Rendiconti
del
Seminario Matematico
della
Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 95 (1996), p. 11-22
<http://www.numdam.org/item?id=RSMUP_1996__95__11_0>

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REND. SEM. MAT. UNIV. PADOVA, Vol. 95 (1996)

On Univalent and Starlike Wright's Hypergeometric Functions.

R. K. RAINA (*)

ABSTRACT - In this paper we establish several results associated with certain classes of univalent, starlike and convex Wright's generalized hypergeometric functions. These results not only yield similar such results studied recently in [6], but would also be applicable to special functions like, the Bessel-Maitland functions and the Mittag-Leffler functions.

1. - Introduction and preliminaries.

Denote by E the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \tag{1.1}$$

which are regular and univalent in the open disk $U = \{z: |z| < 1\}$.

Let $S(A, B)$ represent the class of those functions $f(z) \in E$ satisfying the inequality

$$\left| \frac{(zf'(z)/f(z)) - 1}{A - B(zf'(z)/f(z))} \right| < 1 \quad (z \in U), \tag{1.2}$$

where $-1 \leq B < A \leq 1$, and $-1 \leq B \leq 0$.

Also, we denote by $K(A, B)$ the class of functions $f(z) \in E$, if and only if $zf'(z) \in S(A, B)$. It can easily be verified that the functions belonging to $S(A, B)$ are starlike of order $(A + B)/2B$ and type $|B|$. Further, the functions belonging to $K(A, B)$ are said to be convex.

When $A = 1 - 2\alpha$, $B = -1$, and $S(A, B) = S^*(\alpha)$, then $S^*(\alpha)$ repre-

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sents the class of functions in E which are starlike of order $\alpha(0 \leqslant \alpha < < 1)$, and (1.2) is seen to be equivalent to

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U). \tag{1.3}$$

On the other hand, if we put $K(1-2\alpha, -1) = K^*(\alpha)$, then $K^*(\alpha)$ represents the class of all such functions $f(z) \in E$ which are convex of order α satisfying

$$\text{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > \alpha - 1 \quad (z \in U). \tag{1.4}$$

where $0 \leqslant \alpha < 1$. It being understood that the functions such as $zf'(z)/f(z)$ which have removable singularities at $z = 0$, have had these singularities removed throughout this paper.

For general references to the aferomentioned definitions and statements, we refer to [2] and [3]. The Wright's generalized hypergeometric function [8] (see also [7]) is defined by

$$(1.5) \quad {}_p\psi_q[z] = {}_p\psi_q\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p); \\ (b_1, B_1), \ldots, (b_q, B_q);\end{matrix} \; z\right] =$$

$$= \sum_{n=0}^{\infty}\left\{\prod_{i=1}^{p}\Gamma(a_i + A_i n)\right\}\left\{\prod_{i=1}^{q}\Gamma(b_i + B_i n)\right\}^{-1} \cdot z^n/n!,$$

where the coefficients A_i $(i = 1, \ldots, p)$ and B_i $(i = 1, \ldots, q)$ are positive real numbers such that

$$1 + \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i \geqslant 0. \tag{1.6}$$

If $A_i = 1$ $(i = 1, \ldots, p)$, $B_i = 1$ $(i = 1, \ldots, q)$, then we have the relationship

$$(1.7) \quad {}_p\psi_q = \left[\begin{matrix}(a_1, 1), \ldots, (a_p, 1); \\ (b_1, 1), \ldots, (b_q, 1);\end{matrix} \; z\right] =$$

$$= \frac{\prod_{i=1}^{p}\Gamma(a_i)}{\prod_{i=1}^{p}\Gamma(b_i)} {}_pF_q\left[\begin{matrix}a_1, \ldots, a_p; \\ b_1, \ldots, b_q;\end{matrix} \; z\right].$$

Recently interesting classes of starlike and convex hypergeometric functions were studied in [1] and [5]. Subsequently, Owa and Srivastava [6] gave various useful results concerning univalent, starlike and convex generalized hypergeometric functions. Motivated by these recent works, we aim at presenting certain general classes of univalent, starlike and convex Wright's generalized hypergeometric functions. The Wright's function (1.5) includes the functions like the generalized Bessel-Maitland functions $J_\mu^\lambda(z)$, and the Mittag-Leffler functions E_λ (or $E_{\lambda,\mu}$). Thus the results of this paper would, therefore, possess wider applicability than those considered in [6].

2. – Univalent Wright's functions.

A function $f(z)$ belonging to the class E is said to be close-to-convex if there is a convex function $g(z)$ such that

$$\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0 \qquad (z \in U). \tag{2.1}$$

We first state the following results due to Jack [4] and Duren [2], respectively, which are need in our investigations:

LEMMA 1. *If $w(z)$ is regular in the unit disk U, such that $w(0) = 0$, and $|w(z_1)| = \max_{|z| = r} |w(z)|$, $0 \leq r < 1$, then*

$$z_1 w'(z_1) = kw(z_1), \tag{2.2}$$

where $k \geq 1$ (k is real).

LEMMA 2. *Every close-to-convex function is univalent.*

We now prove the following result:

THEOREM 1. *Let the Wright's generalized hypergeometric function ${}_p\psi_q[z]$ be defined by* (1.5) *such that*

$$\left| {}_p\psi_q'\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p); \\ (b_1, B_1), \ldots, (b_q, B_q);\end{matrix}\; z\right] - \frac{\prod_{i=1}^{p} \Gamma(a_i + A_i)}{\prod_{i=1}^{q} \Gamma(b_i + B_i)} \right|^{1-h} . \tag{2.3}$$

$$\cdot\left|\frac{z\,{}_p\psi''_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p);\\(b_1,B_1),\ldots,(b_q,B_q);\end{matrix}\;z\right]}{{}_p\psi'_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p);\\(b_1,B_1),\ldots,(b_q,B_q);\end{matrix}\;z\right]}\right|^h<\left(\frac{\prod_{i=1}^{p}\Gamma(a_i+A_i)}{\prod_{i=1}^{q}\Gamma(b_i+B_i)}\right)^{1-h}\cdot\left(\frac{1}{2}\right)^h,$$

for some fixed $h \geqslant 0$, and $\forall z \in U$, where

$$\begin{cases}\left(\prod_{i=1}^{p}\Gamma(a_i+A_i)\right)\left(\prod_{i=1}^{q}\Gamma(b_i+B_i)\right)^{-1}>0,\\ \Delta=\left(\prod_{i=1}^{p}\Gamma(a_i)\right)\left(\prod_{i=1}^{q}\Gamma(b_i)\right)^{-1}>0.\end{cases}\tag{2.4}$$

Then ${}_p\psi_q[z]$ is univalent in the disk U.

PROOF. As in [6], we shall invoke Lemmas 1 and 2 to prove Theorem 1 above. Consider the function

$$P(z)=\frac{\prod_{i=1}^{q}\Gamma(b_i+B_i)}{\prod_{i=1}^{p}\Gamma(a_i+A_i)}\left\{{}_p\psi'_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p);\\(b_1,B_1),\ldots,(b_q,B_q);\end{matrix}\;z\right]-\right.$$
$$\left.-\left(\prod_{i=1}^{p}\Gamma(a_i)\right)\left(\prod_{i=1}^{q}\Gamma(b_i)\right)^{-1}\right\},\qquad z\in U.\tag{2.5}$$

Then $P(z) \in E$, and (2.3) implies

$$|P'(z)-1|^{1-h}\left|\frac{zP''(z)}{P'(z)}\right|^h<\left(\frac{1}{2}\right)^h.\tag{2.6}$$

Define now a function $w(z)$ by

$$w(z)=P'(z)-1\quad(z\in U).\tag{2.7}$$

We notice that $w(z)$ is regular in unit disc U, and $w(0)=0$, since $P'(0)=1$ (which can easily be verified from (2.5) in conjunction with the definition (1.5)).

From (2.6) and (2.7), we have

$$|w(z)|\cdot\left|\frac{zw'(z)}{w(z)}\cdot\frac{1}{1+w(z)}\right|^h<\left(\frac{1}{2}\right)^h,\tag{2.8}$$

where the singularity $z = 0$ in $zw'(z)/w(z)$ is removable just as in (1.3). If now $z_1 \in U$ such that

$$\max_{|z| \leqslant |z_1|} |w(z)| = |w(z_1)| = 1, \tag{2.9}$$

and setting by means of Lemma 1:

$$\frac{z_1 w'(z_1)}{w(z_1)} = k \quad (k \geqslant 1),$$

we then get

$$|w(z_1)| \cdot \left| \frac{z_1 w'(z_1)}{w(z_1)} \cdot \frac{1}{1 + w(z_1)} \right|^h \geqslant \left(\frac{k}{2}\right)^h \geqslant \left(\frac{1}{2}\right)^h. \tag{2.10}$$

This assertion contradicts (2.8), and also (2.3); and hence

$$|w(z)| = |P'(z) - 1| < 1 \Rightarrow \operatorname{Re}\{p'(z)\} > 0 \quad (z \in U). \tag{2.11}$$

Also, we observe that $Q(z) = z$ is convex in unit disk U, and, therefore,

$$\operatorname{Re}\left\{\frac{P'(z)}{Q'(z)}\right\} > 0 \quad (z \in U) \tag{2.12}$$

$$\Rightarrow \operatorname{Re}\left\{\frac{{}_p\psi_q'\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p); \\ (b_1, B_1), \ldots, (b_q, B_q);\end{matrix}\; z\right]}{Q'(z)}\right\} > 0 \qquad (z \in U)$$

(provided that (2.4) holds true)

$\Rightarrow_p \psi_q'[z]$ is a closed-to-convex in U

$\Rightarrow {}_p\psi_q'[z]$ is univalent in the unit disk U

(in view of Lemma 2), and the proof is complete.

EXAMPLES. If we put $h = 1$, then Theorem 1 gives the following:

COROLLARY 1. *Let* ${}_p\psi_q[z]$ *be defined by* (1.5), *and satisfy*

$$\left| \frac{z\,{}_p\psi_q''\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p);\\ (b_1, B_1), \ldots, (b_q, B_q);\end{matrix}\; z\right]}{{}_p\psi_q'\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p);\\ (b_1, B_1), \ldots, (b_q, B_q);\end{matrix}\; z\right]} \right| < \frac{1}{2} \qquad (z \in U), \tag{2.13}$$

provided that (2.4) *holds true, then* ${}_p\psi_q[z]$ *is univalent in the unit disk* U.

On the other hand, when $h = 0$, then Theorem 1 yields

COROLLARY 2. *Let* ${}_p\psi_q[z]$ *be defined by* (1.5), *and satisfy*

$$\left| {}_p\psi_q'\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p);\\ (b_1, B_1), \ldots, (b_q, B_q);\end{matrix}\; z\right] - \frac{\prod_{i=1}^{p} \Gamma(a_i + A_i)}{\prod_{i=1}^{q} \Gamma(b_i + B_i)} \right| < \tag{2.14}$$

$$< \frac{\prod_{i=1}^{p} \Gamma(a_i + A_i)}{\prod_{i=1}^{q} \Gamma(b_i + B_i)} \qquad (z \in U),$$

provided that (2.4) *holds true, then* ${}_p\psi_q[z]$ *is univalent in the unit disk* U.

REMARK. If we put $A_i = 1$ $(i = 1, \ldots, p)$, $B_i = 1$ $(i = 1, \ldots, q)$, then by virtue of the relationship (1.7), Theorem 1 (and its above special cases Corollaries 1 and 2) correspond to the results given in [6].

3. - Starlike Wright's functions.

We shall first prove the following result by making use of Lemma 1:

LEMMA 3. *Let* $f(z)$ *be defined by* (1.1), *and satisfy the condition*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right|^{1-h} \left| \frac{zf''(z)}{f'(z)} \right|^{h} < \frac{(A - B)(2 + A + A^2)^h}{(1 + |B|)(1 + A)^{2h}}, \tag{3.1}$$

for fixed constants A, B and h, such that $-1 \leqslant B < A \leqslant 1$, $-1 \leqslant B \leqslant 0$, $h \geqslant 0$, $\forall z \in U$. Then $f(z)$ is in the class $S(A, B)$.

PROOF. The lemma would be established if we show that

$$\left| \frac{zf'(z)/f(z) - 1}{A - B(zf'(z)/f(z))} \right| < 1 \quad (z \in U), \tag{3.2}$$

under the condition (3.1). To this end, we define a function $w(z)$ by

$$\frac{zf'(z)}{f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{3.3}$$

where $-1 \leqslant B < A \leqslant 1$, $-1 \leqslant B \leqslant 0$, and $z \in U$.

Logarithmic differentiation of (3.3) yields

$$\frac{zf''(z)}{f'(z)} = \frac{(A - B)w(z)}{1 + Bw(z)} \left[1 + \frac{zw'(z)}{\{1 + Aw(z)\}w(z)} \right] \tag{3.4}$$

$\Rightarrow$

$$\left| \frac{zf'(z)}{f(z)} - 1 \right|^{1-h} \left| \frac{zf''(z)}{f'(z)} \right|^{h} = \tag{3.5}$$

$$= \left| \frac{(A - B)w(z)}{1 + Bw(z)} \right| \left| 1 + \frac{zw'(z)}{w(z)} \left\{ \frac{1}{1 + Aw(z)} \right\} \right|^{h}.$$

It may be observed that

$$\left| \frac{(zf'(z)/f(z)) - 1}{A - B(zf'(z)/f(z))} \right| < 1 \Leftrightarrow |w(z)| < 1 \quad (z \in U), \tag{3.6}$$

and the inequality holds true at $z = 0$ also, since $w(0) = 0$, and hence $w(z)$ is a Schwarz function.

If now

$$\left| \frac{(zf'(z)/f(z)) - 1}{A - B(zf'(z)/f(z))} \right| = 1, \tag{3.7}$$

for $z = z_1 \in U$, and

$$\left| \frac{(zf'(z)/f(z)) - 1}{A - B(zf'(z)/f(z))} \right| > 1, \tag{3.8}$$

for $|z| < |z_1|$, then we have

$$|w(z)|\,|w(z_1)| = 1 \qquad \text{for } |z| < |z_1|,$$

such that from (3.3), $w(z) \neq -1/B$ $(B \neq 0)$. Applying Lemma 1 to $w(z)$ at $z = z_1 \in U$, and setting

$$z_1 w'(z_1) = kw(z_1) \quad (k \geqslant 1,\ k \text{ is real}), \tag{3.9}$$

we find from (3.5) that

$$\left| \frac{z_1 f'(z_1)}{f(z_1)} - 1 \right|^{1-h} \left| \frac{z_1 f''(z_1)}{f'(z_1)} \right|^{h} \geqslant \tag{3.10}$$

$$\geqslant \frac{A-B}{1+|B|}\left(1 + \frac{k}{1-A}\right)^{h} \geqslant \frac{(A-B)(2+A+A^2)^h}{(1+|B|)(1+A)^{2h}},$$

since $-1 \leqslant B < A \leqslant 1$, $-1 \leqslant B \leqslant 0$, and $1 \geqslant \{(1-A)^2/(1+A)^2\}$. This is a contradiction of the condition (3.1), and hence (3.2) holds true, and so $f(z) \in S(A, B)$.

Now we apply Lemma 3 to prove the following results for the starlike Wright's generalized hypergeometric functions:

THEOREM 2. *Corresponding to* ${}_p\psi_q[z]$ *defined by* (1.5), *let*

$$\left| \frac{z\, {}_p\psi_q'\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p);\\ (b_1, B_1), \ldots, (b_q, B_q);\end{matrix}\ z\right]}{{}_p\psi_q\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p);\\ (b_1, B_1), \ldots, (b_q, B_q);\end{matrix}\ z\right]} \right| < \frac{A-B}{1+|B|}, \tag{3.11}$$

for $-1 \leqslant B < A \leqslant 1$, $-1 \leqslant B \leqslant 0$. *Then* $z\Delta^{-1}{}_p\psi_q[z] \in S(A, B)$, *where* Δ *is defined by* (2.4).

PROOF. Define a function $H(z)$ by

$$H(z) = \Delta^{-1} z\, {}_p\psi_q\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p);\\ (b_1, B_1), \ldots, (b_q, B_q);\end{matrix}\ z\right] \quad (z \in U). \tag{3.12}$$

Then (3.11) becomes

$$\left| \frac{zH'(z)}{H(z)} - 1 \right| < \frac{A-B}{1+|B|}. \tag{3.13}$$

The result follows at once by using (3.1) (for $h = 0$) and (3.12).

THEOREM 3. *Let* ${}_p\psi_q[z]$ *be defined by* (1.5) *and satisfy*

$$\left| \frac{z\,{}_p\psi_q''\begin{bmatrix}(a_1, A_1), \dots, (a_p, A_p); & \\ (b_1, B_1), \dots, (b_q, B_q); & z\end{bmatrix}}{{}_p\psi_q'\begin{bmatrix}(a_1, A_1), \dots, (a_p, A_p); & \\ (b_1, B_1), \dots, (b_q, B_q); & z\end{bmatrix}} \right| < \frac{(A-B)(2+A+A^2)}{(1+|B|)(1+A)^2}, \tag{3.14}$$

for $-1 \leqslant B < A \leqslant 1$, $-1 \leqslant B \leqslant 0$, *then* ${}_p\psi_q[z]$ *is starlike of order* $(A+B)/2B$ *and type* $|B|$ *with respect to*

$$\left(\prod_{i=1}^{p} \Gamma(a_1)\right)\left(\prod_{i=1}^{q} \Gamma(b_1)\right)^{-1}.$$

PROOF. We observe that $P(z)$ defined by (2.5) is in the class E, and satisfies

$$\left| \frac{zP''(z)}{P'(z)} \right| = \left| \frac{z\,{}_p\psi_q''\begin{bmatrix}(a_1, A_1), \dots, (a_p, A_p); & \\ (b_1, B_1), \dots, (b_q, B_q); & z\end{bmatrix}}{{}_p\psi_q'\begin{bmatrix}(a_1, A_1), \dots, (a_p, A_p); & \\ (b_1, B_1), \dots, (b_q, B_q); & z\end{bmatrix}} \right| < \tag{3.15}$$

$$< \frac{(A-B)(2+A+A^2)}{(1+|B|)(1+A)^2}, \qquad z \in U.$$

Therefore, by virtue of Lemma 3 (when $h = 1$), we conclude that $P(z) \in S(A, B)$, and so $P(z)$ is starlike of order $(A+B)/2B$ and type $|B|$ with respect to the origin $(-1 \leqslant B < A \leqslant 1;\ -1 \leqslant B \leqslant 0)$. This implies that ${}_p\psi_q[z]$ is starlike of order $(A+B)/2B$ and type $|B|$ with respect to

$$\left(\prod_{i=1}^{p} \Gamma(a_i)\right) \Big/ \left(\prod_{i=1}^{q} \Gamma(b_i)\right)^{-1}.$$

4. – Convex Wright's functions.

In this section we establish a result on convex Wright's generalized hypergeometric functions. The result is contained in the following:

THEOREM 4. *Let* ${}_p\psi_q[z]$ *be defined by* (1.5) *and satisfy the condi-*

tion (3.11). *Then*

$$(4.1)\quad \Delta^{-1} z\ {}_{p+1}\psi_{q+1}\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p), (1, 1); \\ (b_1, B_1), \ldots, (b_q, B_q), (2, 1);\end{matrix}\quad z\right] \in K(A, B).$$

PROOF. Since $zf'(z) \in S(A, B) \Rightarrow f(z) \in K(A, B)$, therefore, by virtue of condition (3.11) (Theorem 2) we have

$$\Delta^{-1} z\ {}_{p}\psi_{q}\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p); \\ (b_1, B_1), \ldots, (b_q, B_q);\end{matrix}\quad z\right] \in S(A, B).$$

Hence

$$\Delta^{-1} \int_0^z {}_{p}\psi_{q}\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p); \\ (b_1, B_1), \ldots, (b_q, B_q);\end{matrix}\quad t\right] dt \in K(A, B),$$

$$\Rightarrow$$

$$\Delta^{-1} z\ {}_{p+1}\psi_{q+1}\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p), (1, 1); \\ (b_1, B_1), \ldots, (b_q, B_q), (2, 1);\end{matrix}\quad z\right] \in K(A, B),$$

which proves the result.

5. – Applications.

Due to the generality of the class of functions ${}_p\psi_q[z]$, the results obtained in Sections 2-4 can be made applicable to various special functions. To illustrate, we apply Theorems 1 to 4 to the Mittag-Leffler function $E_{\alpha,\beta}(z)$. Noting from (1.5) the relationship

$$(5.1)\qquad {}_1\psi_1\left[\begin{matrix}(1, 1); \\ (\beta, \alpha);\end{matrix}\quad z\right] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)} = z^{(1-\beta)/\alpha} E_{\alpha,\beta}(z),$$

we have, respectively, the following consequences of Theorems 1 to 4:

COROLLARY 3. *Let* $E_{\alpha,\beta}(z)$ *be defined by* (1.5) *such that*

$$(5.2)\qquad \left| \frac{d}{dz}(z^{(1-\beta)/\alpha} E_{\alpha,\beta}(z)) - \frac{1}{\Gamma(\alpha+\beta)} \right|^{1-h} \cdot$$

$$\cdot \left| \frac{z(d^2/dz^2)(z^{(1-\beta)/\alpha} E_{\alpha,\beta}(z))}{(d/dz)(z^{(1-\beta)/\alpha} E_{\alpha,\beta}(z))} \right|^{h} < \left(\frac{1}{\Gamma(\alpha+\beta)}\right)^{1-h} \left(\frac{1}{2}\right)^{h},$$

for some fixed $h \geqslant 0$, and $\forall z \in U$, where $\alpha + \beta > 0$. Then $E_{\alpha,\beta}(z)$ is univalent in the disk U.

COROLLARY 4. *Corresponding to $E_{\alpha,\beta}(z)$ defined by* (5.1), *let*

$$\left| \frac{z^{1+(\beta-1)/\alpha}(d/dz)(z^{(1-\beta)/\alpha}E_{\alpha,\beta}(z))}{E_{\alpha,\beta}(z)} \right| < \frac{A-B}{1+|B|}, \tag{5.3}$$

for $-1 \leqslant B < A \leqslant 1$, $-1 \leqslant B \leqslant 0$. Then $\Gamma(\beta) z^{1+(1-\beta)/\alpha} E_{\alpha,\beta}(z) \in S(A, B)$.

COROLLARY 5. *Let $E_{\alpha,\beta}(z)$ be defined by* (5.1), *and satisfy*

$$\left| \frac{z(d^2/dz^2)(z^{(1-\beta)/\alpha}E_{\alpha,\beta}(z))}{(d/dz)(z^{(1-\beta)/\alpha}E_{\alpha,\beta}(z))} \right| < \frac{(A-B)(2+A+A^2)}{(1+|B|)(1+A)^2}, \tag{5.4}$$

for $-1 \leqslant B < A \leqslant 1$, $-1 \leqslant B \leqslant 0$; then $z^{(1-\beta)/\alpha}E_{\alpha,\beta}(z)$ is starlike of order $(A-B)/2B$ and type $|B|$ w.r.t. $1/\Gamma(\beta)$.

COROLLARY 6. *Let $E_{\alpha,\beta}(z)$ be defined by* (5.1), *and satisfy the condition* (5.3). *Then*

$$\Gamma(\beta)\, z\, {}_2\Psi_2\left[\begin{matrix}(1,1),(1,1); \\ (\beta,\alpha)(2,1);\end{matrix}\; z\right] \in K(A, B). \tag{5.5}$$

We conclude this paper by mentioning few interesting special cases of Theorems 1 and 2, involving the Bessel-Maitland function $J_\nu^\mu(z)$. By virtue of the relation

$${}_0\Psi_1\left[\begin{matrix}- \quad ; \\ (1+\nu,\mu);\end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(1+\nu+\mu n)} = J_\nu^\mu(-z), \tag{5.6}$$

Theorem 1 (when $h = 0$, and $h = 1$) leads to the following results:

COROLLARY 7. *Let $J_\nu^\mu(z)$ be defined by* (5.6) *such that*

$$\left| J_{\nu+\mu}^\mu(-z) - \frac{1}{\Gamma(1+\mu+\nu)} \right| < \frac{1}{\Gamma(1+\mu+\nu)}, \tag{5.7}$$

$\forall z \in U$, $\mu + \nu > -1$; then $J_\nu^\mu(-z)$ is univalent in the disk U.

COROLLARY 8. *Let $J_\nu^\mu(z)$ be defined by* (5.6) *such that*

$$\left| \frac{zJ_{\nu+\mu}^{\mu}(-z)}{J_{\nu+\mu}^{\mu}(-z)} \right| < \left(\frac{1}{2}\right)^h, \tag{5.8}$$

$z \in U$, $\mu > 0$, $\nu > 0$; then $J_\nu^\mu(-z)$ is univalent in the disk U.

Similarly Theorem 2 in view of (5.6) gives

COROLLARY 9. *Corresponding to $J_\nu^\mu(z)$ defined by* (5.6), *let*

$$\left| \frac{zJ_{\nu+\mu}^{\mu}(-z)}{J_{\nu}^{\mu}(-z)} \right| < \frac{A-B}{1+|B|}, \tag{5.9}$$

for $-1 \leqslant B < A \leqslant 1$, $-1 \leqslant B \leqslant 0$. Then $\Gamma(1+\nu)\, zJ_\nu^\mu(-z) \in S(A, B)$.

Acknowledgements. The author would like to express sincerest thanks to the referee for suggestions.

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Manoscritto pervenuto in redazione il 17 febbraio 1994
e, in forma revisionata, il 12 luglio 1994.