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A Result on B_1-Groups.

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We shall work with torsionfree abelian groups H satisfying the condition (iii) in Arnold's paper [A], that means that when localized at any prime p, H becomes completely decomposable and if B is a generalized regular subgroup of H and L is a finite rank pure subgroup of H, then $(L + B)/B$ has a finite number of non-zero p-primary components, only.

In this note we show that a B_1-group has always this property and correct the proof of Theorem I.a — (ii) $\Rightarrow$ (iii) of Arnold [A]. Our argument also corrects and greatly simplifies the proof of Theorem 3.4 (that a countable B_1-group is finitely Butler) of Bican-Salce [BS]. Moreover, the proofs of Proposition 9 and Theorem 11 in [BSS] used the same incorrect argument (see the Remark at the and of this note) as in the proof of Theorem 3.4 of [BS] and the proof of Theorem 1 of Dugas [D] assumed the truth of Theorem I.a — (ii) $\Rightarrow$ (iii) of [A]. So their validity is also assured by our proof presented below.

All the groups that we consider are abelian and we refer to [F] for the general notation and terminology. Recall that a torsionfree group G is called a *B_1-group* if $\mathrm{Bext}^1(G, T) = 0$ for all torsion groups T, where Bext^1 denotes the subfunctor of Ext^1 consisting of all balanced extensions. A subgroup K of a torsionfree group G is said to be *generalized regular*, if G/K is torsion and for any rank one pure subgroup X of G, the p-component $(X/(X \cap K))_p = 0$ for almost all primes p.

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Theorem *Let H be an arbitrary B_1-group and B a generalized regular subgroup of H. Then for any finite rank pure subgroup L of H, the torsion group $(L+B)/B$ has at most finitely many non-zero p-components.*

PROOF. As pointed out by Arnold [A], we can assume, without loss of generality, that H/B is isomorphic to a subgroup of Q/Z, since there is $B \subseteq C \subseteq H$ with H/C isomorphic to a subgroup of Q/Z and, for any prime p, $(H/B)_p = 0$ exactly when $(H/C)_p = 0$. Suppose, by way of contradiction, there is a finite rank pure subgroup L of H with $((L+ + B)/B)_{p_i} \neq 0$, for infinitely many primes p_i. Without loss of generality, we may assume that $(H/B)_p = 0$ for all prime $p \notin \{p_i \mid i < \omega\}$. Write $H/B = \bigoplus_{i<\omega} Z(p_i^{k_i})$ where $0 < k_i \leqslant \infty$. For each i, choose $x_i \in L$ so that $x_i + B$ generates the p_i-socle of H/B. If $\{h_1, \ldots, h_n\}$ is a maximal independent subset of $L \cap B$, then replacing x_i, if necessary, by an integral multiple of x_i, we could assume that there is an integer $s_i \geqslant 1$ such that

$$(*) \qquad p_i^{s_i} x_i = l_{i1} h_1 + \ldots + l_{in} h_n ,$$

where the l_{ij} are integers. If we use the convention that $\infty + s_i = \infty$ and denote $Z(p_i^{k_i+s_i})$ by C_i, then, for each $i < \omega$, we get an exact sequence

$$0 \to C_i[p_i^{s_i}] \to C_i \xrightarrow{\gamma_i} (H/B)_{p_i} \to 0 .$$

Let $C = \oplus C_i$. Then $\gamma = \oplus \gamma_i$ is an epimorphism from C to H/B. Consider the pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & G & \xrightarrow{\varphi} & H & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & T & \longrightarrow & C & \xrightarrow{\gamma} & H/B & \longrightarrow & 0 \end{array}$$

where $\pi: H \to H/B$ is the natural map. We claim that the top row is balanced exact: Suppose R is a rank one group and $\alpha: R \to H$ is a homomorphism, with $K = \ker(\pi\alpha)$. Since B is generalized regular, there is an integer n such that $(R/K)_p = 0$, for all primes $p \notin \{p_1, \ldots, p_n\}$. Then the obvious map $R \to R/(p_1^{s_1} p_2^{s_2} \ldots p_n^{s_n} K)$ induces a $\beta: R \to C$ such that $\gamma\beta =$

$= \pi\alpha$. By the pull-back property, there exists an $\alpha'\colon R \to G$ satisfying $\varphi\alpha' = \alpha$. This establishes our claim. As H is a B_1-group, the top row then splits. Let $\delta\colon H \to G$ be the split map. If we regard $G = \{(c, h) \mid \gamma(c) = \pi(h)\} \subseteq C \oplus H$, then we can write $\delta(h_k) = (y_k, h_k)$, for $k = 1, \ldots, n$, where $y_k \in C$. Since C is torsion, there is an integer m such that $\delta(mh_k) = (0, mh_k)$ for all $k = 1, \ldots, n$. For each $i < \omega$, let $\delta(x_i) = (z_i, x_i)$. From (*) we conclude that

$$(mp_i^{s_i} z_i, mp_i^{s_i} x_i) = \delta(mp_i^{s_i} x_i) = (0, mp_i^{s_i} x_i),$$

so that $mp_i^{s_i} z_i = 0$. Since $\gamma(z_i) = \pi(x_i) = x_i + B \neq 0$ we have $z_i \notin \ker(\gamma_i) = C_i[p_i^{s_i}]$. This means that p_i must be a divisor of m. Since there are infinitely many primes p_i, we obtain a contradiction.

REMARK. We wish to justify the statements made at the beginning of this note by pointing out where exactly the inaccuracies have occured in the referenced articles: In [A], it occurs on page 180 at bottom paragraph in the sentence «Since H has finite rank ...». In [BS], in the proof of Theorem 3.4 on page 187, the index s_i on the right side of the equation (9) should be $s_i + 1$ and this leads to the (corrected) conclusion on line 3 from the bottom that «p_i divides $p_i \varrho \lambda_{ir}$» which does not imply that p_i divides ϱ, as claimed. To correct the proof of Theorem 3.4 of [BS], just delete the entire part of the proof beginning with the sentence «Finally, the factor group ...»: on line 8 on page 186 and insert the proof of our Theorem. We wish to point out that Theorem 3.4 of [BS] has been derived by different methods in [DR], [FM] and [MV]. The proof of our theorem above was obtained by distilling arguments used by Arnold [A] and by Bican-Salce [BS] and some our arguments are similar to those used in [FM].

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