RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

PATRIZIA LONGOBARDI
MERCEDE MAJ
AVINOAM MANN
AKBAR RHEMTULLA

Groups with many nilpotent subgroups

Rendiconti del Seminario Matematico della Università di Padova, tome 95 (1996), p. 143-152

<http://www.numdam.org/item?id=RSMUP_1996__95__143_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 95 (1996)

Groups with Many Nilpotent Subgroups.

PATRIZIA LONGOBARDI (*) - MERCEDE MAJ (*)
AVINOAM MANN (**) (1) - AKBAR RHEMTULLA (***)

The following result was proved by B. H. Neumann [N] and also by Faber, Laver and McKenzie [FLM], answering a question of P. Erdös:

A group G has the property, that each infinite subset of G contains a pair of commuting elements, if and only if the centre $Z(G)$ of G has finite index.

This prompted J. C. Lennox and J. Wiegold [LW] to consider groups in which every infinite subset contains a pair of elements generating an X-subgroup, where X is some given property of groups, e.g. nilpotence. Observing that the characterisation of such groups is possibly very hard, they restrict themselves to finitely generated soluble groups, and among these they characterise the groups with the above property when X is the class of polycyclic, or nilpotent, or coherent, groups (supersoluble groups were handled later by J. R. J. Groves [G]).

A more complex variation on the theorem of Neumann and McKenzie, involving several infinite subsets, is discussed in [LMR], while in [CLMR] the authors impose restrictions on finite subsets with more than 2 elements. The present paper is in a similar spirit. For convenience, we say «of class k» when we mean «of class at most k». Our basic hypothesis on a group G is:

(*) Indirizzo degli AA.: Dipartimento di Matematica e Applicazioni «R. Caccioppoli», Università degli Studi di Napoli, Via Cintia, Napoli 80126, Italy.
(**) Indirizzo dell'A.: Einstein Institute of Mathematics, Hebrew University, Givat Ram, Jerusalem 91904, Israel.
(***) Indirizzo dell'A.: Department of Mathematics, University of Alberta, Edmonton, Canada T6G 2G1.
(1) This author's research was done in part during his visits to the University of Manitoba and to the Università degli Studi di Napoli «Federico II», the latter under a CNR grant.

(N_k) *Any infinite subset of G contains a set of $k+1$ elements, which generates a nilpotent subgroup of class k.*

Or the following more general assumption:

(N) *Any infinite subset of G contains a finite subset X, such that $\langle X \rangle$ is nilpotent, and class $(\langle X \rangle) < |X|$.*

We refer to such groups as (N) or (N_k) groups. If G is an (N_k)-group, and X is an infinite subset of G, then Ramsey's Theorem shows that X contains an infinite subset Y such that all $(k+1)$-tuples of elements of Y generate a nilpotent subgroup of class k. Then $\langle Y \rangle$ itself is nilpotent of class k. This implies that (N_k)-groups are also (N_{k+1})-groups. Moreover, let us denote by (N_k^r) the assumption: *any infinite subset contains r elements generating a nilpotent subgroup of class k.* Then (N_k^2) is the weakest among these hypotheses, while our assumption $(N_k) = (N_k^{k+1})$ is the strongest, implying not only all propertis (N_k^r), but also all properties (N_l^r) with $l > k$. This imposition of a precise quantitative assumption makes it possible to avoid further assumptions, such as solubility. Indeed, once we derive the key fact that (N)-groups have a non-trivial FC-centre, we are able to show that they are hyperabelian by finite, and thus avail ourselves of the techniques and results of [LW]. This yields a satisfactory answer for finitely generated groups.

THEOREM A. *For a finitely generated group G the following are equivalent:*

a) G is an (N_k)-group.

b) G is an extension of a finite group by a nilpotent group of class k.

c) $|G : Z_k(G)|$ is finite.

Moreover, G is an (N)-group if and only if it is an (N_k)-group, for some k.

Part *b*) implies that in an (N)-group G the elements of finite order generate a locally finite subgroup T, and that G/T is torsionfree. For these factors of G we have

THEOREM B. *a) A torsion free (N)-group is hypercentral.*

b) A torsion free (N_k)-group is nilpotent of class k.

c) A torsion (N)-group is hypercentral by finite.

Along the way we derive the following characterisation for FC-groups:

THEOREM C. *An* FC-*group satisfies* (N_k), *for some* k, *if and only if it contains a subgroup* H *of finite index such that, for some* n, *any two elements of* H *generate a nilpotent subgroup of class* n. *If we choose* l *and* n *as minimal possible, then* $|k-n| \leq 1$. *Moreover, in an* FC-*group which is an* (N)-*group the* ω-*th term of the upper central series has finite index.*

The authors are grateful to the referee for his many helpful remarks.

The following result will be used time and again:

NEUMANN'S COVERING LEMMA. *If a group* G *is the union of finitely many subgroups, then some of these subgroups have finite indices, and* G *is already the union of these finite index subgroups.*

For the proof, see e.g. [T2, 2.2].

We start our proofs by noting a very useful fact.

PROPOSITION 1. *Let* G *satisfy* (N). *Then the* FC-*centre of* G *is not trivial.*

PROOF. We consider subsets A of G such that if B is a subset of A, and $|B| = k$, then $\langle B\rangle$ is not a nilpotent group of class less than k. By (N), there exists, among such sets, one which is finite and maximal. Let C be this maximal set, and let $x \notin C$. Then $C \cup \{x\}$ contains a subset B such that $\langle B\rangle$ is nilpotent of class less than k, where $k = |B|$. Then $B = D \cup \{x\}$, for some subset D of C. But then $|D| = k-1$, and $\langle D\rangle$ is not a nilpotent group of class less than $k-1$, so there exists a commutator w of weight $k-1$ in the elements of D, such that $w \neq 1$. But $[w, x] = 1$, so x commutes with w. Since C is finite and $k \leq |C|$, the number of possibilities for w is finite, and G is covered by the centralisers of these elements, hence one of these centralisers has a finite index, so that the corresponding w is an FC element.

COROLLARY 2. *An* (N)-*group is* FC-*hypercentral. An infinite* (N)-*group has an infinite* FC-*centre.*

PROPOSITION 3. *If* G *is an* (N_k)-*group, then* $|G : F_k(G)|$ *is finite, where* $F_k(G)$ *is the* k-*th* FC-*centre of* G.

PROOF. Suppose that $|G : F_k(G)|$ is infinite. We will construct an infinite sequence (x_i) such that

1) $x_i \notin F_k(G)$.

2) $[x_{i(1)}, x_{i(2)}] \notin F_{k-1}(G)$, for any $i(1) < i(2)$.

$\vdots$

k) $[x_{i(1)}, \ldots, x_{i(k)}] \notin F(G)$, for $i(1) < \ldots < i(k)$.

$k+1$) $[x_{i(1)}, \ldots, x_{i(k+1)}] \neq 1$, for $i(1) < \ldots < i(k+1)$.

Assume that we have already found $x_1, \ldots, x_n$ satisfying these restraints. We consider the subgroups

$$C_{i(1), i(2), \ldots, i(t)} := C_G([x_{i(1)}, \ldots, x_{i(t)}] \bmod F_{k-t}(G)),$$

for any indices $i(1) < i(2) < \ldots < i(t) \leq n$. The restraints 1), ..., k) imply that these subgroups are each of infinite index, as is $F_k(G)$, so we can find an element x which belongs neither to one of these subgroups nor to $F_k(G)$, and then take $x_{n+1} = x$. This completes the construction of the sequence (x_n), which obviously contradicts the assumption (N_k).

PROPOSITION 4. *An* (N)*-group is hyperabelian by finite.*

PROOF. We may assume that G is infinite, and it suffices to show that G contains a non-trivial normal abelian subgroup. Suppose no such normal abelian subgroup exists, and let x be a non-identity element with only finitely many conjugates, let $N = \langle x \rangle^G$ be the normal subgroup generated by these conjugates, and let $C = C_G(N)$. Then $|G : C|$ is finite, and $N \cap C = Z(N)$ is a normal abelian subgroup of G. Hence $N \cap C = 1$. Therefore N is finite, and having a trivial centre, it is certainly not nilpotent. Write $N_1 = N$, $C_1 = C$, and in C find an element y with finitely many conjugates. Then y has also only finitely many conjugates in G. Let $N_2 = \langle y \rangle^G$. In the same way, we see that N_2 is a finite non-nilpotent group. Moreover, $N_2 \subseteq C$, so that N_1 and N_2 generate their direct product in G. Write $C_2 = C_G(N_1 N_2)$, and continue the process by choosing an FC element in C_2, etc. We thus find in G an infinite direct product $N_1 \times N_2 \times \ldots \times N_k \times \ldots$ of finite non-nilpotent groups. Being finite, N_i contains a pair of elements x_i and y_i which generate a non-nilpotent subgroup. Now let $z_1 = x_1$, $z_2 = y_1 x_2$, $z_3 = y_1 y_2 x_3$, ..., $z_n = y_1 y_2 \ldots y_{n-1} x_n$, ... If $i < j$, then z_i and z_j project on the elements x_i and y_i of N_i, hence $\langle z_i, z_j \rangle$ is not nilpotent, so that the set $\{z_i\}$ violates (N).

At this point acquaintance with [LW] is recommended. First, for completeness, we perform one of that paper's «finger exercises».

PROPOSITION 5. *Let G be a finitely generated hyperabelian group in which every infinite subset contains a pair of elements generating a polycyclic group. Then G is polycyclic.*

PROOF. If we change «hyperabelian» to «soluble», this is Theorem B of [LW]. To prove the present version we assume that G violates the Proposition. Since polycyclic groups are finitely presented, we can find a normal subgroup N such that G/N is not polycyclic, but all proper factor groups of G/N are polycyclic. We may replace G by G/N. Since G is hyperabelian, it contains a proper normal abelian subgroup A. Then G/A is polycyclic, and now the proof of Theorem B of [LW] applies.

THEOREM 6. *A finitely generated group G satisfies* (N) *if and only if it is finite by nilpotent, and in that case it satisfies* (N_k), *for some k.*

PROOF. Let G be a finitely generated (N)-group. By Proposition 4, G contains a hyperabelian subgroup H of finite index. Then H is also finitely generated, so Proposition 5 shows that it is polycyclic, and G itself satisfies the maximum condition. If G is not finite by nilpotent, we find in it a normal subgroup N such that G/N is not finite by nilpotent, but all proper factor groups of it are finite by nilpotent. Replace G by G/N. Naturally G is infinite, so the proof of Proposition 4 shows that G contains a normal abelian subgroup. Now again a proof from [LW], this time of Theorem A, applies to show that G is finite by nilpotent, a contradiction.

Conversely, if G is finite by nilpotent (not necessarily finitely generated), it is well known that some term $Z_k(G)$ of the upper central series of G has finite index in G [R, 4.25], and it is immediate that then G satisfies (N_k).

COROLLARY 7. *In an* (N)*-group the set of elements of finite order forms a locally finite subgroup.*

Thus an (N)-group is an extension of a locally finite group by a torsion free one. For the top factor we can now prove parts *a*) and *b*) of Theorem B.

COROLLARY 8. *A torsion free group G satisfying* (N) *is hypercentral.*

PROOF. Theorem 6 shows that G is locally nilpotent, and Corollary 1 to [R, 4.38] and our Corollary 2 show that it is hypercentral.

REMARK. The fact that a locally nilpotent (N)-group is hypercentral will be used on several occasions below.

COROLLARY 9. *A torsion free* (N_k)*-group is nilpotent of class* k.

PROOF. Since G is locally nilpotent, Proposition 3 combined with [R, Corollary 2 to 4.38] shows that $|G : Z_k(G)|$ is finite. Then G is a torsion free nilpotent group, and in such a group all factor groups $G/Z_i(G)$ are also torsion free. Therefore $G = Z_k(G)$.

COROLLARY 10. *The class* (N_k) *is properly contained in* (N_{k+1}).

PROOF OF THEOREM A. The implication $a) \Rightarrow b)$ follows by combining Theorem 6 and Corollary 9, $c) \Rightarrow a)$ is clear, and the equivalence of $b)$ and $c)$ is a combination of [R, 4.24 and Corollary 2 to 4.21].

We consider locally finite groups next. In view of Proposition 1, it is of interest to look at FC-groups. If G is an FC-group, then $G/Z(G)$ is a residually finite torsion group. In such groups the property (N_k) turns out to be closely related to a property sometimes denoted by $(2 \to k)$, namely: *the subgroup generated by any two elements of* G *is nilpotent of class at most* k. With this notation, we have[2]

THEOREM 11. *Let* G *be a residually finite torsion* FC*-group. Then* G *is an* (N_k)*-group if and only if* G *contains a* $(2 \to k)$ *subgroup of finite index.*

PROOF. Let G be an (N_k)-group, and suppose it does not contain a $(2 \to k)$ subgroup of finite index. Then we can find elements x_1, y_1, such that $\langle x_1, y_1 \rangle$ is not nilpotent of class k. The normal closure $N_1 = \langle x_1, y_1 \rangle^G$ is finite, therefore there exists a normal finite index subgroup M_1 satisfying $M_1 \cap N_1 = 1$. We can find in M_1 another pair x_2, y_2, such that $\langle x_2, y_2 \rangle$ is not nilpotent of class k, and proceeding in the same way we repeat the contradiction in the proof of Proposition 4.

Let G be a $(2 \to k)$ group. We will prove by induction on r that G satisfies (N_k^r). Let X be an infinite subset of G, let $x \in X$, and let M be a fi-

(2) Our original proof of Theorem 11 and Theorem C relied on a structure theorem of M. J. Tomkinson for FC-groups [T1, 2.24]. The present elementary proofs are due to the referee.

nite index normal subgroup satisfying $M \cap \langle x \rangle^G = 1$. There is a coset My for which $My \cap X$ is infinite, and by induction there are elements $m_1 y, \ldots, m_{r-1} y \in X$ such that $\langle m_1 y, \ldots, m_{r-1} y \rangle$ is of class k. Let $L = \langle x, m_1 y, \ldots, m_{r-1} y \rangle$. Then $LM/M = \langle x, y \rangle M/M$ is of class k, as is $L\langle x \rangle^G / \langle x \rangle^G$, so L itself has class k.

To conclude the proof we show that if G contains an (N_k)-subgroup H of finite index, then G is an (N_k)-group. We first find a normal finite subgroup F such that $G = HF$, and then a finite index normal subgroup $M \leq H$ such that $M \cap F = 1$. Let X be an infinite subset of G, and find y so that $X \cap My$ is infinite. Write $y = hf$, $h \in H$, $f \in F$. Then $X \cap My$ contains elements $m_1 hf, \ldots, m_{k+1} hf$ such that $\langle m_1 h, \ldots, m_{k+1} h \rangle$ is of class k. Let $L = \langle m_1 hf, \ldots, m_{k+1} hf \rangle$. Then LF/F is of class k and LM/M is cyclic, so L is of class k.

Proof of Theorem C. The claims about (N_k)-groups follow by applying Theorem 11 to $G/Z(G)$. Now let G be an (N)-group that is an FC-group. To show that the ω-th centre has a finite index, it suffices to show that for $G/Z(G)$, so we assume also that G is torsion and residually finite. We first show that G is (locally nilpotent)-by-finite. If not, we find elements x_1, y_1, such that $\langle x_1, y_1 \rangle$ is not nilpotent. Then $N_1 = \langle x_1, y_1 \rangle^G$ is finite, and there exists a normal subgroup of finite index M_1 such that $N_1 \cap M_1 = 1$, M_1 contains two elements generating a non-nilpotent subgroup, etc.

Let H be a locally nilpotent subgroup of finite index. By [R, 4.38], H is its own ω-th centre. There exists a finite normal subgroup N such that $G = HN$, and it follows that $Z_n(H) \cap C_G(N) \leq Z_n(G)$. This concludes the proof.

We can now prove Theorem B *c*). We need a lemma first.

Lemma 12. *Let G be an* (N)*-group whose derived subgroup G' is a π'-group, for some set of primes π. Then each π-element of G is an* FC*-element.*

Proof. Let x be a π-element, and suppose that x has infinitely many conjugates, say $\{x_n\}$. Then two of these conjugates, say x_i and x_j, generate a nilpotent group H, which is then a π-group. Then $x_i^{-1} x_j \in H \cap G' = 1$, so $x_i = x_j$, a contradiction.

Proof of Theorem B *c*). We first prove the theorem in the special case that G' is hypercentral. We claim the following:

If π is a set of primes, let $O := O_\pi(G)$ and $L := L_\pi(G)$ be the largest normal π-subgroup of G and the subgroup generated by all π-elements of G, respectively. Then L/O is finite.

PROOF. Let $H = G/O$. Since H' is hypercentral, it has a normal Hall π-subgroup, so by the definition of O we see that H' is a π'-group. By Lemma 12, the π-elements of H are in its FC-centre K. Since K is also hypercentral by finite, by Theorem C, and its hypercentre is a π'-group, by definition of O, we see that all π-subgroups of H are finite and of bounded order. We note also that H is hypercentral by abelian, so a finite subgroup of H is soluble. Let T be a maximal π-subgroup of H. Since T consists of FC-elements, it is contained in a finite normal subgroup N. Let x be a π-element of H. Then T is a Hall subgroup of $\langle T, x\rangle$, so x is conjugate to some element of T, hence $x \in N$, and $L/O \leq N$.

Having established this, let S be the product of all the normal Sylow subgroups of G, and let $\boldsymbol{Q}$ be the set of primes for which there is no normal Sylow subgroup. Then S is contained in the Hirsch-Plotkin radical R of G, which is a hypercentral group. The previous claim, applied to each prime individually, shows that the Sylow subgroups of G/R are finite, so if $\boldsymbol{Q}$ is finite, so is G/R. We thus assume that $\boldsymbol{Q}$ is infinite.

Let $q \in \boldsymbol{Q}$, and write L_1 for the subgroup L_q defined above. Then L_1 is not hypercentral, by the choice of q (note that $L_1 \neq 1$), and $\pi(L_1)$, the set of primes occuring as orders of elements in L_1, is finite. Suppose that we have already found non-hypercentral normal subgroups $L_1, \ldots, L_n$, which generate their direct product in G, and such that $\pi :=$ $= \pi(L_1 \times \ldots \times L_n) =$ is finite. Let $K = L_{\pi'}(G)$, $U = O_{\pi'}(G)$. Then K/U is finite. Let $\boldsymbol{P} = \pi' - \pi(K/U)$. Then all $\boldsymbol{P}$-elements of G lie in U. In particular, we can find a prime $r \in \boldsymbol{Q} \cap \boldsymbol{P}$, so $L_r \leq U$, and L_r is not hypercentral, so we can choose $L_{n+1} = L_r$.

We have thus an infinite direct product of non-hypercentral subgroups L_i. Then L_i is not locally nilpotent either. Being locally finite, this means that we can find in L_i a pair of elements generating a non-nilpotent subgroup, and this leads again to the same contradiction as in Proposition 4.

Now remove the assumption that G' is hypercentral, and let G be any (N)-torsion group. Let N be the Hirsch-Plotkin radical of G, which is hypercentral, and suppose that $|G:N|$ is infinite. Then Corollary 2 and Theorem C show that G contains a normal subgroup M, such that M/N is an infinite hypercentral group. Then M/N contains an infinite normal abelian subgroup A/N. Thus A' is hypercentral, so the special case implies that A has a normal hypercentral subgroup B of finite index. Since B is subnormal in G, we have $B \leq N$, contradicting the infiniteness of A/N. This proves our theorem.

Note that if G is a hypercentral (N_k)-group (for some k) then its hypercentral height is at most ωk, by Proposition 3 and [R, 4.38].

REMARKS. 1) As seems natural for a problem posed by P. Erdös, the paper [N, FLM] apply Ramsey's Theorem. But this result was hardly used in this paper, nor in [LW]. We wish to point out that its use in [N] and [FLM] is also not essential. Thus, Ramsey's Theorem is used in these papers to prove that if a group G has the property that each infinite subset contains a commuting pair, then G is an FC-group. We prove this alternatively as follows. Let H be the FC-centre of G. Choose a subset A which is maximal with respect to having an empty intersection with H, and not containing a commuting pair. Then A is finite, and each element of G belongs either to H or to a centraliser of one of the elements of A. Since these centralisers have infinite indices, it follows that $G = H$ (3).

Note that we have replaced Ramsey's Theorem by Neumann's covering lemma. On the other hand, in [T1, 7.3] this lemma is proved by applying Ramsey's Theorem (which was not used in the original proof).

2) A variation on the problem of this paper, also suggested by [LMR], is to ask in (N) only that some commutator of weight $|X|$ in the elements of X is the identity. We were able to handle only a stronger assumption

(N*) For any infinite sequence $x_1, x_2, \ldots, x_n, \ldots$ of distinct elements of G there exist indices $i(1) < i(2) < \ldots < i(k)$, such that $[x_{i(1)}, x_{i(2)}, \ldots, x_{i(k)}] = 1$.

It turns out that at least our preliminary results hold also under this weaker assumption. We indicate the necessary changes in the proofs.

PROPOSITION 1*. *A group G satisfying* (N*) *has a non trivial* FC-*centre.*

PROOF. We assume that G is infinite, and well order it according to the least ordinal of cardinality $|G|$. We denote the order relation by «$<$». We can find a finite subset X of G maximal with respect to the property: if $x_1 < x_2 < \ldots < x_k$ are elements of X, then $[x_1, x_2, \ldots, x_k] \neq 1$. Let z be the largest element of X. It then follows as in the proof of Proposition 1 that each element $x > z$ centralises one of finitely many commutators in the elements of X.

(3) The referee made the following comment at this point: «This argument is known to a number of people but has not appeared in print... it came to me at the end of a chain in which the earliest link that I know of is Passman».

If G is countable, there are only finitely many elements which precede z. Then G is the union of these finitely many elements and finitely many centralisers. Considering the individual elements as cosets of 1, we see that they can be omitted, G is the union of finitely many centralisers, and one of these centralisers has a finite index. Suppose that G is uncountable. Then the elements preceding z generate a subgroup H of cardinality smaller than $|G|$, and therefore $|G:H|$ is infinite. The group G being the union of H and finitely many centralisers, again H can be omitted.

(We see that this proposition can be proved also under the following weaker condition: we first well order G as in the proof, and then in (N*) consider only sequences which are ordered by this ordering.)

PROPOSITION 4*. *A group satisfying* (N*) *is hyperabelian by finite.*

PROOF. We repeat the proof of Proposition 4, only choosing x_i, y_i differently. As N_i is finite and not nilpotent, it is not an Engel group either, so we choose x_i and y_i to satisfy $[x_i, y_i, \ldots, y_i] \neq 1$, for any number of occurences of y_i. With this choice the proof of Proposition 4 applies.

REFERENCES

[CLMR] M. CURZIO - P. LONGOBARDI - M. MAJ - A. RHEMTULLA, *Groups with many rewritable products*, Proc. Amer. Math. Soc., **115** (1992), pp. 931-934.

[FLM] V. FABER - R. LAVER - R. MCKENZIE, *Coverings of groups by abelian subgroups*, Can. J. Math., **30** (1978), pp. 933-945.

[G] J. R. J. GROVES, *A conjecture of Lennox and Wiegold concerning supersoluble groups*, J. Austral. Math. Soc. (A), **35** (1983), pp. 218-220.

[LMR] P. LONGOBARDI - M. MAJ - A. RHEMTULLA, *Infinite groups in a given variety and Ramsey's theorem*, Comm. Alg., **20** (1992), pp. 127-139.

[LW] J. C. LENNOX - J. WIEGOLD, *Extensions of a problem of Paul Erdös on groups*, J. Austral. Math. Soc. (A), **31** (1981), pp. 459-463.

[N] B. H. NEUMANN, *A problem of Paul Erdös on groups*, J. Austral. Math. Soc., **21** (1976), pp. 467-472.

[R] D. J. S. ROBINSON, *Finiteness Conditions and Generalized Soluble Groups*, Springer-Verlag, Berlin (1972).

[T1] M. J. TOMKINSON, FC-*Groups*, Pitman Advanced Publishing Program, London (1984).

[T2] M. J. TOMKINSON, *Groups covered by finitely many cosets of subgroups*, Comm. Alg., **15** (1987), pp. 845-859.

Manoscritto pervenuto in redazione il 23 novembre 1993
e, in forma revisionata, il 10 gennaio 1995.