)$, whence (ii) by (5.4)(i). For (ii) $\Rightarrow$ (i), let $P^{-1} = S(P)$. Then $(P^{-1})^2 = P^{-1}$ so P is not U-invertible. Let $a \in S(P)$, whence $aP \subseteq R$ and $(R :_R a) \supseteq P$. If $a \notin R_{[P]}$ then $(R :_R a) \subseteq P$ whence $P = (R :_R a)$, contradicting P not U-invertible. (ii) $\Rightarrow$ (iii) by (2.3). Assuming (iii), we may take $P = \operatorname{rad}(L^{-1})$ where $L = \sum_{i=1}^{n} Ra_i$, so that

$$P = \operatorname{rad}\left(\bigcap_i (R :_R a_i)\right) = \bigcap_i \operatorname{rad}(R :_R a_i).$$

Then $P = (R :_R a_i) = S^{-1}(a_i)$ for some i, giving (ii) by (2.3). ■

(5.8) REMARK. One verifies that, for a prime ideal P, the three conditions: $P^{-1} = (P:P)$, $R_{[P]} \supseteq P^{-1}$, and P is not U-invertible, are equivalent. This extends [12, Cor. 3.6] since for a valuation domain in its quotient field $(P:P) = R_P$. ■

(5.9) PROPOSITION. The following conditions are equivalent.

(i) $S(P) = U$ for all prime ideals P which are not U-regular.

(ii) For every subalgebra $A \neq U$, each maximal A-ideal is U-regular.

(iii) $(U, (R:U))$ is a local ring.

PROOF. By (2.1), $S(P) = U$ is equivalent to P being contained in every U-regular prime ideal. For (iii) $\Rightarrow$ (i), P not U-regular implies $PU \neq U$ so that $P \subseteq (R:U)$, and $R:U$ is contained in every U-regular ideal. Assuming (i), let $A \neq U$ be a subalgebra and H a non-U-regular prime A-ideal. For some U-regular prime R-ideal Q we have $A \subseteq R_{[Q]}$ (see proof of (5.4)). Then $QA \neq A$, and QA is a U-regular A-ideal since $QAU = QU = U$. The prime R-ideal $H \cap R$ is not U-regular, so $S(H \cap R) = U$ whence $H \cap R \subseteq Q$. But, from the proof of $(5a) \Rightarrow (5b)$ in [18, Theorem 2.1], $H = (H \cap R)A$ and so $H \subseteq QA$. Thus (ii) holds. Assuming (ii), let M be a U-regular maximal R-ideal. Then, by [18, Theorem 3.4.3], we obtain that $(R_{[M]}, M_{[M]})$ is local and so, by $(1) \Rightarrow (2)$ of [8, Prop. 5], $(U, (R_{[M]} : U))$ is local. By (5.10), $(R:U) = (R_{[M]} : U)$ which gives (iii). ■

(5.10) REMARK. Let $\mathbb{M}$ denote the set of all U-regular maximal R-ideals. By [8, Prop. 9],

$$(R:U) = \bigcap_{a \in U}\left(\left(\bigcap_{M \in \mathbb{M}} R_{[M]}\right) : a\right) = \bigcap_{a}\left(\bigcap_{M}(R_{[M]} : a)\right) = \bigcap_{M}(R_{[M]} : U).$$

Hence $(R:U)$ is radical as a U- and R-ideal. ■

6. - Property (#) in a Prüfer extension.

We say that our Prüfer extension $R \subsetneq U$ satisfies the property (#) *if* $\bigcap_{M \in \mathbb{L}} R_{[M]} \neq \bigcap_{M \in \mathbb{N}} R_{[M]}$ *for all distinct non-empty subsets* $\mathbb{L}$ *and* $\mathbb{N}$ *of the set* $\mathbb{M}$ *of U-regular maximal ideals.* A deep study of this property for Prüfer domains was made in [6]. In considering connections between S^{-1}-transforms and the condition (#), we shall find two properties of valuation domains, which, at least under the ACC on prime ideals, extend to Prüfer domains (and extensions) only to the extent that (##) *holds for* R, *i.e. that for each subalgebra* $A \neq U$, *the extension* $A \subset U$ *satisfies* (#). (The deletion of «U-regular» in our definition of (#) would give an alternative generalisation of the Gilmer-Heinzer definition. One verifies that the effect on property (##) of taking this alternative would be to impose additionally the conditions of (5.9).)

(6.1) LEMMA. If (##) holds and P is a branched U-regular prime ideal then $P = S^{-1}(a)$ for some a.

PROOF. Adapting the proof of [6, Theorem 3 $(a) \Rightarrow (b)$] we obtain a U-invertible ideal $I \subsetneq P$ such that each maximal ideal containing I also contains P. Also, there is a prime ideal Q_0 which is maximal subject to $Q_0 \subset P$. Choose $x \in (P \backslash Q_0)$ and set $I + Rx = J$. Let Q be a prime ideal containing J, and hence U-regular. Then each maximal ideal containing Q must contain P. Hence, by (5.1), the ideals P, Q and Q_0 form a chain. It follows that $Q \supsetneq P$. Hence $\operatorname{rad}(J) = P$ and so, by (5.7) and (2.3), $P = S^{-1}(a)$ for some a. ■

(6.2) LEMMA. For the following conditions on R, we have $(a) \Rightarrow (b)$.

(*a*) Every U-regular prime ideal which is not an S^{-1}-transform is unbranched.

(*b*) Every U-regular prime ideal which is not an S^{-1}-transform is a G-ideal.

PROOF. Let P be a U-regular prime ideal which is not a G-ideal, and let Q be a prime ideal such that $Q \supset P$. Now Q is the union of the set,

$\{Q_\alpha\}$ say, of all prime ideals within it that are minimal over principal ideals. Clearly, each Q_α is either branched or minimal over (0). Also, by (5.1), each Q_α is either part of the chain of U-regular prime ideals within Q or satisfies $Q_\alpha \subsetneq (R_{[Q]} :_R U) \subset P \subset Q$. Thus $Q \supseteq Q_\alpha \supset P$, for some α, where Q_α must be branched and so, by (a), be an S^{-1}-transform. Since P is the intersection of all prime ideals Q such that $Q \supset P$, it is an intersection of S^{-1}-transforms and so is itself an S^{-1}-transform. ■

For R a valuation domain and U its quotient field, the next result is [9, Cor. 2.10].

(6.3) THEOREM. A U-regular prime ideal P is not an S^{-1}-transform if and, when (##) holds, only if P is an unbranched G-ideal.

PROOF. By (5.1)(iii), the «if» part is a special case of (2.5). Combine (6.1) and (6.2) to obtain the «only if» part. ■

The domain case of the equivalence of (ii) and (iii) in the next result is in [6]. The ACCRP (ascending chain condition for U-regular prime ideals) in R implies the ACCRP in every subalgebra A, since contraction to R of prime A-ideals is one-one (by [18]) and maintains U-regularity (since $A_{[H]} \supseteq R_{[H \cap R]}$ for H prime in A).

(6.4) PROPOSITION. The following conditions are equivalent:

(i) In every subalgebra A, each U-regular prime A-ideal is an S_A^{-1}-transform.

(ii) The conditions (##) and ACCRP hold.

(iii) The ACCRP holds and each U-invertible ideal has only finitely many minimal prime ideals.

PROOF. (ii) ⇒ (i) By (5.1) a U-regular prime A-ideal is non-minimal, hence is branched by ACCRP, and hence is an S_A^{-1}-transform by (6.1). (i) ⇒ (ii) For a U-regular prime ideal P, $(R_{[P]}, P_{[P]})$ is (MV) Prüfer in U. By [18, Theorem 3.4.3], $P_{[P]}$ is a maximal U-regular $R_{[P]}$-ideal, hence a G-ideal. By (2.5) and (5.1)(iii), $P_{[P]}$ is branched. Hence P is branched, by the correspondence between prime ideals within P and prime $R_{[P]}$-ideals within $P_{[P]}$. Thus the ACCRP holds. Let H be a maximal U-regular A-ideal. By (2.3), $A_{[H]} \not\supseteq S_A(H)$. Hence $A_{[H]} \not\supseteq \bigcap A_{[K]}$ with the intersection taken over all maximal U-regular A-ideals K such that $K \neq H$; it is easy to see that this property for all H is equivalent to (#) for the extension $A \subset U$. Thus (##) holds.

(iii) ⇒ (ii) holds by natural adaptation of the proof of $(b) \Rightarrow (c)$ in [6, Theorem 4], but for (ii) ⇒ (iii) some deviation from the method in [6] is necessary. Let $\{M_\lambda\}$ be the set of prime ideals minimal over a U-invertible ideal I. As in [6, Prop. 3] we can assume that $\{M_\lambda\}$ is the set M of (5.10). Let $M \in \mathrm{M}$. Then M is branched and so, by (6.1), $M = \mathrm{rad}(J)$ for some U-invertible ideal J. Put $A = J + I$ and $B = IA^{-1}$. Then $B \subseteq R$ and $BA = I \subseteq M_\lambda$ for all λ. If $M_\lambda \neq M$ then $M_\lambda \not\supseteq J$ whence $M_\lambda \not\supseteq A$, and so $M_\lambda \supseteq$ $\supseteq B$. In the local ring R_M, $M_M = \mathrm{rad}(I_M)$ and so $J^n R_M \subseteq I_M$ for some n. Replacing J by J^n we have $A_M = I_M = B_M A_M$. Hence $B_M = R_M$, so $B \not\subseteq M$ and $M + B = R$. For some $x \in M$, $b \in B$ we have $1 - x = b \in M_\lambda$ for all $M_\lambda \neq M$. Finally, that $\{M_\lambda\}$ is finite follows from [4, Lemma 8] applied to the ring R/I. ∎

(6.5) REMARKS. Consider the conditions: (1) Condition (6.4)(i). (2) condition (##). (3) For all subalgebras A, each branched U-regular prime A-ideal is $\mathrm{rad}(A :_A b)$ for some $b \in U$. (4) Condition (a) of (6.2) for all subalgebras. We have (1) ⇒ (2) ⇒ (3) ⇒ (4) by (6.4) and (6.1). Also, (4) ⇒ (1) under the ACCRP since, by (5.1)(iii), no U-regular prime ideal is unbranched. Assuming the ACCRP, (4) ⇒ (2) shows that the validity of the «only if» part of (6.3) for all subalgebras requires (##).

Example 2 in [6] gives a 2-dimensional Prüfer domain such that condition (2) fails. Hence (1) and, since the ACCRP holds, (3) and (4) also fail. In particular, there exists an overring A containing a branched U-regular prime A-ideal which is not an S_A^{-1}-transform.

Finally, for R a Prüfer domain with quotient field U, condition (3) is effectively the equivalence, for all prime ideals $P \neq 0$, of the conditions «P is branched» and «P is the radical of an invertible ideal» proved in [5, Theorem 17.3(e)] for a valuation domain. By the foregoing remarks, this equivalence fails in general Prüfer domains but holds if and, assuming the ACCRP, only if (##) holds. ∎

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