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Optimal Segmentation of Unbounded Functions.

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ABSTRACT - We study a variational problem connected to image segmentation in computer vision. More precisely, assuming that Ω is open in $\mathbb{R}^n$, $g \in L^p(\Omega) \cap \cap L^{np}_{\mathrm{loc}}(\Omega)$ and $\lambda > 0$, we prove that the functional $F(C, u) = \lambda \int_{\Omega \setminus C} |u - - g|^p dx + \mathcal{H}^{n-1}(C \cap \Omega)$ achieves its minimum on pairs (C, u) with C closed and u constant on each connected component of $\Omega \setminus C$. Moreover, we show that the family of connected components of $\Omega \setminus K$ is locally finite in Ω, for any minimizer (K, w) of F.

1. - Introduction.

Given an open set Ω of $\mathbb{R}^n$, a function $g \in L^p(\Omega)$ and a positive coefficient λ, one can consider for C closed in $\mathbb{R}^n$ and u smooth in $\Omega \setminus C$, with vanishing gradient everywhere in $\Omega \setminus C$, the following functional:

$$F(C, u) = \lambda \int_{\Omega \setminus C} |u - g|^p dx + \mathcal{H}^{n-1}(C \cap \Omega).$$

Notice that $\Omega \setminus C$ is partitioned into open connected sets on which u is constant, so that the possible discontinuities of u are included in C. The minimizers of F thus yield «optimal approximations» of g by «piecewise constant functions» having «not too many discontinuities» in Ω ($\mathcal{H}^{n-1}$ denotes Hausdorff $(n-1)$-dimensional measure).

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Variational problems of this type, with jumping solutions, have interesting motivations: they are encountered for instance in Computer Vision Theory, where a basic problem is to obtain «optimal segmentations» of a given image. In the usual 2-dimensional setting, given a function g defined on a plane domain Ω (describing the «grey-level» at points of the image), one would appropriately decompose Ω in uniform regions (the main «objects» in the image) where g is replaced by constant values u, separated by sharp edges (the «contours» of the objects), by means of a closed set of curves C. The piecewise constant function u obtained in this way is a segmentation of the original image g. The minimization of F as a criterion to sort out «optimal segmentations» was essentially proposed by D. Mumford and J. Shah: we refer the reader to the paper [10] for further information on the subject. See also [9].

The investigation of the functional F in general dimension began with work of the authors and U. Massari. Precisely, we proved in [3] the existence of minimizers of F, under the additional assumption of a *bounded* datum: $g \in L^{\infty}(\Omega)$. Regularity results of the solutions are presented in [8]. See also [6, 7, 11, 12].

The aim of the present paper is to remove that boundedness assumption: we will show in particular that if $g \in L^{np}_{\mathrm{loc}}(\Omega)$ then there exists (K, w) minimizing F, and the family of connected components of $\Omega \setminus K$ is locally finite in Ω (see Theorem 2.1 below); on the other hand, we exhibit a function $g \in L^{q}(\Omega)$ for all $q < np$, for which no minimizer of F can exist (see Example 1).

The two main tools employed in the sequel are: (i) an *elimination lemma* (Lemma 5.3 below), which extends to unbounded functions an analogous result of [8]; and (ii) a *convergence result* (Theorem 6.1), which is used in connection with a blow-up procedure. Of course, we rely heavily on results and methods developed in our preceding papers (among them, the recourse to a weak formulation of the minimum problem, and the local finiteness of *minimal partitions*, i.e. partitions of Ω with locally least-area interfaces). However, the general setting for the *weak formulation* of the minimization problem is slightly different here - we work essentially with *pairs* $(\mathcal{U}, u)$ where $\mathcal{U}$ is a *partition* of Ω and u an associated *piecewise constant function* (i.e., u is constant on each element of $\mathcal{U}$).

The resulting treatment is even simpler, in that we can avoid any reference to functions of generalized bounded variation and related concepts, the analysis of which takes a large part of the paper [3].

We now give an outline of the paper. Section 2 is devoted to the statement of the Main Theorem and the construction of the counterexample. Notation and known facts about partitions of locally finite total perimeter are reviewed in Section 3. The weak formulation is introduced in Section 4, while in Section 5 we prove the Elimination Lemma and the existence of minimizers of F. The convergence result is derived in Section 6 and the proof of the Main Theorem is then concluded in Section 7. In the last section 8 a comparison is made between the weak formulation used here and that of the preceding papers [3], [8].

The results of the present paper were announced in [4].

2. - Statement of the main result.

Throughout this paper, we denote by n an integer ≥ 2, by p and λ two real numbers with $p \geq 1$ and $\lambda > 0$, by Ω on open subset of the Euclidean n-dimensional space $\mathbb{R}^n$, by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$, and by g a measurable, real valued function defined on Ω.

For C closed in $\mathbb{R}^n$ and $u \in C^1(\Omega \backslash C)$ s.t. $\nabla u(x) = 0$ for all $x \in \Omega \backslash C$ (i.e., u is constant on the connected components of $\Omega \backslash C$), we define

$$F(C, u) = \lambda \int_{\Omega \backslash C} |u - g|^p dx + \mathcal{H}^{n-1}(C \cap \Omega). \tag{2.1}$$

Any pair (C, u) satisfying the preceding requirements will be called *admissible* for the functional F.

We are going to prove the following:

THEOREM 2.1. *When $g \in L^p(\Omega) \cap L^{np}_{\mathrm{loc}}(\Omega)$, the functional F in* (2.1) *achieves its minimum, i.e. there exists an admissible pair (K, w) s.t. $F(K, w) \leq F(C, u)$ for all admissible pairs (C, u). In this case moreover the function w takes on a finite number of values in $D \backslash K$, for all compact subsets D of Ω.*

Actually, we will prove something more, in particular that the family of connected components of $\Omega \backslash K$ is locally finite in Ω: see Propositions 5.4 and 7.1. We remark that when $g \in L^p(\Omega) \cap L^\infty(\Omega)$, the preceding result is proved in [3, 8]. The choice $g \in L^{np}_{\mathrm{loc}}(\Omega)$ is optimal, as the following example shows:

EXAMPLE 1. With n, Ω, λ and p as above, we choose a sequence of points $\{x_h\}$ of Ω, dense in Ω, and put for $h \in \mathbb{N}$:

$a = (2n/\lambda)^{1/p}$, $r_h = e^{-hp}$,

$B_h = B_{x_h, r_h}$ (the ball of centre x_h and radius r_h),

$g_h = ae^h \chi_{B_h}$ (χ_B is the characteristic function of B),

$g = \sum_{h=1}^{\infty} g_h .$

It is easily seen that $g \in L^q(\Omega)$ for all q with $1 \leqslant q < np$. Arguing by contradiction, let us assume the existence of a minimizing pair (K, w) of F; being $F(\emptyset, 0) = \lambda \int_\Omega |g|^p dx < +\infty$, we obtain in particular that $\mathcal{H}^{n-1}(K \cap \Omega) < +\infty$, hence $|K \cap \Omega| = 0$ and $w \in L^p(\Omega)$ ($|B|$ denotes the Lebesgue measure of $B \subset \mathbb{R}^n$).

Let now A be a connected component of $\Omega \backslash K$ and denote by t the (constant, positive) value of w on A. Since $\{x_h\}$ is dense in A, we can certainly find a positive integer k s.t.

$$B_k \subset\subset A \quad \text{and} \quad te^{-k} < a(1 - 2^{-1/p}). \tag{2.2}$$

Setting $C = K \cup \partial B_k$ and

$$u = \begin{cases} ae^k & \text{in } B_k , \\ t & \text{in } A \backslash \overline{B}_k , \\ w & \text{in } \Omega \backslash (A \cup K), \end{cases}$$

we find that (C, u) is admissible for F. However:

$$F(K, w) - F(C, u) = \lambda \int_{B_k} (|t - g(x)|^p - |ae^k - g(x)|^p)\, dx - \mathcal{H}^{n-1}(\partial B_k) \geqslant$$

$$\geqslant \lambda (ae^k - t)^p |B_k| - \mathcal{H}^{n-1}(\partial B_k) > 0$$

by virtue of (2.2), thus contradicting the minimality of (K, w).

It follows that in this case functional F has no minimizer at all.

3. - Caccioppoli partitions.

In order to prove Theorem 2.1, we find it convenient to introduce a certain *weak formulation* of functional (2.1), in terms of a class of partitions of Ω with *locally finite total perimeter*.

Let $\mathcal{U}$ be a countable family of (measurable) subsets of $\mathbb{R}^n$; $\mathcal{U}$ is a *Caccioppoli partition* of Ω (shortly, $\mathcal{U} \in CP(\Omega)$) if and only if one can find a sequence $\{U_i\}$ s.t.

$$\text{(3.1)}\quad \begin{cases} \mathcal{U} = \{U_i\colon i \in \mathbb{N}\}, & \left|\Omega \setminus \bigcup_{i=1}^{\infty} U_i\right| = 0, \\ U_i = U_i(1) \quad \forall i \in \mathbb{N}, & U_i \cap U_j = \emptyset \quad \text{whenever } i \neq j, \\ \sum_{i=1}^{\infty} P(U_i, A) < \infty & \forall A \text{ open } \subset\subset \Omega. \end{cases}$$

Here, $|U|$ is the Lebesgue measure of $U \subset \mathbb{R}^n$, $P(U, A)$ is the perimeter of U in the open set A, and $U(1)$ is the set of points of density 1 for U, i.e.

$$P(U, A) = \sup\left\{\int_U \operatorname{div}\phi(x)\,dx\colon \phi \in C_0^1(A; \mathbb{R}^n),\ |\phi(x)| \leq 1\ \forall x \in A\right\},$$

$$U(\alpha) = \left\{x \in \mathbb{R}^n\colon \lim_{r \to 0} |U \cap B_{x,r}|/|B_{x,r}| = \alpha\right\} \qquad \forall \alpha \in [0, 1],$$

Of course, $B_{x,r}$ denotes the open Euclidean ball of centre x and radius r in $\mathbb{R}^n$. When the center is at the origin we simply write B_r and put $\omega_n = |B_1|$. The notation $A \subset\subset \Omega$ means that A is a relatively compact subset of Ω.

It is often convenient to assume that the empty set belongs to the Caccioppoli partition $\mathcal{U}$. Furthermore, $\emptyset$ (and only it!) can appear repeatedly among the terms of any sequence $\{U_i\}$ representing $\mathcal{U}$ as in (3.1)—this being the case for instance when $\mathcal{U}$ is finite. Any such sequence $\{U_i\}$ will be called an *arrangement* of $\mathcal{U}$.

General properties of Caccioppoli partitions are presented in [3]; among them we have that if $\mathcal{U} \in CP(\Omega)$ and $\{U_i\}$ is an arrangement of $\mathcal{U}$, then:

$$\text{(3.2)}\qquad 2\mathcal{H}^{n-1}\left(A \setminus \bigcup_{i=1}^{\infty} U_i\right) = \sum_{i=1}^{\infty} P(U_i, A),$$

$$\text{(3.3)}\qquad \mathcal{H}^{n-1}\left[\left(A \setminus \bigcup_{i=1}^{\infty} U_i\right) \setminus \bigcup_{i \neq j}^{\infty}\left(U_i\left(\frac{1}{2}\right) \cap U_j\left(\frac{1}{2}\right)\right)\right] = 0,$$

for all A open $\subset \Omega$. The value of $\sum_{i=1}^{\infty} P(U_i, A)$ is thus independent of the particular sequence $\{U_i\}$ chosen to represent $\mathcal{U}$; we put

$$\text{(3.4)}\qquad P(\mathcal{U}, A) = \frac{1}{2} \sum_{i=1}^{\infty} P(U_i, A),$$

for any open $A \subset \Omega$ and any arrangement $\{U_i\}$ of $\mathcal{U}$, and call this quantity the *total perimeter* of $\mathcal{U}$ in A.

Given $\mathcal{U}$, $\mathcal{V} \in CP(\Omega)$ and A open $\subset \Omega$, we say that $\mathcal{U}$ *coincides with* $\mathcal{V}$ *in* A (written $\mathcal{U} = \mathcal{V}$ in A) if and only if arrangements $\{U_i\}$ of $\mathcal{U}$ and $\{V_i\}$ of $\mathcal{V}$ can be found s.t. $U_i \cap A = V_i \cap A$, $\forall i \in \mathbb{N}$.

Given $\mathcal{U}_h \in CP(\Omega)$ $\forall h \in \mathbb{N} \cup \{\infty\}$, we say that $\mathcal{U}_h$ *converges to* $\mathcal{U}_\infty$ *locally in* Ω (written $\mathcal{U}_h \to \mathcal{U}_\infty$ in $L^1_{\mathrm{loc}}(\Omega)$) if and only if arrangements $\{U_{h,i}\}$ of $\mathcal{U}_h$ can be found s.t.

$$(3.5)\qquad \lim_{h\to\infty} \int_A |\chi_{U_{h,i}} - \chi_{U_{\infty,i}}|\,dx = 0 \qquad \forall i \in \mathbb{N} \text{ and } \forall A \text{ open } \subset\subset \Omega,$$

or equivalently

$$(3.6)\qquad \lim_{h\to\infty}\left(\sum_{i=1}^{\infty}\int_A |\chi_{U_{h,i}} - \chi_{U_{\infty,i}}|\,dx\right) = 0 \qquad \forall A \text{ open } \subset\subset \Omega.$$

The total perimeter is lower-semicontinuous with respect to this convergence, i.e.

$$(3.7)\qquad \text{if } \mathcal{U}_h \to \mathcal{U}_\infty \text{ in } L^1_{\mathrm{loc}}(\Omega), \text{ then } P(\mathcal{U}_\infty, A) \le \liminf_{h\to\infty} P(\mathcal{U}_h, A)$$

for all A open $\subset \Omega$.

Now consider a Caccioppoli partition $\mathcal{U}$ of Ω and a function

$$u\colon \cup\mathcal{U} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$$

(u is thus defined a.e. on Ω). We say that $(\mathcal{U}, u)$ is a *weighted Caccioppoli partition* of Ω (shortly: $(\mathcal{U}, u) \in WCP(\Omega)$) if and only if u is constant on each element of $\mathcal{U}$; we denote by $u(U) \in \overline{\mathbb{R}}$ the (common) value of u at all points of $U \in \mathcal{U}$. We point out explicitly that $u(U)$ might coincide with $u(U')$ even if $U \cap U' = \emptyset$ $(U, U' \in \mathcal{U})$.

The following result is a consequence of Theorem 1.6 of [3]:

THEOREM 3.1 (Compactness). *Suppose that* $(\mathcal{U}_h, u_h) \in WCP(\Omega)$ $\forall h \in \mathbb{N}$ *be such that* $\sup_{h\in\mathbb{N}} P(\mathcal{U}_h, A) < \infty$ $\forall A$ *open* $\subset\subset \Omega$. *Then we can find* $(\mathcal{U}_\infty, u_\infty) \in WCP(\Omega)$ *and a subsequence* $\{(\mathcal{U}_{k(h)}, u_{k(h)})\}$ *of the given sequence* $\{(\mathcal{U}_h, u_h)\}$ *s.t.*

$$\mathcal{U}_{k(h)} \to \mathcal{U}_\infty \quad \text{in } L^1_{\mathrm{loc}}(\Omega),$$

$$u_{k(h)} \to u_\infty \quad \text{a.e. in } \Omega,$$

as $h \to \infty$.

4. – Weak solutions.

For $(\mathcal{U}, u) \in WCP(\Omega)$ we define

$$G(\mathcal{U}, u) = P(\mathcal{U}, \Omega) + \lambda \int_\Omega |u - g|^p\, dx\,. \tag{4.1}$$

Clearly, $0 \leqslant \inf\{G(\mathcal{U}, u): (\mathcal{U}, u) \in WCP(\Omega)\} \leqslant \lambda \int_\Omega |g|^p\, dx$ (by considering $\mathcal{U} = \{\mathbb{R}^n, \emptyset\}$ and $u = 0$). On the account of the Compactness Theorem 3.1 and of the semicontinuity (3.7), the direct method of the calculus of variations yields immediately the following existence result:

THEOREM 4.1. *If $g \in L^p(\Omega)$, then there exists $(\mathcal{W}, w) \in WCP(\Omega)$ s.t. $G(\mathcal{W}, w) \leqslant G(\mathcal{U}, u)$ for all $(\mathcal{U}, u) \in WCP(\Omega)$. In addition, $P(\mathcal{W}, \Omega) < \infty$, $w \in L^p(\Omega)$ and $\int_\Omega |w|^p\, dx \leqslant 2^p \int_\Omega |g|^p\, dx$.*

The following section is devoted to showing how to obtain a minimizer of F in (2.1) from a minimizer of G.

5. – Local minimizers and the elimination lemma.

We now introduce a *localized version* of the functional G of the preceding section. Thus, for $g \in L^p_{loc}(\Omega)$, A open $\subset\subset \Omega$ and $(\mathcal{U}, u) \in WCP(\Omega)$ we define

$$G_g(\mathcal{U}, u; A) = P(\mathcal{U}, A) + \lambda \int_A |u - g|^p\, dx \tag{5.1}$$

and say that $(\mathcal{W}, w) \in WCP(\Omega)$ is a *local minimizer* of G_g *in* Ω if and only if for all A open $\subset\subset \Omega$ it holds:

$$G_g(\mathcal{W}, w; A) < \infty \tag{5.2}$$

and

$$G_g(\mathcal{W}, w; A) \leqslant G_g(\mathcal{U}, u; A)\,, \tag{5.3}$$

for all $(\mathcal{U}, u) \in WCP(\Omega)$ satisfying $\mathcal{U} = \mathcal{W}$ and $u = w$ in $A \backslash D$, for some compact subset D of A. Any such $(\mathcal{U}, u)$ is a *compact variation* of $(\mathcal{W}, w)$ in A.

Clearly, any (global) minimizer of G (given by Theorem 4.1 for a fixed $g \in L^p(\Omega)$), is a local minimizer of G_g in the preceding sense.

We now begin to prove some useful facts about local minimizers.

LEMMA 5.1. *Let $(\mathcal{W}, w)$ be a local minimizer of G_g in Ω and $B \equiv B_{x,r} \subset\subset \Omega$. Then*

$$(5.4) \qquad G_g(\mathcal{W}, w; B) \leq \mathcal{H}^{n-1}(\partial B) + \lambda \int_B |g|^p dx,$$

$$(5.5) \qquad |w(W)| \cdot |W \cap B|^{1/p} \leq (\lambda^{-1} \mathcal{H}^{n-1}(W \cap \partial B))^{1/p} + 2\|g\|_{L^p}(W \cap B),$$

for all $W \in \mathcal{W}$ (recall that $w(W)$ is the constant value of w on W).

PROOF. Without loss of generality, we can assume that $\emptyset \in \mathcal{W}$. Let $\{W_i\}$ be an arrangement of $\mathcal{W}$, with $W_1 = \emptyset$. Put $U_1 = B$, $U_i = W_i \setminus \overline{B}$ if $i \geq 2$, and $\mathcal{U} = \{U_i \colon i \in \mathbb{N}\}$. Moreover, put $u(U_1) = 0$ and $u(U_i) = w(W_i)$ if $i \geq 2$ and $U_i \neq \emptyset$. Fix A open with $B \subset\subset A \subset\subset \Omega$. Then $(\mathcal{U}, u)$ is a compact variation of $(\mathcal{W}, w)$ in A, since clearly $\mathcal{U} = \mathcal{W}$ and $u = w$ in $A \setminus \overline{B}$, and from (5.3) we get easily (5.4).

As for (5.5), fix $W \in \mathcal{W}$ with $W \cap B \neq \emptyset$ and put $t = w(W)$; since $g \in L^p_{loc}(\Omega)$, we have $t \in \mathbb{R}$, by (5.2). Let $\{W_i\}$ be an arrangement of $\mathcal{W}$ with $W_1 = \emptyset$, $W_2 = W$. Put $U_1 = W \cap B$, $U_2 = W \setminus \overline{B}$, $U_i = W_i$ for $i \geq 3$, and $\mathcal{U} = \{U_i \colon i \in \mathbb{N}\}$. Moreover, put $u(U_1) = 0$ and $u(U_i) = w(W_i)$ if $i \geq 2$ and $U_i \neq \emptyset$. Then $(\mathcal{U}, u)$ is a compact variation of $(\mathcal{W}, w)$ in A, for all A open s.t. $B \subset\subset A \subset\subset \Omega$, hence from (5.3) we get easily

$$\lambda \int_{W \cap B} |t - g(y)|^p dy \leq \mathcal{H}^{n-1}(W \cap \partial B) + \lambda \int_{W \cap B} |g(y)|^p dy$$

and (5.5) follows at once. ■

When $\Omega = \mathbb{R}^n$ and $g = 0$, we have an interesting consequence of Lemma 5.1. First of all, we introduce a new functional defined for $\mathcal{V} \in CP(\Omega)$ and A open $\subset\subset \Omega$:

$$(5.6) \qquad \Psi(\mathcal{V}, A) = P(\mathcal{V}, A) - \inf\{P(\mathcal{U}, A) \colon \mathcal{U} \in CP(\Omega),$$
$$\mathcal{U} = \mathcal{V} \text{ in } A \setminus D \text{ for some } D \text{ compact } \subset A\}.$$

Clearly, $\Psi \geq 0$; when $\Psi(\mathcal{V}, A) = 0$ for all A open $\subset\subset \Omega$, we say that $\mathcal{V}$ is a *minimal* (or *least-area*) partition of Ω—any compact variation increases its total perimeter. Various properties of Ψ are studied in [8].

PROPOSITION 5.2. *Let $(\mathcal{W}, w)$ be a local minimizer of G_0 in $\mathbb{R}^n$ (i.e., with $g = 0$). Then $\mathcal{W}$ is a minimal partition of $\mathbb{R}^n$ and $w = 0$ a.e. in $\mathbb{R}^n$.*

PROOF. Choose $W \in \mathcal{W}$ and $R > 0$ s.t. $W \cap B_R \neq \emptyset$ and define

$$\alpha(r) = |W \cap B_r| .$$

Then α is non-decreasing on $(0, +\infty)$, with $\alpha'(r) = \mathcal{H}^{n-1}(W \cap \partial B_r)$ for almost all $r > 0$ and

$$0 \leqslant \alpha(r) \leqslant \omega_n r^n .$$

From (5.5) we get for almost all $r > R$:

$$\lambda |w(W)|^p \cdot \alpha(r) \leqslant \alpha'(r) .$$

Since $\alpha(r) \geqslant \alpha(R) > 0$, we deduce from this that $w(W) = 0$. Thus $w \equiv 0$ on $\cup \mathcal{W}$. Now, if A is open $\subset\subset \mathbb{R}^n$, if $\mathcal{U} \in CP(\mathbb{R}^n)$ is s.t. $\mathcal{U} = \mathcal{W}$ outside some compact $D \subset A$, and if $u \equiv 0$, we find from (5.3)

$$P(\mathcal{W}, A) \leqslant P(\mathcal{U}, A)$$

thus proving that $\Psi(\mathcal{W}, A) = 0$. ∎

Our next result is of basic importance.

LEMMA 5.3 (Elimination Lemma). *Let $(\mathcal{W}, w) \in WCP(\Omega)$ be a local minimizer of G_g in Ω, and assume $g \in L^{np}_{loc}(\Omega)$. Fix an arrangement $\{W_i\}$ of $\mathcal{W}$, and put $t_i = w(W_i)$. For $m \in \mathbb{N}$, define*

$$\eta = \eta(n, m) = \omega_n 4^{-n}(m+1)^{-n} \tag{5.7}$$

and for $x \in \Omega$ select $\bar{s} \in (0, \operatorname{dist}(x, \partial\Omega))$ s.t.

$$2m\lambda \cdot \max_{1 \leqslant i \leqslant m} \left\{ \left(\int_{B_{x,\bar{s}}} |t_i - g(y)|^{np} \, dy \right)^{1/n} \right\} \leqslant n\omega_n^{1/n} . \tag{5.8}$$

If for $s \in (0, \bar{s}]$ it holds $\left| B_{x,s} \setminus \bigcup_{i=1}^m W_i \right| \leqslant \eta s^n$, then $\left| B_{x,s/2} \setminus \bigcup_{i=1}^m W_i \right| = 0$.

PROOF. A similar result has been proved in [8], Theorem 1, in the case when $g \in L^\infty(\Omega)$; here, with a simpler proof (based on pp. 255-256 of [2]) we cover a more general situation.

For fixed $x \in \Omega$ and $m \in \mathbb{N}$ define

$$V_m = \bigcup_{i=m+1}^{\infty} W_i \quad \text{and} \quad \alpha(r) = |V_m \cap B_{x,r}|. \tag{5.9}$$

We have to prove that if $\alpha(s) \leq \eta s^n$ for a certain $s \in (0, \bar{s}]$, then $\alpha(s/2) = 0$. Notice that $\alpha(r)$ is a non-decreasing, absolutely continuous function in $[s/2, s]$, with

$$\alpha'(r) = \int_{\partial B_{x,r}} \chi_{V_m} \, d\mathcal{H}^{n-1} \quad \text{for almost all } r \in (s/2, s). \tag{5.10}$$

In addition, for almost all $r \in (s/2, s)$ we have

$$P(W_i, \partial B_{x,r}) = 0 \quad \forall i \in \mathbb{N}. \tag{5.11}$$

We therefore assume

$$\alpha(r) > 0 \quad \forall r \in (s/2, s), \tag{5.12}$$

$$\alpha(s) \leq \eta s^n, \tag{5.13}$$

fix $r \in (s/2, s)$ such that (5.11) holds, and select W_q, $q \in \{1, \ldots, m\}$, as anyone of the W_i's, $i = 1, \ldots, m$, having the *greatest contact* with V_m in $B_{x,r}$, so that

$$\begin{aligned} m\mathcal{H}^{n-1}\left(W_q\left(\frac{1}{2}\right) \cap V_m\left(\frac{1}{2}\right) \cap B_{x,r}\right) \geq \\ \geq \sum_{i=1}^{m} \mathcal{H}^{n-1}\left(W_i\left(\frac{1}{2}\right) \cap V_m\left(\frac{1}{2}\right) \cap B_{x,r}\right). \end{aligned} \tag{5.14}$$

Now we put

$$\begin{aligned} &U_q = (W_q \cup (V_m \cap B_{x,r}))(1), \\ &U_i = W_i \quad \text{if } i \in \{1, \ldots, m\},\ i \neq q, \\ &U_i = W_i \setminus \overline{B}_{x,r} \quad \text{if } i \geq m+1. \end{aligned}$$

Then, see (3.1), $\mathcal{U} = \{U_i \colon i \in \mathbb{N}\} \in CP(\Omega)$ and $\mathcal{U} = \mathcal{W}$ in $\Omega \setminus \overline{B}_{x,r}$. We also put $u(U_i) = w(W_i)$ for all $i \in \mathbb{N}$ s.t. $U_i \neq \emptyset$. $(\mathcal{U}, u)$ is then a compact variation of $(\mathcal{W}, w)$ in $B_{x,s}$, and from (5.3) we get

$$0 \leq P(\mathcal{U}, B_{x,s}) - P(\mathcal{W}, B_{x,s}) + \lambda \int_{B_{x,s}} (|u-g|^p - |w-g|^p)\, dy. \tag{5.15}$$

We compute (see (3.4)):

$$P(\mathcal{U}, B_{x,s}) = \frac{1}{2} \sum_{i=1}^{\infty} P(U_i, B_{x,s}) =$$

$$= \frac{1}{2}\left\{ \sum_{\substack{i=1 \\ i \neq q}}^{m} P(W_i, B_{x,s}) + P(W_q \cup V_m, B_{x,r}) + P(W_q, B_{x,s} \setminus \overline{B}_{x,r}) + \right.$$

$$\left. + \sum_{i=m+1}^{\infty} P(W_i, B_{x,s} \setminus \overline{B}_{x,r}) + 2 \int_{\partial B_{x,r}} \chi_{V_m} d\mathcal{H}^{n-1} \right\},$$

$$P(\mathcal{W}, B_{x,s}) = \frac{1}{2}\left\{ \sum_{\substack{i=1 \\ i \neq q}}^{m} P(W_i, B_{x,s}) + P(W_q, B_{x,r}) + P(W_q, B_{x,s} \setminus \overline{B}_{x,r}) + \right.$$

$$\left. + \sum_{i=m+1}^{\infty} P(W_i, B_{x,r}) + \sum_{i=m+1}^{\infty} P(W_i, B_{x,s} \setminus \overline{B}_{x,r}) \right\},$$

(recall (5.11)). Simplifying and recalling that

$$P(E \cup F, A) - P(E, A) = P(F, A) - 2\mathcal{H}^{n-1}\left(E\left(\frac{1}{2}\right) \cap F\left(\frac{1}{2}\right) \cap A \right),$$

whenever E, F have finite perimeter in the open set $A \subset \mathbb{R}^n$ and $E \cap \cap F \cap A = \emptyset$, we get

$$P(\mathcal{U}, B_{x,s}) - P(\mathcal{W}, B_{x,s}) =$$

$$\frac{1}{2}\left\{ P(V_m, B_{x,r}) - 2\mathcal{H}^{n-1}\left(W_q\left(\frac{1}{2}\right) \cap V_m\left(\frac{1}{2}\right) \cap B_{x,r} \right) + \right.$$

$$\left. - \sum_{i=m+1}^{\infty} P(W_i, B_{x,r}) + 2 \int_{\partial B_{x,r}} \chi_{V_m} d\mathcal{H}^{n-1} \right\}$$

$$\leqslant -\frac{1}{m} \sum_{i=1}^{m} \mathcal{H}^{n-1}\left(W_i\left(\frac{1}{2}\right) \cap V_m\left(\frac{1}{2}\right) \cap B_{x,r} \right) +$$

$$+ \int_{\partial B_{x,r}} \chi_{V_m} d\mathcal{H}^{n-1} \qquad \text{(by (5.14) and (5.9))} =$$

$$= -\frac{1}{m} P(V_m, B_{x,r}) + \int_{\partial B_{x,r}} \chi_{V_m} d\mathcal{H}^{n-1} =$$

$$= -\frac{1}{m} P(V_m \cap B_{x,r}, \mathbb{R}^n) + \left(1 + \frac{1}{m}\right) \int_{\partial B_{x,r}} \chi_{V_m} d\mathcal{H}^{n-1} \leqslant$$

$$\leqslant -\frac{n\omega_n^{1/n}}{m} |V_m \cap B_{x,r}|^{(n-1)/n} + \left(1 + \frac{1}{m}\right) \int_{\partial B_{x,r}} \chi_{V_m} d\mathcal{H}^{n-1}$$

thanks to the isoperimetric inequality. On the other hand, by the definition of u:

$$\lambda \int_{B_{x,s}} (|u-g|^p - |w-g|^p)\, dy \leqslant \lambda \int_{V_m \cap B_{x,r}} |t_q - g(y)|^p dy \leqslant$$

$$\leqslant \frac{n\omega_n^{1/n}}{2m} |V_m \cap B_{x,r}|^{(n-1)/n},$$

thanks to Hölder's inequality and (5.8). In conclusion, by (5.15), (5.9), (5.10) we get:

$$0 \leqslant 2(m+1)\,\alpha'(r) - n\omega_n^{1/n} \alpha(r)^{(n-1)/n}$$

that is, owing to (5.12):

$$\omega_n^{1/n}/2(m+1) \leqslant (\alpha^{1/n}(r))'$$

which holds for almost all $r \in (s/2, s)$. A straightforward integration then gives, recalling (5.13) and (5.7): $\alpha(s/2) = 0$, as was to be proved. ■

With the aid of Lemma 5.3 we can prove the first half of Theorem 2.1. For, assume that $g \in L^p(\Omega) \cap L^{np}_{\mathrm{loc}}(\Omega)$, and call $(\mathcal{W}, w)$ a (global) minimizer of G—see Theorem 4.1.

Fix $W \in \mathcal{W}$ s.t. $W \cap \Omega \neq \emptyset$, and fix an arrangement $\{W_i\}$ of $\mathcal{W}$ s.t. $W_1 = W$. By (3.1), $W_1 = W_1(1)$; thus, if $x \in \Omega \cap W$, we can find $s \in (0, \bar{s})$ s.t. $|B_{x,s} \setminus W_1| < \eta s^n$. By Lemma 5.3 (with $m = 1$), $B_{x,s/2} \subset W$, and we conclude that:

$$W \cap \Omega \quad \text{is open, for all } W \in \mathcal{W}.$$

We define

$$K = \overline{\Omega \setminus \cup \mathcal{W}}.$$

By restricting w to $\Omega \cap \bigcup \mathcal{W} = \Omega \backslash K$, we get an admissible pair (K, w) for functional F (see Section 2), with

$$F(K, w) = G(\mathcal{W}, w) \tag{5.16}$$

since $K \cap \Omega = \overline{\Omega \backslash \bigcup \mathcal{W}}$ (see (2.1), (3.2), (3.4), (4.1)).

Let now (C, u) be another admissible pair for F, with

$$\mathcal{H}^{n-1}(C \cap \Omega) < +\infty, \qquad u \in L^p(\Omega)$$

and denote by $\mathcal{U}$ the family of connected components of $\Omega \backslash C$.

Put $\mathcal{V} = \mathcal{U}(1)$, i.e. $V \in \mathcal{V}$ iff $V = U(1)$ with $U \in \mathcal{U}$, and put $v(V) = u(U)$ in this case. It is easily seen that $(\mathcal{V}, v) \in WCP(\Omega)$, with

$$P(\mathcal{V}, \Omega) \leq \mathcal{H}^{n-1}(C \cap \Omega).$$

Therefore $F(K, w) = G(\mathcal{W}, w) \leq G(\mathcal{V}, v) \leq F(C, u)$, showing that (K, w) is a minimizer of F. Thus, from any minimizer of G we obtain a minimizer of F.

Reciprocally, let (K, w) be a minimizer of F (we are still assuming that $g \in L^p(\Omega) \cap L^{np}_{\text{loc}}(\Omega)$); by comparison with the admissible pair $(\emptyset, 0)$ we get

$$\mathcal{H}^{n-1}(K \cap \Omega) < +\infty, \qquad w \in L^p(\Omega).$$

As before, denote by $\mathcal{U}$ the family of connected components of $\Omega \backslash K$, and put $\mathcal{V} = \mathcal{U}(1)$, $v(V) = w(U)$ iff $V = U(1)$. Then $(\mathcal{V}, v) \in WCP(\Omega)$ and $P(\mathcal{V}, \Omega) \leq \mathcal{H}^{n-1}(K \cap \Omega)$, so that

$$G(\mathcal{V}, v) \leq F(K, w). \tag{5.17}$$

According to Theorem 4.1, call $(\mathcal{W}_0, w_0)$ a minimizer of G: from the preceding considerations, the pair (K_0, w_0) with $K_0 = \overline{\Omega \backslash \bigcup \mathcal{W}_0}$ is a minimizer of F, whence:

$$F(K_0, w_0) = G(\mathcal{W}_0, w_0), \tag{5.18}$$

(see (5.16)). Since evidently $F(K, w) = F(K_0, w_0)$, we deduce from (5.17) and (5.18) that $(\mathcal{V}, v)$ minimizes G; in particular, $P(\mathcal{V}, \Omega) = \mathcal{H}^{n-1}(\Omega \cap K)$. Applying the preceding argument once more, we obtain a *new* minimizer (K_1, w_1) of F, with

$$K_1 = \overline{\Omega \backslash \bigcup \mathcal{V}}, \qquad w_1 = v \qquad \text{(restricted to } \Omega) \tag{5.19}$$

which satisfies

$$(5.20) \qquad K_1 \subset K, \qquad K_1 = \overline{K_1 \cap \Omega}, \qquad \mathcal{H}^{n-1}((K \setminus K_1) \cap \Omega) = 0,$$

$$w_1 = w \quad \text{on } \Omega \setminus K.$$

We have thus proved the following:

PROPOSITION 5.4 *If $g \in L^p(\Omega) \cap L^{np}_{loc}(\Omega)$, then F achieves its minimum on the admissible pairs (C, u). Furthermore, if $(\mathcal{W}, w)$ minimizes G, then (K, w) with $K = \overline{\Omega \setminus \cup \mathcal{W}}$ minimizes F, and* (5.16) *holds. Reciprocally, from any minimizer (K, w) of F we obtain* (*through the consideration of the connected components of $\Omega \setminus K$*) *a minimizer $(\mathcal{V}, v)$ of G, which in turn gives rise to a new minimizer (K_1, w_1) of F* (*see* (5.19)) *for which* (5.20) *holds.*

6. - Convergence of local minimizers.

A (simple) convergence result for minimal partitions (see (5.6) above) has been proved in [8], Lemma 4. The following theorem, which concerns convergent sequences of local minimizers of G_g, has a much more technical proof.

THEOREM 6.1. *Let $g_h \in L^p_{loc}(\Omega)$ be s.t. $g_h \to g_\infty$ in $L^p_{loc}(\Omega)$, and assume that $\forall h \in \mathbb{N}$, $(\mathcal{W}_h, w_h) \in WCP(\Omega)$ is a local minimizer of G_{g_h} in Ω* (*see* (5.2), (5.3)). *If*

$$(6.1) \qquad \mathcal{W}_h \to \mathcal{W}_\infty \qquad \text{in } L^1_{loc}(\Omega),$$

$$(6.2) \qquad w_h(x) \to w_\infty(x) \qquad \text{for a.e. } x \in \Omega,$$

with $(\mathcal{W}_\infty, w_\infty) \in WCP(\Omega)$, then $(\mathcal{W}_\infty, w_\infty)$ is a local minimizer of G_{g_∞} in Ω.

PROOF. Fix A open $\subset\subset \Omega$. We cover A by a finite number of balls $\subset\subset \Omega$, and use (5.4) and semicontinuity (i.e. (3.7) and Fatou's lemma) to obtain:

$$G_{g_\infty}(\mathcal{W}_\infty, w_\infty; A) < +\infty.$$

Next, fix a compact variation $(\mathcal{U}, u)$ of $(\mathcal{W}_\infty, w_\infty)$ in A, i.e. $(\mathcal{U}, u) \in WCP(\Omega)$ and $\mathcal{U} = \mathcal{W}_\infty$, $u = w_\infty$ in $A \setminus C$, for a certain compact subset C of A. We have to prove that

$$(6.3) \qquad G_{g_\infty}(\mathcal{W}_\infty, w_\infty; A) \leq G_{g_\infty}(\mathcal{U}, u; A).$$

We can certainly assume that $u \in L^p(A)$, and that the empty set belongs to all partitions under consideration.

We begin by fixing some special arrangements:

$$\mathcal{W}_h = \{W_{h,i}\}, \qquad \mathcal{W}_\infty = \{W_{\infty,i}\}, \qquad \mathcal{U} = \{U_i\} \qquad (i \in \mathbb{N})$$

satisfying:

$$U_i \cap (A \backslash C) = W_{\infty,i} \cap (A \backslash C) \quad \forall i \in \mathbb{N}, \tag{6.4}$$

$$W_{h,i} \to W_{\infty,i} \quad \text{in } L^1(A)\ \forall i \in \mathbb{N}, \tag{6.5}$$

$$W_{h,2j} = W_{\infty,2j} = U_{2j} = \emptyset \quad \forall j \in \mathbb{N}. \tag{6.6}$$

Here, (6.4) and (6.5) follow from the assumptions $\mathcal{U} = \mathcal{W}_\infty$ in $A \backslash C$ and from (6.1)—see Section 3, especially (3.5) and (3.6)—while (6.6) is a technical requirement needed for the subsequent construction.

We now select a regular, open set D with $C \subset D \subset\subset A$, s.t.

$$P(W_{h,i}, \partial D) = 0 \quad \forall i \in \mathbb{N},\ \forall h \in \mathbb{N} \cup \{\infty\}, \tag{6.7}$$

$$\lim_{h \to \infty} \sum_{i=1}^{\infty} \int_{\partial D} |\chi_{W_{h,i}} - \chi_{W_{\infty,i}}|\, d\mathcal{H}^{n-1} = 0. \tag{6.8}$$

Moreover, we fix $k \in \mathbb{N}$ and $\forall h \in \mathbb{N}$ we construct a compact variation $\mathcal{U}_h^{(k)}$ of $\mathcal{W}_h$, as follows:

$$\mathcal{U}_h^{(k)} = \{U_{h,i}^{(k)}\colon i \in \mathbb{N}\},$$

where for all $h, j \in \mathbb{N}$ we define (as explained below):

$$a_{hj} = \mathcal{H}^{n-1}(W_{h,2j-1} \cap W_{\infty,2j-1} \cap \partial D), \tag{6.9}$$

$$I_h^{(k)} = \{j < k \text{ s.t. } a_{hj} > 0\}, \tag{6.10}$$

$$J_h^{(k)} = \{j < k \text{ s.t. } a_{hj} = 0\}, \tag{6.11}$$

$$U_{h,2j-1}^{(k)} = \tag{6.12}$$

$$= \begin{cases} ((W_{h,2j-1} \backslash \overline{D}) \cup (U_{2j-1} \cap D))(1) & \text{if } j \in I_h^{(k)}, \\ W_{h,2j-1} \backslash \overline{D} & \text{if either } j \in J_h^{(k)}, \text{ or } j \geq k, \end{cases}$$

$$(6.13)\qquad U^{(k)}_{h,2j} = \begin{cases} U_{2j-1} \cap D & \text{if } j \in J^{(k)}_h, \\ \emptyset & \text{if either } j \in I^{(k)}_h, \text{ or } j > k, \\ D \cap \Big(\bigcup_{i \geq 2k-1} U_i\Big)(1) & \text{if } j = k. \end{cases}$$

The general idea is to construct a new partition by an appropriate combination of the partitions $\mathcal{W}_h$ of $\Omega \setminus \overline{D}$ and $\mathcal{U}$ of D. However, in order to preserve the L^p-integrability of modified functions, we first *truncate* $\mathcal{U}$ in D, by taking essentially a whole *tail* of $\mathcal{U}$ (all terms of the sequence $\{U_i\}$ with index $\geq 2k-1$) as the single term $U^{(k)}_{h,2k}$. More precisely, we have to form the set of points of density one, according to requirement (3.1). Secondly, when $i < 2k$, the two pieces $W_{h,i} \setminus \overline{D}$ and $U_i \cap D$ are taken either together or separately to form corresponding terms of $\mathcal{U}^{(k)}_h$, according to whether they are *in contact* along ∂D or not. Precisely, when $j \in I^{(k)}_h$ we see that $W_{h,2j-1} \setminus \overline{D}$ and $W_{\infty,2j-1} \cap D$ (hence, $U_{2j-1} \cap \cap D$: recall that $U_i = W_{\infty,i}$ outside C, see (6.4)) are in contact along ∂D, so we essentially glue them together to form $U^{(k)}_{h,2j-1}$ (see (6.12)) and put $U^{(k)}_{h,2j} = \emptyset$ (see (6.13)). If instead $j \in J^{(k)}_h$ (no contact), then we put

$$U^{(k)}_{h,2j-1} = W_{h,2j-1} \setminus \overline{D}, \qquad U^{(k)}_{h,2j} = U_{2j-1} \cap D.$$

One checks easily that $\mathcal{U}^{(k)}_h \in CP(\Omega)$, with

$$\mathcal{U}^{(k)}_h = \mathcal{W}_h \quad \text{in } A \setminus \overline{D}$$

for all $h, k \in \mathbb{N}$. Moreover, $\mathcal{U}^{(k)}_h$ is a *finite* partition of D.

Finally, we put

$$(6.14)\qquad \begin{cases} t_{h,i} = w_h(W_{h,i}), \\ t_{\infty,i} = w_\infty(W_{\infty,i}), \\ s_i = u(U_i), \end{cases}$$

and define $s^{(k)}_{h,i} \equiv u^{(k)}_h(U^{(k)}_{h,i})$ as follows:

$$(6.15)\qquad \begin{cases} s^{(k)}_{h,2j-1} = t_{h,2j-1}, \\ s^{(k)}_{h,2j} = \begin{cases} s_{2j-1} & \text{if } j \in J^{(k)}_h, \\ 0 & \text{if } j = k. \end{cases} \end{cases}$$

We see easily that $(\mathcal{U}^{(k)}_h, u^{(k)}_h) \in WCP(\Omega)$, with

$$u^{(k)}_h = w_h \quad \text{in } A \setminus \overline{D}$$

for all $h, k \in \mathbb{N}$. In addition, $u^{(k)}_h \in L^p(D)$.

It follows that $G_{g_h}(\mathcal{W}_h, w_h; A) \leq G_{g_h}(\mathcal{U}_h^{(k)}, u_h^{(k)}; A)$, which simplifies to:

$$(6.16)\quad P(\mathcal{W}_h, D) + \lambda \int_D |w_h - g_h|^p\, dx \leq P(\mathcal{U}_h^{(k)}, D) + + P(\mathcal{U}_h^{(k)}, \partial D) + \lambda \int_D |u_h^{(k)} - g_h|^p\, dx,$$

(recall (6.7)). We check without difficulty that

$$(6.17)\quad P(\mathcal{U}_h^{(k)}, D) \leq P(\mathcal{U}, D),$$

$$(6.18)\quad P(\mathcal{U}_h^{(k)}, \partial D) \leq \leq \sum_{i=1}^{\infty} \int_{\partial D} |\chi_{W_{h,i}} - \chi_{W_{\infty,i}}|\, d\mathcal{H}^{n-1} + \mathcal{H}^{n-1}\left(\partial D \cap \bigcup_{i=2k-1}^{\infty} U_i\right),$$

$$(6.19)\quad \int_D |u_h^{(k)} - g_h|^p\, dx \leq \leq (\|u_h^{(k)} - u\|_{L^p(D)} + \|u - g_\infty\|_{L^p(D)} + \|g_\infty - g_h\|_{L^p(D)})^p .$$

To estimate the first term in the rigth-hand side of (6.19) we notice that:

$$(6.20)\quad \int_D |u_h^{(k)} - u|^p\, dx =$$

$$= \sum_{i=1}^{2k} \int_{D \cap U_{h,i}^{(k)}} |u_h^{(k)} - u|^p\, dx = \sum_{j \in I_h^{(k)}} \int_{D \cap U_{h,2j-1}^{(k)}} |u_h^{(k)} - u|^p\, dx +$$

$$+ \sum_{j \in J_h^{(k)}} \int_{D \cap U_{h,2j}^{(k)}} |u_h^{(k)} - u|^p\, dx + \int_{D \cap U_{h,2k}^{(k)}} |u_h^{(k)} - u|^p\, dx =$$

$$= \sum_{j \in I_h^{(k)}} |t_{h,2j-1} - t_{\infty,2j-1}|^p\, |U_{2j-1} \cap D| + \int_{D \cap \left(\bigcup_{i=2k-1}^{\infty} U_i\right)} |u(x)|^p\, dx,$$

(see (6.9)-(6.15), and notice that $|W_{h,2j-1} \cap (D \backslash C)| > 0$ and $|W_{\infty,2j-1} \cap \cap (D \backslash C)| > 0$ when $j \in I_h^{(k)}$, whence $s_{2j-1} = t_{\infty,2j-1}$, since $\mathcal{U} = \mathcal{W}_\infty$ and $u = w_\infty$ on $A \backslash C$).

Combining (6.16)-(6.20) we obtain:

$$(6.21)\qquad P(\mathcal{W}_h, D) + \lambda \int_D |w_h - g_h|^p\, dx \leq P(\mathcal{U}, D) +$$

$$+ \sum_{i=1}^{\infty} \int_{\partial D} |\chi_{W_{h,i}} - \chi_{W_{\infty,i}}|\, d\mathcal{H}^{n-1} + \mathcal{H}^{n-1}\left(\partial D \cap \bigcup_{i=2k-1}^{\infty} U_i\right) +$$

$$+\lambda\left[\left(\sum_{j\in I_h^{(k)}} |t_{h,2j-1} - t_{\infty,2j-1}|^p\, |U_{2j-1}\cap D| + \int_{D\cap\left(\bigcup_{i=2k-1}^{\infty} U_i\right)} |u(x)|^p\, dx\right)^{1/p} +\right.$$

$$\left. + \|u - g_\infty\|_{L^p(D)} + \|g_\infty - g_h\|_{L^p(D)}\right]^p .$$

On letting $h \to \infty$ (with k fixed) we obtain (by (3.7), Fatou's lemma, (6.8), and the hypotheses of the theorem):

$$(6.22)\qquad P(\mathcal{W}_\infty, D) + \lambda \int_D |w_\infty - g_\infty|^p\, dx \leq P(\mathcal{U}, D) +$$

$$+\mathcal{H}^{n-1}\left(\partial D \cap \bigcup_{i=2k-1}^{\infty} U_i\right) + \lambda\left[\left(\int_{D\cap\left(\bigcup_{i=2k-1}^{\infty} U_i\right)} |u(x)|^p\, dx\right)^{1/p} + \|u - g_\infty\|_{L^p(D)}\right]^p .$$

Finally, on letting $k \to \infty$ in (6.22) we obtain

$$G_{g_\infty}(\mathcal{W}_\infty, w_\infty; D) \leq G_{g_\infty}(\mathcal{U}, u; D),$$

(recall that $u \in L^p(A)$), thus proving (6.3). ∎

7. – Conclusion of the proof of Theorem 2.1.

We recall that the first part of Theorem 2.1 (i.e., the existence of minimizers of F) has been proved in Section 5, see especially Proposition 5.4. Let then (K, w) be a minimizer of F; we known that $(\mathcal{V}, v)$, with

$$\mathcal{V} = \{V\colon\ V = U(1),\ U \text{ a connected component of } \Omega\backslash K\},$$

$$v(V) = w(U),$$

is a (global) minimizer of G. Thus, if we prove that $\mathcal{V}$ is locally finite in Ω, then we are done, i.e. $w(D\setminus K)$ is finite for all D compact $\subset \Omega$. Actually, this result holds for *local minimizers* of G_g as well:

PROPOSITION 7.1. Let $(\mathcal{W}, w) \in WCP(\Omega)$ be a local minimizer of G_g in Ω. If $g \in L^{np}_{\text{loc}}(\Omega)$, then $\mathcal{W}$ is locally finite in Ω. i.e. for all $x \in \Omega$ one can find $r > 0$, $m \in \mathbb{N}$ and an arrangement $\{W_i\}$ of $\mathcal{W}$ s.t. $W_i \cap B_{x,r} = \emptyset$ $\forall i > m$.

We recall from [8] that whenever $\mathcal{W}$ is a minimal partition of $\mathbb{R}^n$ (i.e. $\Psi(\mathcal{W}, A) = 0$ $\forall A$ open $\subset\subset \mathbb{R}^n$; see (5.6) above), then $\mathcal{W}$ is locally finite: this follows from Theorem 10 of [8], since $w = \sum_{i=1}^{\infty} i^{-1}\chi_{W_i}$, $\{W_i\}$ any arrangement of $\mathcal{W}$, evidently satisfies (2.18) of [8] with $g = w$ (see also the following Section 8).

To prove Proposition 7.1, let us define for fixed $E \subset \mathbb{R}^n$ and $\varepsilon > 0$:

$$E_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon x \in E\}.$$

If $\mathcal{U} \in CP(\Omega)$, put $\mathcal{U}_\varepsilon = \{U_\varepsilon : U \in \mathcal{U}\} \in CP(\Omega_\varepsilon)$ (for $\varepsilon \to 0$, we are thus «blowing-up» the partition $\mathcal{U}$). Similarly, if f: $E \to \overline{\mathbb{R}}$ put

$$f_\varepsilon(x) = \varepsilon^{1/p} f(\varepsilon x) \qquad \forall x \in E_\varepsilon .$$

Clearly, if $(\mathcal{U}, u) \in WCP(\Omega)$, then $(\mathcal{U}_\varepsilon, u_\varepsilon) \in WCP(\Omega_\varepsilon)$ and if $(\mathcal{W}, w)$ is a local minimizer of G_g in Ω, then $(\mathcal{W}_\varepsilon, w_\varepsilon)$ is a local minimizer of G_{g_ε} in Ω_ε. We have in addition

$$\|g_\varepsilon\|_{L^{np}(B_{x,r})} = \|g\|_{L^{np}(B_{\varepsilon x, \varepsilon r})} . \tag{7.1}$$

We now fix $g \in L^{np}_{\text{loc}}(\Omega)$, a local minimizer $(\mathcal{W}, w)$ of G_g in Ω, a point $x \in \Omega$, and a sequence $\{\varepsilon_h\}$ of positive real numbers, decreasing to 0. Without loss of generality, we assume $x = 0$. Then $(\mathcal{W}_{\varepsilon_h}, w_{\varepsilon_h})$ is a local minimizer of $G_{g_{\varepsilon_h}}$ in Ω_{ε_h}, for all $h \in \mathbb{N}$, and (7.1) gives: $g_{\varepsilon_h} \to 0$ in $L^p_{\text{loc}}(\mathbb{R}^n)$. By compactness (Theorem 3.1 and (5.4)) we find $(\mathcal{W}_\infty, w_\infty) \in WCP(\mathbb{R}^n)$ s.t., passing possibly to a subsequence:

$$\mathcal{W}_{\varepsilon_h} \to \mathcal{W}_\infty \qquad \text{in } L^1_{\text{loc}}(\mathbb{R}^n),$$

$$w_{\varepsilon_h} \to w_\infty \qquad \text{a.e. in } \mathbb{R}^n .$$

By Theorem 6.1, $(\mathcal{W}_\infty, w_\infty)$ is a local minimizer of G_0 (i.e., with $g_\infty = 0$) in $\mathbb{R}^n$; Proposition 5.2 then gives: $w_\infty = g_\infty = 0$ a.e. in $\mathbb{R}^n$, and $\mathcal{W}_\infty$ is a minimal partition of $\mathbb{R}^n$. By [8], $\mathcal{W}_\infty$ is locally finite, i.e. we can find

$R > 0$, $m \in \mathbb{N}$ and an arrangement $\{W_{\infty, i}\}$ of $\mathcal{W}_\infty$ s.t.

$$W_{\infty, i} \cap B_R = \emptyset \qquad \forall i > m .$$

If $\{W_{\varepsilon_h, i}\}$ is a corresponding arrangement of $\mathcal{W}_{\varepsilon_h}$, satisfying

$$W_{\varepsilon_h, i} \to W_{\infty, i} \quad \text{ in } L^1_{\text{loc}}(\mathbb{R}^n), \text{ as } h \to \infty, \ \forall i ,$$

then clearly

$$\left| B_R \setminus \bigcup_{i=1}^{\infty} W_{\varepsilon_h, i} \right| < \eta(n, m) R^n ,$$

provided h is big enough. An application of Lemma 5.3 then yields $r > 0$ such that $W_i \cap B_r = \emptyset$ $\forall i > m$. The proof of Proposition 7.1 is thus concluded. As we have seen, this concludes the proof of Theorem 2.1 as well.

8. - Comparison with preceding work.

Since the weak formulation of the minimum problem adopted here is not the same of the preceding papers [3] and [8], we now proceed to point out the main differences and stress the advantages of the present approach. First notice that functional F is the same in all three works. However, this is no longer true for the functional G intervening in the weak formulation. In fact, the leading assumption of [3] and [8] being the boundedness of g, G was there defined by

$$G(u) = \lambda \int_\Omega |u - g|^p \, dx + \mathcal{H}^{n-1}(S_u), \tag{8.1}$$

on functions $u \in SBV_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ s.t. $\nabla u = 0$ a.e. in Ω and $\mathcal{H}^{n-1}(S_u) < \infty$—or equivalently on functions u of the type

$$u = \sum_i s_i \chi_{U_i} , \tag{8.2}$$

with $\mathcal{U} = \{U_i\} \in CP(\Omega)$, $\sup |s_i| < \infty$, $s_i \neq s_j$ if $i \neq j$. The asserted equivalence follows from Lemma 1.10 and 1.11 of [3], where it is also shown that in the preceding assumptions the jump set S_u of u satisfies $\mathcal{H}^{n-1}(S_u) = P(\mathcal{U}, \Omega)$.

Notice that $\mathcal{H}^{n-1}(S_u) \leq P(\mathcal{U}, \Omega)$ for a general u as in (8.2), with $s_i \in \mathbb{R}$ and $\{U_i\} \in CP(\Omega)$; equality holds if and only if $s_i \neq s_j$ whenever U_i and U_j $(i \neq j)$ are *in contact*, i.e.

$$\mathcal{H}^{n-1}(U_i(1/2) \cap U_j(1/2) \cap \Omega) > 0 .$$

Now recall the definition of G given in (4.1) of the present paper:

$$G(\mathcal{U}, u) = P(\mathcal{U}, \Omega) + \lambda \int_{\Omega} |u - g|^p \, dx \tag{8.3}$$

for $(\mathcal{U}, u) \in WCP(\Omega)$; u is again of the type (8.2), but now (i) the values s_i can be unbounded, and (ii) they can coincide on different U_i's.

Clearly, (i) is designed to keep into account that g itself can now be unbounded, but this could have been done by still defining G by (8.1) on a more general space of functions than SBV (e.g., the space $GSBV(\Omega)$ of functions all of which truncations $u_a \equiv (u \wedge a) \vee -a \in SBV_{\text{loc}}(\Omega)$, $a > 0$; see [1] and [5]. Actually, the function g considered in the enlightening Example 2.8 of [3] does belong to $GSBV(\Omega)$!).

What is more important is however (ii)—a fact that can be fully appreciated only when *local* minimizers are considered (because if $(\mathcal{U}, u)$ is a global minimizer of (8.3) then values of u necessarily differ on sets in contact). We see this as follows. Consider the «standard trisector» $\mathcal{W} = \{W_1, W_2, W_3, \emptyset\}$ of the unit disc $\Omega \subset \mathbb{R}^2$, where in polar coordinates

$$W_i = \{(\varrho, \theta): 0 < \varrho < 1, \ (i-1)2\pi/3 < \theta < i2\pi/3\} \qquad (i = 1, 2, 3).$$

It is well-known that $\mathcal{W}$ is a minimal partition of Ω, i.e. $\Psi(\mathcal{W}, A) = 0$ $\forall A$ open $\subset\subset \Omega$, see (5.6) above, hence $(\mathcal{W}, g)$ is a local minimizer of G_g in Ω, whenever $g = \sum_{i=1}^{3} s_i \chi_{W_i}$, *any* $s_i \in \mathbb{R}$ (see (5.1)-(5.3) above).

On the other hand, if e.g. $s_1 = s_2 = 0$, $s_3 = 1$, then g is *not* a local minimizer of (8.1) with the same datum $g = \chi_{W_3}$ (as defined in (2.18) of [8]): we have indeed $G(u) < G(g)$ if $u = g$ in $\Omega \setminus T_\varrho$ and $u = 0$ in T_ϱ, where $0 < \varrho < 1$ is such that $\lambda\varrho < 8(3^{-1/2} - 2^{-1}) \cong 0.6188$ and T_ϱ is the triangle with vertices $(\varrho, 0)$, $(\varrho, 4\pi/3)$, and the origin.

This fact has striking implications on the convergence of local minimizers. For, if $g_h = \sum_{i=1}^{3} s_{h,i} \chi_{W_i}$ with $s_{h,1} = 0$, $s_{h,2} = h^{-1}$, $s_{h,3} = 1$, then for all h, g_h is a local minimizer of (8.1) with datum g_h, but tends to the preceding g as $h \to \infty$: according to Theorem 6.1 above, the usefullness of the present approach is thus apparent.

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