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Propagation of analytic and Gevrey regularity for a class of semi-linear weakly hyperbolic equations

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Propagation of Analytic and Gevrey Regularity for a Class of Semi-Linear Weakly Hyperbolic Equations.

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1. Introduction and notations.

Let Ω be an open set in $\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}^n_x$ (t the «time variable»), $\Omega_+ = \Omega \cap \{t > 0\}$, $\overline{\Omega}_+ = \Omega \cap \{t \geqslant 0\}$, $\Omega_0 = \Omega \cap \{t = 0\}$ and let $u(t, x)$ be a real solution of a semilinear equation

$$(1.1)\qquad P_m(t, x, \partial_{t,x})u + G(t, x, u^{(\alpha)})_{|\alpha| \leqslant m-1} = 0 \quad \text{in } \Omega_+ \, (u^{(\alpha)} = \partial^\alpha_{t,x} u)$$

where G is an analytic function of its arguments and $P_m(t, x, \partial_{t,x})$ is a homogenuous differential operator of order $m \geqslant 2$ with analytic coefficients in Ω which is hyperbolic with respect to the hypersurfaces $t = t_0$.

We are concerned with the problem of the propagation of the analytic regularity of u in a domain of influence $D \subset \overline{\Omega}_+$ provided that the Cauchy data are analytic functions in $\bar{\omega}$, ω a open bounded subset of $\mathbb{R}^n$ such that $\bar{\omega} \subset \Omega_0$ and $D \cap \{t = 0\} \subset \omega$.

From the results of Alinhac and Metivier [2] we know that if P_m is strictly hyperbolic and u is C^∞, then u is analytic in D.

Weakly hyperbolic equations has been considered by Spagnolo [8].

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He proved that if (1.1) is of the type

$$\partial_t^2 u - \sum_{h,k=1}^{n} \partial_{x_k}(a_{hk}(t,x)\,\partial_{x_h} u) = G(t,x,u) \tag{1.2}$$

then u is analytic in D under one of the following conditions:

a) the coefficients a_{hk} have the form $a_{hk}(t,x) = b(t)a^0_{hk}(x)$ and u is of class C^1;

b) the solution u is assumed to belong to some Gevrey class of order less than two.

Here we consider the case where (1.1) is a weakly hyperbolic equation of the form $\partial_t^m u + G(t, x, u^{(\alpha)})_{|\alpha| \leq m-1} = 0$, $m \geq 2$, and prove that the solution is analytic in every cylinder $[0, T] \times \omega$ contained in $\overline{\Omega}_+$ if u is assumed to be in some Gevrey class of order σ_1 smaller than $1/\varrho$ with a index $\varrho \leq 1 - 1/m$ which is determined by the derivatives of u that really appear as arguments of G. In fact we shall prove a more general result (see Theorem 1 below) considering also the propagation of the regularity of u in Gevrey classes when G and the Cauchy data are not analytic but Gevrey functions of order $\sigma \in]1, \sigma_1[$.

Note that our result with $m = 2$ is not covered by [8] since the derivatives of u do not appear in the non linear terms of (1.2).

We denote by $\mathcal{G}^\sigma(\mathcal{O})$, $1 \leq \sigma < \infty$, $\mathcal{O}$ an open subset of $\mathbb{R}^\nu$, the space of Gevrey functions of index σ, i.e. the space of all functions in $C^\infty(\mathcal{O})$ which satisfy for every compact subset K of $\mathcal{O}$:

$$|\partial^\alpha v(x)| \leq CA^{|\alpha|}\alpha!^\sigma, \qquad x \in K, \qquad \alpha \in \mathbb{Z}_+^\nu,$$

C, A constants depending on K (and v).

Moreover we denote by $\gamma^\sigma(\mathcal{O})$, $1 < \sigma < \infty$, the space of all functions v in $C^\infty(\mathcal{O})$ satisfying the following condition: for every $\varepsilon > 0$, for every compact subset K of $\mathcal{O}$ there exists a constants c_ε such that:

$$|\partial^\alpha v(x)| \leq c_\varepsilon \varepsilon^{|\alpha|}\alpha!^\sigma, \qquad x \in K, \qquad \alpha \in \mathbb{Z}_+^\nu,$$

It is $\mathcal{G}^\sigma(\mathcal{O}) \subset \gamma^{\sigma_1}(\mathcal{O}) \subset \mathcal{G}^{\sigma_1}(\mathcal{O})$ for every $1 \leq \sigma < \sigma_1 < \infty$. We write $v \in \mathcal{G}^\sigma(K)$, $v \in \gamma^\sigma(K)$ if $v \in \mathcal{G}^\sigma(\mathcal{O})$, $v \in \gamma^\sigma(\mathcal{O})$ respectively for some open neighbourhood $\mathcal{O}$ of the compact set K.

Consider a function $G(t, x, u^{(\alpha)})_{\alpha \in \mathcal{A}}$, where $(t, x) \in \Omega$ (Ω an open set in $\mathbb{R}^{n+1}$ containing the origin), $\mathcal{A} \subset \{(\alpha_0, \alpha') \in \mathbb{Z}_+ \times \mathbb{Z}_+^n, |\alpha| \leq m-1\}$, m a positive integer, $m \geq 2$. Let $\varrho = \max_{\alpha \in \mathcal{A}} |\alpha'|/(m - \alpha_0)$ and assume that G is a Gevrey function of index σ of its arguments for some $\sigma \in [1, 1/\varrho[$. Moreover assume that g_j, $0 \leq j \leq m-1$, are given Gevrey functions of index σ in $\bar{\omega}$, ω and open bounded subset of $\mathbb{R}^n$ such that $\bar{\omega} \subset \Omega_0$. Then we have:

THEOREM 1. *Let u be a solution of the problem:*

$$(1.3) \qquad \begin{cases} \partial_t^m u + G(t, x, u^{(\alpha)})_{\alpha \in \mathcal{A}} = 0 & \text{in } \Omega_+ , \\ \partial_t^k u_{|t=0} = g_k & \text{in } \omega, k = 0, 1, \ldots, m-1 . \end{cases}$$

If $u \in \mathcal{G}^{\sigma_1}(\overline{\Omega}_+)$ for some $\sigma_1 \in]\sigma, 1/\varrho[$ then $u \in \mathcal{G}^{\sigma}(\mathcal{C})$ for every cylinder

$$\mathcal{C} = [0, T_1] \times \omega \subset \overline{\Omega}_+ .$$

In particular if G, $g_0, \ldots, g_{m-1}$ are analytic functions and $u \in \mathcal{G}^{\sigma_1}(\overline{\Omega}_+)$ for some $\sigma_1 \in]1, 1/\varrho[$, then u is analytic in $\mathcal{C}$.

Note that the Cauchy problem for the linearized equation

$$P = \partial_t^m + \sum_{\alpha \in \mathcal{A}} \frac{\partial G}{\partial u^{(\alpha)}}(t, x, u^{(\alpha)}) \partial_{t,x}^{\alpha}$$

of (1.3) at a solution u may present phenomena of non existence or non uniqueness if u is in a Gevrey class of order greater or equal than $1/\varrho$ (see Komatsu [5], Mizohata [7], Agliardi [1]). Thus it seems difficult to weaken the hypotheses of Theorem 1 as it concerns the a propri regularity of u (cf. the above condition *a*) and *b*) for the equation (1.2) in [8]: if the coefficients are as in condition *a*) then the Cauchy problem for the linearized equation of (1.2) at a C^∞ solution u is well posed in C^∞. In the case of general coefficients, condition *b*) ensures that the Cauchy problem for the linearized equation at $u \in G^{\sigma_1}$ is well posed in G^{σ_1} as in our Theorem 1).

We shall give the proof of Theorem 1 in section 3 after some preliminary lemmas which are the subject of next section 2.

2. Preliminary lemmas.

Let $\mu > n/2$, $0 < \varrho < 1$ be two fixed real numbers. For every $\tau > 0$ we denote by $\mathcal{H}_\tau(\mathbb{R}^n)$ the space of all $u \in L^2(\mathbb{R}^n)$ such that:

$$\|\langle D\rangle^\mu \exp(\tau\langle D\rangle^\varrho) u\|_{L^2(\mathbb{R}^n)} < \infty ,$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, $\xi \in \mathbb{R}^n$.

$\mathcal{H}_\tau(\mathbb{R}^n)$ is a Hilbert space with respect to the inner product:

$$\langle u, v\rangle = (2\pi)^{-n} \int \langle\xi\rangle^{2\mu} \exp(2\tau\langle\xi\rangle^{\varrho})\, \hat{u}(\xi)\, \bar{\hat{v}}(\xi)\, d\xi\,,$$

$\hat{u}$ the Fourier transform of u.

We denote the corresponding norm by $\|\cdot\|_\tau$, i.e.

$$\|u\|_\tau = \|\langle D\rangle^{\mu} \exp(\tau\langle D\rangle^{\varrho})\, u\|_{L^2(\mathbb{R}^n)}\,.$$

Since $\mu > n/2$ and $0 < \varrho < 1$, it is easy to prove, as for the usual Sobolev spaces, that $\mathcal{H}_\tau(\mathbb{R}^n)$ is an algebra. More precisely we have:

PROPOSITION 2.1. *There exists a constant c_0, depending only on n and μ, such that*

$$\|uv\|_\tau \leq c_0 \|u\|_\tau \|v\|_\tau\,, \qquad u, v \in \mathcal{H}_\tau(\mathbb{R}^n)\,. \tag{2.1}$$

For ω an open ball of $\mathbb{R}^n$, we introduce the space $\mathcal{H}_\tau(\bar{\omega})$ of the restrictions to ω of the elements in $\mathcal{H}_\tau(\mathbb{R}^n)$:

$$\mathcal{H}_\tau(\bar{\omega}) = \{v \in L^2(\mathbb{R}^n);\ \exists u \in \mathcal{H}_\tau(\mathbb{R}^n),\ u = v \text{ in } \omega\}$$

endowed with the norm

$$\|v\|_{\tau,\omega} = \|E_\tau(v)\|_\tau\,, \tag{2.2}$$

$E_\tau(v)$ the element of minimum norm in the closed convex subset $\mathcal{E}(v) = \{u \in \mathcal{H}_\tau(\mathbb{R}^n);\ u = v \text{ in } \omega\}$ of the Hilbert space $\mathcal{H}_\tau(\mathbb{R}^n)$.

Thus, $\mathcal{H}_\tau(\bar{\omega})$ is the quotient space of $\mathcal{H}_\tau(\mathbb{R}^n)$ with the closed subspace $M = \{u \in \mathcal{H}_\tau(\mathbb{R}^n);\ u = 0 \text{ in } \omega\}$.

Note that the Paley Wiener Theorem implies $\mathcal{G}^\sigma(\bar{\omega}) \subset \mathcal{H}_\tau(\bar{\omega})$ with continuous injection for $\sigma < 1/\varrho$ and every $\tau > 0$.

In view of Proposition 2.1, $\mathcal{H}_\tau(\bar{\omega})$ is a normed algebra and (2.1) is valid with the same constant c_0 (and $\|\cdot\|_{\tau,\omega}$ instead of $\|\cdot\|_\tau$) for u, $v \in \mathcal{H}_\tau(\bar{\omega})$.

LEMMA 2.2. *Let $w \in \gamma^{1/\varrho}(\bar{\omega})$ be a real valued function and let $\phi \in \mathcal{G}^\sigma(w(\bar{\omega}))$ for a $\sigma \in [1, 1/\varrho[$. Then we can find positive constants τ_0, C, R such that for every $0 < \tau \leq \tau_0$*

$$\|\phi^{(q)}(w)\|_{\tau,\omega} \leq CR^q q!^\sigma\,, \qquad q \in \mathbb{Z}_+\,, \tag{2.2}$$

where R depends only on ϕ and ω, τ_0 depends on ϕ, w and ω, whereas C is a majorant of $\|w\|_{\tau_0,\omega}$.

PROOF. Let K be a compact subset of $\mathbb{R}^n$ such that $\mathring{K} \supset \bar{\omega}$ and $w \in \gamma^{1/\varrho}(K)$ and let us denote $H = w(K)$. then we have:

$$\sup_H |\phi^{(q)}| \leqslant R_0 R_1^q q!^\sigma \qquad (\exists R_0, R_1, \forall q),$$

$$\sup_K |\partial^\alpha w| \leqslant c_h h^{|\alpha|} \alpha!^{1/\varrho} \qquad (\forall h \exists c_h, \forall \alpha).$$

By using Faa-De Bruno's formula, we obtain

$$(2.3) \qquad |\partial^\gamma \phi^{(q)}(w(x))| \leqslant 2^\sigma R_0 (2^\sigma R_1)^q q!^\sigma ((2d)^\sigma R_1 h c_h)^{|\gamma|} |\gamma|^{|\gamma|/d}$$

for every $x \in K$, $\gamma \in \mathbb{Z}_+^n$, $q \in \mathbb{Z}_+$, where the constant d depends only on σ and n.

Let $\chi \in \gamma^{1/\varrho}(\mathbb{R}^n)$, $\operatorname{supp} \chi \subset K$, $\chi = 1$ in a neighbourhood of $\bar{\omega}$, and

$$\sup |\partial^\alpha \chi| \leqslant l_h h^{|\alpha|} \alpha!^{1/\varrho} \qquad (\forall h \exists l_h, \forall \alpha).$$

From (2.3) it follows:

$$|\xi^\gamma (\widehat{\chi \phi^{(q)}(w)})(\xi)| \leqslant C_h (A_h h)^{|\gamma|} |\gamma|^{|\gamma|/\varrho} R^q q!^\sigma, \qquad \xi \in \mathbb{R}^n,$$

where $C_h = 2^\sigma R_0 l_h$ meas (K), $A_h = (2d)^\sigma R_1 c_h + 1$, $R = 2^\sigma R_1$.

Hence, by the arbitrariness of γ:

$$(2.4) \qquad |(\widehat{\chi \phi^{(q)}(w)})(\xi)| \leqslant C_1 \exp(-k_1 \langle \xi \rangle^\varrho) R^q q!^\sigma$$

for a constant $k_1 \geqslant d' A_1^{-\varrho}$, d' depending only on n, σ, ϱ.

From (2.4) it follows (2.2) for every $\tau \leqslant k_1/2 = \tau_0$, with

$$C = C_1 \left(\int \langle \xi \rangle^{2\mu} \exp(-2\tau_0 \langle \xi \rangle^\varrho)\, d\xi \right)^{1/2}$$

and the proof is complete.

Now we introduce some notations: we consider the sequence $m_p = a(p!^\sigma/(p+1)^2)$, where $\sigma \geqslant 1$ and the constant a is chosen in order to satisfy:

$$\sum_{0 \leqslant \beta \leqslant \alpha} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} m_{|\beta|} m_{|\alpha-\beta|} \leqslant m_{|\alpha|},$$

$$\sum_{0 < \beta \leqslant \alpha} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} m_{|\beta|} m_{|\alpha-\beta|+1} \leqslant |\alpha| m_{|\alpha|}.$$

For $\varepsilon > 0$, $p \geqslant 1$ we define $M_p = \varepsilon^{1-p} m_p$ and for $w \in \gamma^{\varrho}(\bar{\omega})$, $p \geqslant 1$ we let

$$|w|_{p,\tau} = \sup_{|\alpha| = p} \|\partial_x^{\alpha} w\|_{\tau, w},$$

$$[w]_{p,\tau} = \sup_{0 < q \leqslant p} \frac{|w|_{q,\tau}}{M_q}.$$

As in [2], from Proposition 2.1 and Lemma 2.2 we can prove the following lemma by the method of majorant series:

LEMMA 2.3. *If w and ϕ satisfy the hypotheses of Lemma 2.2, then there exist τ_0, L, $\delta_0 > 0$ such that for every $p \geqslant 1$, $\varepsilon > 0$, $0 \leqslant \tau \leqslant \tau_0$ the condition*

$$\varepsilon [w]_{p,\tau} \leqslant \delta_0 \tag{2.5}$$

implies:

i) $[\phi(w)]_{p,\tau} \leqslant L[w]_{p,\tau}$,

ii) $|\phi(w)|_{p+1,\tau} \leqslant L(|w|_{p+1,\tau} + M_{p+1}[w]_{p,\tau})$,

iii) *for* $|\alpha| = p+1$, $j = 1, \ldots, n$,

$$\|\partial_x^{\alpha}(\phi(w)\,\partial_j w) - \phi(w)\,\partial^{\alpha}\partial_j w\|_{\tau,\omega} \leqslant$$
$$\leqslant (p+1) L\{|w|_{p+1,\tau} + M_{p+1}(1 + [w]_{p,\tau})^2\},$$

where τ_0 is the constant found in Lemma 2.2, L and δ_0 depend only on ϕ and $\|\mathrm{grad}\, w\|_{\tau_0,\omega}$.

Lemma 2.3 can be extended to functions $w = (w_1, \ldots, w_\nu)$ with values in $\mathbb{R}^{\nu}$, by defining $|w|_{p,\tau} = \max_{1 \leqslant j \leqslant \nu} |w_j|_{p,\tau}$. In fact we have:

LEMMA 2.4. *Let $w = (w_1, \ldots, w_\nu)$, w_j real functions in $\gamma^{1/\varrho}(\bar{\omega})$, and let $\phi(x, w) \in \mathcal{G}^{\sigma}(\bar{\omega} \times w(\bar{\omega}))$ for a $\sigma \in [1, 1/\varrho[$. There exist four positive constants τ_0, ε_0, δ_0, L such that for $p \geqslant 1$, $\varepsilon \in]0, \varepsilon_0]$, $\tau \in [0, \tau_0]$, condition* (2.5) *implies:*

j) $[\phi(\cdot, w)]_{p,\tau} \leqslant L(1 + [w]_{p,\tau})$,

jj) $|\phi(\cdot, w)|_{p+1,\tau} \leqslant L\{|w|_{p+1,\tau} + M_{p+1}(1 + [w]_{p,\tau})\}$,

jjj) *for* $|\alpha| = p+1$, $j = 1, \ldots, n$, $k = 1, \ldots, \nu$ *it is*

$$\|\partial_x^{\alpha}(\phi(\cdot, w)\,\partial_j w_k) - \phi(\cdot, w)\,\partial^{\alpha}\partial_j w_k\|_{\tau,\omega} \leqslant$$
$$\leqslant (p+1) L\{|w|_{p+1,\tau} + M_{p+1}(1 + [w]_{p,\tau})^2\}.$$

In next section we apply Lemma 2.4 to functions $w(t, x)$, $\phi(t, x, w(t, x))$, where $w \in \mathcal{G}^{\sigma_1}([0, T_1]; \mathcal{G}^{\sigma_1}(\bar{\omega})^\nu)$, $\phi \in \mathcal{G}^{\sigma}([0, T_1] \times \bar{\omega} \times \times w([0, T_1] \times \bar{\omega}))$, $1 \leqslant \sigma < \sigma_1 < 1/\varrho$ for which t is considered as a parameter. By Lemma 2.4 there exist positive constants $\tau_0, \varepsilon_0, \delta_0, L$ such that for every integer $p \geqslant 1$, $\varepsilon \in]0, \varepsilon_0]$, $0 \leqslant \tau \leqslant \tau_0$, if $\varepsilon[w(t, \cdot)]_{p, \tau} \leqslant \delta_0$ for every $t \in [0, T_1]$ then j), jj), jjj) are true uniformly with respect to t in this interval.

For the function $w(t, x)$ we use also the seminorms we are going to introduce. Let c, T be positive constants such that $cT \leqslant \tau_0$, $T \leqslant T_1$ and let

$$M_p^t = (\varepsilon \exp(-\lambda t))^{1-p} m_p = \exp(-\lambda t(1-p)) M_p ,$$

where $\lambda \in \mathbb{R}$ is a parameter which will be choosen in a suitable way at the end of the proof of Theorem 1.1. For $0 < \varepsilon \leqslant \varepsilon_0$, $0 \leqslant t \leqslant T$, $p \geqslant 1$ we define:

$$|w|_p^t = \max_{|\beta| = p} \|\partial_x^\beta w(t, \cdot)\|_{c(T-t), \omega} ,$$

$$[w]_p^t = \max_{1 \leqslant q \leqslant p} \frac{|w|_q^t}{M_q^t} .$$

Moreover, for the solution u of (1.3) we write:

$$\Phi_p^t(u) = \max_{\alpha \in \mathcal{A}_1} |\partial_{t,x}^\alpha u|_p^t ,$$

$$\Psi_p(u) = \sup_{0 \leqslant t \leqslant T} \max_{1 \leqslant q \leqslant p} \frac{\Phi_q^t(u)}{M_{q-1}^t} , \qquad p \geqslant 2 ,$$

where

$$\mathcal{A}_1 = \{(\alpha_0, \alpha') \in \mathbb{Z}_+ \times \mathbb{Z}_+^n ; (\alpha_0, \alpha' + e_j) \in \mathcal{A}, j = 1, \dots, n\} \cup$$
$$\cup \{(\alpha_0, 0) \in \mathbb{Z}_+ \times \mathbb{Z}_+^n , \alpha_0 \leqslant m-1\} ,$$

$\{e_j\}_{1 \leqslant j \leqslant n}$ is the canonical basis of $\mathbb{R}^n$.

3. Proof of Theorem 1.

By deriving the equations (1.3) with respect to x_j we get:

$$(3.1) \quad \begin{cases} \partial_t^m \partial_j u + \sum_{\alpha \in \mathcal{A}} \dfrac{\partial G}{\partial u^\alpha} \partial_{t,x}^\alpha \partial_j u = -\dfrac{\partial G}{\partial_{x_j}} & \text{in }]0, T_1] \times \omega , \\ \partial_t^k \partial_j u_{|t=0} = \partial_j g_k & \text{in } \omega, \ k = 0, 1, \dots, m-1 . \end{cases}$$

Now we apply the operator ∂_x^β to (3.1), so obtaining:

$$\begin{cases} Pv_{\beta,j} = f_{\beta,j} & \text{in }]0, T_1] \times \omega, \\ \partial_t^k v_{\beta,j\,|t=0} = g_{k,\beta,j} & \text{in } \omega, \ k = 0, 1, \dots, m-1, \end{cases} \tag{3.2}$$

where P is the linear operator

$$\partial_t^m + \sum_{\alpha \in \mathcal{A}} \frac{\partial G}{\partial u^{(\alpha)}}(t, x, u^{(\alpha)})\, \partial_{t,x}^\alpha,$$

$$v_{\beta,j} = \partial_x^\beta \partial_j u,$$

$$f_{\beta,j} = -\partial_x^\beta \left(\frac{\partial G}{\partial x_j}(t, x, u^{(\alpha)}) \right) +$$

$$- \sum_{\alpha \in \mathcal{A}} \left\{ \partial_x^\beta \left[\frac{\partial G}{\partial u^{(\alpha)}}(t, x, u^{(\alpha)})\, \partial_{t,x}^\alpha \partial_j u \right] - \frac{\partial G}{\partial u^{(\alpha)}}(t, x, u^{(\alpha)})\, \partial_{t,x}^\alpha \partial_x^\beta \partial_j u \right\},$$

and

$$g_{k,\beta,j} = \partial_x^\beta \partial_j g_k(x), \qquad k = 0, 1, \dots, m-1.$$

From the hypotheses of Theorem 1 it follows that P can be written in the form $P = \partial_t^m + \sum_{j=1}^m P_j(t, x, D_x) D_t^{m-1}$, where P_j is a linear operator of order less or equal to ϱj with coefficients in $\mathcal{G}^{\sigma_1}(\overline{\Omega}_+)$; also $f_{\beta,j} \in \mathcal{G}^{\sigma_1}(\overline{\Omega}_+)$ whereas $g_{k,\beta,j} \in \mathcal{G}^\sigma(\bar{\omega})$.

We extend the coefficients in the lower order terms of P outside a neighborhood of $[0, T_1] \times \bar{\omega}$ to functions in $\mathcal{G}^\sigma(\mathbb{R}^{n+1})$ with compact support and denote by $\widetilde{P}$ the linear operator in $[0, T_1] \times \mathbb{R}^n$ obtained in this way. Moreover, we set $\tilde{f}_{\beta,j}(t) = E_{c(T-t)}(f_{\beta,j}(t))$, $\tilde{g}_{k,\beta,j} = E_{cT}(g_{k,\beta,j})$, where E_τ is the extension operator defined at the beginning of section 2.

By letting

$$\Lambda = \langle D_x \rangle^\varrho, \qquad U_{\beta,j} = {}^t(\Lambda^{m-1} \tilde{v}_{\beta,j}, \Lambda^{m-2} \partial_t \tilde{v}_{\beta,j}, \dots, \partial_t^{m-1} \tilde{v}_{\beta,j}),$$

$$F_{\beta,j} = {}^t(0, \dots, \tilde{f}_{\beta,j}),\ G_{\beta,j} = {}^t(\Lambda^{m-1} \tilde{g}_{0,\beta,j}, \Lambda^{m-2} \tilde{g}_{1,\beta,j}, \dots, \tilde{g}_{m-1,\beta,j}),$$

the problem

$$\begin{cases} \widetilde{P} \tilde{v}_{\beta,j} = \tilde{f}_{\beta,j} & \text{in }]0, T_1] \times \mathbb{R}^n, \\ \partial_t^j \tilde{v}_{|t=0} = \tilde{g}_{k,\beta,j} & \text{in } \mathbb{R}^n, \ k = 0, 1, \dots, m-1, \end{cases} \tag{3.3}$$

is equivalent to a system

$$\left\{\begin{array}{ll}\partial_t U_{\beta,j} + AU_{\beta,j} = F_{\beta,j} & \text{in }]0, T_1] \times \mathbb{R}^n\,, \\ U_{\beta,j}|_{t=0} = G_{\beta,j} & \text{in } \mathbb{R}^n\,,\end{array}\right. \tag{3.4}$$

where $A = (a_{i,k}(t, x, D_x))$ is a suitable $m \times m$ matrix of pseudo-differential operators with symbols that satisfy:

$$|\partial_t^l \partial_x^\gamma \partial_\xi^\alpha a_{i,k}(t, x, \xi)| \leq C^{l+|\alpha|+|\gamma|+1} \alpha!(\gamma!\, l!)^\sigma \langle \xi \rangle^{\varrho - |\alpha|} \tag{3.5}$$

$(\exists C, \forall l, \alpha, \gamma, \forall (t, x, \xi) \in [0, T_1] \times \mathbb{R}^n \times \mathbb{R}^n)$.

In [3] Cattabriga-Mari proved that the Cauchy problem (3.4) is well posed in $\mathcal{G}^{\sigma_1}$ and constructed for it a fundamental solution $M(t, s)$, $t, s \in [0, T_2]$, $T_2 \leq T_1$ a suitable positive number depending only on the constant C in (3.5). It is $M(t, s) = M_1(t, s) R(t, s)$ where $R(t, s)$ is an operator applying continuously $(\mathcal{H}_\tau(\mathbb{R}^n))^m$ into itself for every τ, $M_1(t, s)$ is a matrix of pseudo-differential operators of infinite order on $\mathbb{R}^n$ with symbol $M_1(t, s; x, \xi)$ satisfying:

$$\begin{aligned}|\partial_t^k \partial_x^\gamma \partial_\xi^\alpha M_1(t, s; x, \xi)| \leq{}& \\ \leq C_1^{k+|\alpha|+|\gamma|+1} \alpha!(\gamma!\, k!)^{\sigma_1} \langle \xi \rangle^{-|\alpha|} &\exp(c|t-s|\langle \xi \rangle^\varrho),\end{aligned} \tag{3.6}$$

$t, s \in [0, T_2]$, $x, \xi \in \mathbb{R}^n$.

We can prove the following lemma (cf. Propositions 2.8 and 2.12 in [4]):

LEMMA 3.1. *Let $Q(x, D_x)$ be a pseudo-differential operator of infinite order with symbol q satisfying*

$$|\partial_x^\beta \partial_\xi^\alpha q(x, \xi)| \leq C_\alpha L^{-|\beta|} (\beta!)^{1/\varrho} \langle \xi \rangle^{-|\alpha|} \exp(\delta \langle \xi \rangle^\varrho), \qquad \delta \geq 0, \tag{3.7}$$

for every $x, \xi \in \mathbb{R}^n$. There exist two positive constants A and δ_1 depending only on μ, ϱ, n and L in (3.7) such that

$$\|Qu\|_\tau \leq A \|u\|_{\tau+\delta} \tag{3.8}$$

for every $u \in \mathcal{H}_{\tau+\delta}(\mathbb{R}^n)$ with $\tau \leq \delta_1$.

PROOF. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\psi(\xi) = 1$ for $|\xi| \leq 1/2$ and set $\psi_j(\xi) = \psi(\xi/j)$ for $j \in \mathbb{Z}_+$. Let us consider the operators

$$R_j(x, D_x) =$$

$$= \exp(\tau \langle D_x \rangle^\varrho) \langle D_x \rangle^\mu \psi_j(D_x) Q(x, D_x) \psi_j(D_x) \langle D_x \rangle^{-\mu} \exp(-(\tau+\delta) \langle D_x \rangle^\varrho)\,.$$

Our aim is to prove that the symbols $r_j(x, \xi)$ satysfy

$$|\partial_x^\beta \partial_\xi^\alpha r_j(x, \xi)| \leq C_{\alpha, \beta} \tag{3.9}$$

for every $x, \xi \in \mathbb{R}^n$ and every $j \in \mathbb{Z}_+$, provided that $\tau \leq \delta_1$, with constants $C_{\alpha, \beta}$ depending on α, β, n and μ, and δ_1 depending on μ, ϱ, n and L in (3.7). Then (3.8) will follow as a consequence of the Calderón-Vaillancourt theorem about the L^2 boundness of pseudo-differential-operators with symbols in the space $S^0_{0,0}$.

The symbol $r_j(x, \xi)$ is represented by means of an oscillatory integral

$$r_j(x, \xi) = Os - (2\pi)^{-n} \int\int \exp(i(x-y)(\eta - \xi))\, a_j(y, \eta, \xi)\, dy\, d\eta,$$

$a_j(y, \eta, \xi) = \exp(\tau\langle\eta\rangle^\varrho)\langle\eta\rangle^\mu \psi_j(\eta) q(\gamma, \xi) \psi_j(\xi) \langle\xi\rangle^{-\mu} \exp(-(\tau + \delta)\langle\xi\rangle^\varrho)$.

For $N = 0, 1, \ldots$, and δ_1 a positive constant to be fixed later on, we put $\Omega_N = \{\eta; (N/\delta_1)^{1/\varrho} \leq |\xi - \eta| \leq ((N+1)/\delta_1)^{1/\varrho}\}$ and we write $r_j(x, \xi) = \sum_{N=0}^{\infty} r_{j,N}(x, \xi)$ with

$$r_{j,N}(x, \xi) = \tag{3.10}$$

$$= \int_{\Omega_N} \left[\int \exp(i(x-y)(\eta - \xi)) \langle x - y\rangle^{-2l} (1 - \Delta_\eta)^l a_j(y, \eta, \xi)\, dy\right] (2\pi)^{-n}\, d\eta,$$

l a fixed integer greater than $n/2$.

Integrating by parts N times with respect to y in (3.10) we get

$$|r_{j,N}(x, \xi)| \leq C_1 (L_1 \delta_1^{-1/\varrho})^{-N} \exp((\tau/\delta_1) N)$$

with C_1 depending on n and μ, L_1 depending on n, μ, ϱ and L in (3.7). Then we can choose $\delta_1 = (L_1/2e)^\varrho$ to have (3.9) with $\alpha = \beta = 0$. In the same way, we can achieve (3.9) for $|\alpha|$, $|\beta| > 0$, completing the proof.

Returning to the fundamental solution $M(t, s)$ of (3.4) constructed in [3], from Lemma 3.1 it follows that there exist constants $A_0 > 0$, $T_3 \in]0, T_2[$, T_3 depending on the constant C_1 in (3.6), such that

$$\|M(t, s) V\|_{c(T-t)} \leq A_0 \|V\|_{c(T-s)}, \tag{3.11}$$

for $0 \leq s < t \leq T \leq T_3$ and every $V \in (\mathcal{H}_{c(T-s)}(\mathbb{R}^n))^m$ with c the same constant that appears in the right side of (3.6).

Hence, for the solution

$$U_{\beta,j}(t,x)=\int_0^t (M(t,s)F_{\beta,j}(s))(x)\,ds+M(t,0)\,G_{\beta,j}(x)$$

of (3.4), we have for $t\in[0,T]$:

$$\|U_{\beta,j}(t)\|_{c(T-t)}\leqslant A_0\left(\int_0^t \|F_{\beta,j}(s)\|_{c(T-s)}\,ds+\|G_{\beta,j}\|_{cT}\right).$$

If $\alpha=(\alpha_0,\alpha')\in\mathcal{C}_1$ (see the definition at the end of section 2), then $\alpha_0+|\alpha'|/\varrho\leqslant m-1$. Hence:

$$\|\partial^\alpha_{t,x}\tilde v_{\beta,j}(t)\|_{c(T-t)}\leqslant \|\Lambda^{m-1-\alpha_0}\partial_t^{\alpha_0}\tilde v_{\beta,j}(t)\|_{c(T-t)}\leqslant \|U_{\beta,j}(t)\|_{c(T-t)},$$

$0\leqslant t\leqslant T\leqslant T_3$.

Since the solution of (3.2) is unique ([3], [4], [7], [9]), we have $v_{\beta,j}(t,x)=\tilde v_{\beta,j}(t,x)$ for $(t,x)\in[0,T_3]\times\omega$. By $\|\partial^\alpha_{t,x}v_{\beta,j}(t)\|_{c(T-t),\,\omega}\leqslant$ $\leqslant\|\partial^\alpha_{t,x}\tilde v_{\beta,j}(t)\|_{c(T-t)}$ and $\|f_{\beta,j}(s)\|_{c(T-s),\,\omega}=\|F_{\beta,j}(s)\|_{c(T-s)}$ from the above inequalities we obtain:

$$\max_{\alpha\in\mathcal{C}_1}\|\partial^\alpha_{t,x}v_{\beta,j}(t)\|_{c(T-t),\,\omega}\leqslant A_0\left(\int_0^t\|f_{\beta,j}(s)\|_{c(T-s),\,\omega}\,ds+\|G_{\beta,j}\|_{cT}\right), \tag{3.12}$$

$0\leqslant t\leqslant T\leqslant T_3$.

Taking the maximum value for $|\beta|=p$ and $j=1,\ldots,n$ in the left side of (3.12) we have the seminorm $\Phi^t_{p+1}(u)$ of the solution u of (1.3). Our next aim is to estimate $\|f_{\beta,j}(s)\|_{c(T-s),\,\omega}$ from above by seminorms $\Phi^t_{p+1}(u)$ in order to deduce from (3.12) an inequality to which we are able to apply Gronwall's Lemma, so obtaining estimates for $\Phi^t_{p+1}(u)$. We state the following Lemma, which is an easy consequence of Lemma 2.4. (See [2] for details).

LEMMA 3.2. *There exist positive constants* $\varepsilon_0,\tau_0,\delta_0,L$ *such that for* $p\geqslant 2$, $0<\varepsilon\leqslant\varepsilon_0$, $cT\leqslant\tau_0$, $\lambda\geqslant 0$ *the condition*

$$\varepsilon\Psi_p(u)\leqslant\delta_0 \tag{3.13}$$

implies:

$$\|f_{\beta,j}(t,\cdot)\|_{c(T-t),\,\omega}\leqslant Lp\{\Phi^t_{p+1}(u)+M^t_p(1+\Psi_p(u))^2\}$$

for every $|\beta|=p$, $j=1,\ldots,n$.

Next we fix $T = \min\{T_3, \tau_0/c\}$, where τ_0 is the constant in Lemma 3.2 and T_3, c are the constants for which (3.11) holds. From (3.12) and Lemma 3.2 it follows that for $p \geqslant 2$, $0 < \varepsilon \leqslant \varepsilon_0$, $\lambda \geqslant 0$ the condition (3.13) implies:

$$\Phi^t_{p+1}(u) \leqslant LA_0 p \int_0^t \{\Phi^s_{p+1}(u) + M^s_p(1 + \Psi_p(u))^2\} ds + G_p , \tag{3.14}$$

$0 \leqslant t \leqslant T$, where $G_p = A_0 \max_{|\beta| = p,\, 1 \leqslant j \leqslant n} \|G_{\beta, j}\|_{cT}$.

By using Gronwall's inequality and letting $LA_0 = L_0$, we obtain from (3.14)

$$\Phi^t_{p+1}(u) \leqslant \tag{3.15}$$

$$\leqslant G_p \exp[L_o pt] + L_o p(1 + \Psi_p(u))^2 \int_0^t \exp(L_0 p(t-s)) M^s_p ds .$$

For $\lambda \geqslant 6 L_0$, $p \geqslant 2$ we have:

$$p \int_0^t \exp(L_0 p(t-s)) M^s_p ds \leqslant pM_p \frac{\exp(\lambda(p-1)t)}{\lambda(p-1) - L' p} \leqslant (3/\lambda) M^t_p . \tag{3.16}$$

Since $g_k \in \mathcal{G}^\sigma(\bar{\omega})$, it is $G_p \leqslant A_1 \varepsilon_1^{1-p} m_p$ for suitable positive constants A_1, ε_1. Hence for $0 < \varepsilon < \varepsilon_1$, $\lambda \geqslant 2L_0$, $p \geqslant 2$

$$G_p \exp(L_0 pt) \leqslant A_1 M^t_p . \tag{3.17}$$

From (3.15), (3.16), (3.17) we obtain for $p \geqslant 2$, $0 < \varepsilon < \tilde{\varepsilon}_0 = \min(\varepsilon_1, \varepsilon_0)$, $\lambda \geqslant \lambda_0 = 6L_0$:

$$\Phi^t_{p+1}(u) \leqslant L_1(1 + (\Psi_p(u))^2/\lambda) M^t_p , \qquad 0 \leqslant t \leqslant T . \tag{3.18}$$

Summing up, we have proved that:

(3.19) Condition (3.13) implies inequality (3.18)

for $p \geqslant 2$, $\varepsilon \in]0, \tilde{\varepsilon}_0]$, $\lambda \geqslant \lambda_0$.

Now we let $H = \max(2L_1, \Psi_2(u))$, $\lambda = \max(\lambda_0, 2HL_1)$ and fix $\varepsilon \in]0, \tilde{\varepsilon}_0]$ such that $\varepsilon H < \delta_0$, where δ_0 is the same constant as in Lemma 3.2. In this way we have $L_1(1 + H^2/\lambda) \leqslant H$.

Since $\Psi_{p+1}(u) = \max(\Psi_p(u), \sup_{t \in [0, T]} (\Phi_{p+1}^t(u)/M_p^t))$, by using (3.19) it is easy to prove inductively that

$$\Psi_p(u) \leq H, \qquad p \geq 2.$$

This means

$$\|\partial_x^\beta u(t, \cdot)\|_{c(T-t), \omega} \leq H M_{q-1}^t$$

for every β, $|\beta| = q \geq 2$, $0 \leq t \leq T$. Hence $u(t, \cdot) \in \mathcal{G}^\sigma(\omega)$ for $0 \leq t \leq T$. Moreover, from equations (1.3), by using the method of majorant series (see for example [6] we can prove that u is a Gevrey function of index σ also with respect to $t \in [0, T]$. By applying this result a finite number of times in the cylinders $[T, 2T] \times \omega, \ldots, [k_0 T, T_1] \times \omega$, we obtain $u \in \mathcal{G}^\sigma([0, T_1] \times \omega)$.

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