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Singular limits in fluidynamics

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Singular Limits in Fluidynamics.

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Introduction.

In this paper we consider the equations of motion of a compressible fluid depending on the viscosity coefficients ν and μ and on the Mach number λ^{-1}. We assume that $\lambda \geq 1$ (we are interested on letting λ go to ∞) and that $\nu \in [0, \nu_0]$, $\mu \in [0, \mu_0]$, where the constants ν_0 and μ_0 are fixed. Viscous and inviscid fluids are treated indistinctly, since 0 is an admissible value to the parameters ν and μ.

We denote by v the velocity field, by ϱ the density and by $p(\lambda, \varrho)$ the law of state that links the pressure p to density ϱ. We are interested on studying the behaviour of the solution (ϱ, v) as, simultaneously, the Mach number goes to zero (i.e., as $\lambda \to \infty$) and $\nu \to \bar{\nu} \geq 0$. Our proofs hold if $p(\lambda, \varrho)$ satisfies the general assumptions described in our previous papers [BV1,2]. The crucial point is to assume that $\lim p'(\lambda, \bar{\varrho}_0) = \infty$, as $\lambda \to \infty$, where $\bar{\varrho}_0$ denotes the «mean density» of the fluid. Nevertheless we will consider the particular case

$$p(\lambda, \varrho) = \lambda^2 p(\varrho), \tag{1.1}$$

where $p(\cdot)$ is a fixed function. The extension of our results to general $p(\lambda, \varrho)$'s can be done by following devices similar to those used in references [BV1,2]. For convenience we assume that the external forces vanish since the manipulations needed to treat the corresponding extra terms are quite obvious. We consider the case where

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x belongs to the n-dimensional thorus, $n \geq 2$, identified here to the set $\Omega \equiv [0, a[^n$. Without loss of generality we set $a = 1$.

In the sequel $\mathbb{R}^+$ denotes the set of positive reals, moreover $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$. We denote by k_0 the smallest integer larger than $n/2$ and by k a fixed integer such that $k \geq k_0 + 1$. We assume that $p \in C^{k+1}(\mathbb{R}^+; \mathbb{R})$ and that $p'(s) > 0$ for each $s \in \mathbb{R}^+$. We denote by $|\cdot|_p$ and $\|\cdot\|_l$ the usual norms in $L^p = L^p(\Omega)$, $1 \leq p \leq \infty$, and in the L^2-Sobolev space $H^l = H^l(\Omega)$, respectively. We set $\|\cdot\| = \|\cdot\|_0$. By $\|\cdot\|_{l,T}$ and $[\cdot]_{l,T}$ we denote the canonical norms in $L^\infty(0, T, H^l)$ and $L^2(0, T; H^l)$ respectively. A function $f(t, \cdot)$ of the space variable x (for a fixed t) is sometimes denoted by $f(t)$.

The equations of motion of a compressible fluid in Ω under the costitutive relation (1.1) are

$$
\begin{cases}
\varrho_t + v \cdot \nabla \varrho + \varrho \nabla \cdot v = 0\,, \\
\varrho(v_t + (v \cdot \nabla) v) + \lambda^2 p'(\varrho) \nabla \varrho = \nu \Delta v + \mu \nabla(\nabla \cdot v)\,, \\
\varrho(0) = \bar{\varrho}_0 + \varrho_0(x)\,, \qquad v(0) = v_0(x)\,,
\end{cases}
\tag{1.2}
$$

where $\bar{\varrho}_0$ is a fixed positive constant. We assume that $\bar{\varrho}_0 + \varrho_0(x) \geq c_{0'}$, where c_0 is a fixed constant. We study the above problem by making the change of variable

$$
g = \log \varrho / \bar{\varrho}_0 \,. \tag{1.3}
$$

Since the general case can be brought back to the case $\bar{\varrho}_0 = 1$, we assume, for convenience, that $\bar{\varrho}_0 = 1$. The equations (1.2) are equivalent to the equations

$$
\begin{cases}
g_t + v \cdot \nabla g + \nabla \cdot v = 0\,, \\
v_t + \lambda^2 \phi'(g) \nabla g + (v \cdot \nabla) v = e^{-g}[\nu \Delta v + \mu \nabla(\nabla \cdot v)]\,, \\
g(0) = g_0(x)\,, \qquad v(0) = v_0(x)\,,
\end{cases}
\tag{1.4}
$$

where, by definition $\phi'(s) = p'(e^s)$, for each $s \in \mathbb{R}$. Hence $\phi' \in C^k(\mathbb{R}; \mathbb{R}^+)$.

Our results will be expressed in terms of the unknown g. In order to get them in terms of ϱ the only rules to keep in mind are that $|g|_\infty$ is bounded if and only if $|\varrho|_\infty$ and $|\varrho^{-1}|_\infty$ are bounded and that convergence of g to 0 in H^m is equivalent to convergence of ϱ to $\bar{\varrho}_0$ in H^m; see Lemma 3.5, in appendix.

Finally we recall the equations of motion of an incompressible fluid

$$
(1.5) \qquad \begin{cases} \nabla \cdot w = 0\,, \\ w_t + (w \cdot \nabla) w + \nabla \pi = \bar{\nu} \Delta w\,, \\ w(0) = w_0(x)\,, \end{cases}
$$

where $\nabla \cdot w_0 \equiv 0$ and $\bar{\nu} \geqslant 0$ is a constant.

The Lemmas 1.1 and 1.2 below are just minor improvements of results stated in [KM1]. In spite of that, we give here complete proofs of the a priori estimates. The Theorem 1.5 improves, in some aspects, the Theorem 2 in reference [KM1] (these improvements are due to a careful use of the standard techniques). As a matter of fact, this paper is intended just as a preparation to reference [BV7]. One has the following results.

LEMMA 1.1. *Assume that*

$$
(1.6) \qquad \lambda \|g_0\|_{k_0+1} \leqslant c_1\,, \qquad \|v_0\|_{k_0+1} \leqslant c_1\,.
$$

Then, there is a positive constant T, that depends only on c_1, such that the problem (1.4) *has a (unique) solution $(g, v) \in C(0, T; H^{k_0+1})$. Moreover, $g_t \in C(0, T; H^{k_0})$ and $v_t \in C(0, T; H^{k_0-1})$. If $\nu = \mu = 0$ then $v_t \in$ $\in C(0, T; H^{k_0})$. Furthermore,*

$$
(1.7) \qquad \lambda^2 \|g\|^2_{k_0+1, T} + \|v\|^2_{k_0+1, T} + \nu[\nabla v]^2_{k_0+1, T} + \mu[\nabla \cdot v]^2_{k_0+1, T} \leqslant C_1\,,
$$

where C_1 depends only on c_1.

We point out that k, n, μ_0, and ν_0 are fixed once and for all (hence, eventual dependence of other quantities on these constants is understood). Constants that depend only on the above fixed parameters are denoted by c. Constants denoted by C_1 depend only on c_1. As a rule, the same symbol is used to denote distinct constants provided that they are of the same type (i.e., depend on the same parameters).

In the sequel the constant T is always the same one appearing in Lemma 1.1, hence it depends only on c_1.

LEMMA 1.2. *Assume that the hypotheses of Lemma* 1.1 *hold and that g_0 and v_0 belong to $H^k (k \geqslant k_0 + 1)$. Then, the solution (g, v) of* (1.4) *belongs to $C(0, T; H^k)$, moreover $g_t \in C(0, T; H^{k-1})$ and $v_t \in$ $\in C(0, T; H^{k-2})$. If $\nu = \mu = 0$, then $v_t \in C(0, T; H^{k-1})$. Finally,*

$$
(1.8) \qquad \lambda^2 \|g\|^2_{k, T} + \|v\|^2_{k, T} + \nu[\nabla v]^2_{k, T} + \mu[\nabla \cdot v]^2_{k, T} \leqslant C_1 (\lambda^2 \|g_0\|^2_k + \|v_0\|^2_k)\,.
$$

In particular, if

$$\lambda\|g_0\|_k \leqslant c_2\,, \qquad \|v_0\|_k \leqslant c_2\,, \tag{1.9}$$

then

$$\lambda^2\|g\|^2_{k,T} + \|v\|^2_{k,T} + \nu[\nabla v]^2_{k,T} + \mu[\nabla\cdot v]^2_{k,T} \leqslant C_2 \tag{1.10}$$

where C_2 depends only on c_1 and c_2.

Note that the above lemmas show that g_t and v_t are regular on $[0, T]$ but do not furnish uniform estimates with respect to the parameters. In this direction, one has the following result.

LEMMA 1.3. *Assume* (1.6) *and* (1.9). *Let l be an integer, $0 \leqslant l \leqslant$ $\leqslant k-1$. If $l = k-1$ also assume that $\nu\nabla v_0$ and $\mu\nabla\cdot v_0$ belong to H^k. Under these hypotheses one has*

$$\lambda^2\|g_t\|^2_{l,T} + \|v_t\|^2_{l,T} + \nu[\nabla v_t]^2_{l,T} + \mu[\nabla\cdot v_t]^2_{l,T} \leqslant \tag{1.11}$$

$$\leqslant C_2(\lambda^4\|\nabla g_0\|^2_l + \lambda^2\|\nabla\cdot v_0\|^2_l + \|v_0\|^2_l + \nu^2\|\nabla v_0\|^2_{l+1} + \mu^2\|\nabla\cdot v_0\|^2_{l+1})\,.$$

The symbol C_2 denote (possibly distinct) constants depending only on c_1 and c_2.

COROLLARY 1.4. *Assume* (1.6), (1.9) *and*

$$\lambda\|\nabla\cdot v_0\|_l \leqslant c_3\,, \qquad \lambda^2\|\nabla g_0\|_l \leqslant c_3\,. \tag{1.12}$$

If $l = k-1$ also assume that

$$\nu\|\nabla v_0\|_k + \mu\|\nabla\cdot v_0\|_k \leqslant c_4\,. \tag{1.13}$$

Then

$$\lambda^2\|g_t\|^2_{l,T} + \|v_t\|^2_{l,T} + \nu[\nabla v_t]^2_{l,T} + \mu[\nabla\cdot v_t]^2_{l,T} \leqslant C_3\,, \tag{1.14}$$

where C_3 depends only on c_1, c_2 and c_3 (and on c_4, if $l = k-1$).

In the sequel we drop the symbol $(0, T)$ from the notations denoting functional spaces. For instance we write $C(H^k)$ instead of $C(0, T; H^k)$, and so on.

To each set of *fixed* constants $k, l, \nu_0, \mu_0, c_1, c_2, c_3$, where $k \geqslant k_0 +$ $+ 1$, $0 \leqslant l \leqslant k-1$, $\nu_0 \geqslant 0$, $\mu_0 \geqslant 0$, $c_1, c_2, c_3 \in \mathbb{R}^+$, we associate the set

$$\chi \equiv \{(v_0, g_0, \lambda, \nu, \mu) \in \tag{1.15}$$

$$\in H^k \times H^k \times [1, \infty[\times [0, \nu_0] \times [0, \mu_0] \colon (1.\ 6), (1.\ 9), (1.\ 12) \text{ hold}\}\,;$$

If $l = k - 1$ we also fix a constant $c_4 \in \mathbb{R}^+$ and assume that (1.13) holds. The results stated above show that to each $(v_0, g_0, \lambda, \nu, \mu) \in \chi$ it corresponds a solution $(v, g) \equiv S(v_0, g_0, \lambda, \nu, \mu)$ of problem (1.4) that satisfies the estimates (1.7), (1.10), (1.14). One has the following result.

THEOREM 1.5. *Let* $(v_0, g_0, \lambda, \nu, \mu)$ *run over* χ *and let* $(v, g) = S(v_0, g_0, \lambda, \nu, \mu)$. *Then*

$$\lim_{(v_0, \lambda, \mu) \to (w_0, \infty, \bar{\nu})} (v, g, v_t, g_t, \nabla \cdot v, \lambda^2 \nabla \phi(g)) = (w, 0, w_t, 0, 0, \nabla \pi) \tag{1.16}$$

where w *is the solution of problem* (1.5) *and the convergence of* v_0 *to* w_0 *is in the* H^k *norm (hence, by* $(1.12)_1$ *it must be* $\nabla \cdot w_0 \equiv 0$). *In* (1.16) *the convergence of the six terms on the left hand side is, respectivelly, in:* $L^\infty(H^k)$ *weak-** *and* $C(H^{k-\varepsilon})$, $\varepsilon > 0$; $C(H^k)$; $L^\infty(H^l)$ *weak-**; $C(H^l)$; $C(H^l)$; $L^\infty(H^{k-2})$ *weak-** *if* $l = k - 1$ *and* $L^\infty(H^l)$ *weak-** *if* $0 \leq l \leq k - 2$.

In addition: If $\bar{\nu} > 0$, *then* $v \to w$ *weakly in* $L^2(H^{k+1})$ *and (strongly) in* $L^2(H^{k+1-\varepsilon})$, *and* $v_t \to w_t$ *weakly in* $L^2(H^l)$; *If* μ *runs over* $[\bar{\mu}, \mu_0]$ *for some* $\bar{\mu} > 0$, *then* $\nabla \cdot v \to 0$ *weakly in* $L^2(H^k)$, *and (strongly) in* $L^2(H^{k-\varepsilon})$, *and* $\nabla \cdot v_t \to 0$ *weakly in* $L^2(H^l)$.

It is worth noting that convergence with respect to topologies stronger than the above ones, for g, for g_t, for $\nabla \cdot v$, and for $\nabla\phi(g)$, follows now immediately by using the equations (1.4). For instance, g_t converges in $C(H^{k-1-\varepsilon})$, moreover (for each $\varepsilon > 0$)

$$\lim_{\lambda \to \infty} \lambda^{1-\varepsilon}(\|g\|_{k,T} + \|g_t\|_{l,T} + \|\nabla \cdot v\|_{l,T}) = 0\,. \tag{1.17}$$

Let us just prove the estimate (1.18) below, useful in the part II (see [BV7]) of this work. Equation $(1.4)_2$, together with (1.8) and (1.14), shows that $\lambda^2 \|\nabla g\|_{0,T} \leq C_3$. Since $\lambda \|\nabla g\|_{k-1,T} \leq C_2$ it follows, by interpolation, that

$$\lambda^{1+1/(k-1)} \|\nabla g\|_{k-2} \leq C_3\,. \tag{1.18}$$

The Theorem 1.5 follows easily by using the uniform estimates (1.7), (1.10), (1.14), well known compact embedding theorems, the uniqueness of the solution of problem (1.5) and the equation (1.4). Note that v is bounded in $W^{1,\infty}(H^0)$, since $\|v\|_{k,T}$ and $\|v_t\|_{0,T}$ are bounded. Moreover v is bounded in $L^\infty(H^k)$ and H^k is compactly embedded in H^0. Hence, compactness in $C(H^0)$ follows by Ascoli-Arzelà's theorem. Compactness in $C(H^{k-\varepsilon})$ follows immediately since $\|\cdot\|_{k-\varepsilon,T} \leq \|\cdot\|_{0,T}^{\vartheta} \|\cdot\|_{k,T}^{1-\vartheta}$, $\vartheta = \varepsilon/k$. Details are left to the reader. The estimates (1.7), (1.10), and (1.14) will be proved in section 2.

The incompressible limit for compressible fluids was studied by many authors under different hypotheses. In references [Eb1, 2], [KM1, 2] [Ag], [M], [Sc1, 2, 3], [U], [As], [BV6] the authors consider inviscid fluids. Viscous, stationary fluids were studied in [BV1, 2]. Viscous, nonstationary fluids were studied in [KM1]. In this last reference it is assumed that $k \geq k_0 + 2$, that $\lambda \|v_0 - w_0\|_k + \lambda^2 \|g_0\|_k \leq c_3$ and that $\nu\lambda \|v_0 - w_0\|_{k+1} \leq c$, hence that $\lambda \|\nabla \cdot v_0\|_k \leq c_3$ (see $(1.12)_1$) and that $\|v_0 - w_0\|_k + \lambda \|g_0\|_k \leq c_3/\lambda$ (which implies that $(v_0, \lambda g_0) \to (w_0, 0)$ in H^k as $\lambda \to \infty$; see eq. (1.19), below).

It is worth noting that the above results are not completely satisfactory. In fact, the solutions $(v(t), g(t))$ describe continuous trajectories in the Hilbert space H^k (the data space). Hence, the natural and optimal result is to prove convergence in the strong norm $C(0, T; H^k)$. Note that convergence of g to 0 (hence of the density ϱ to ϱ_0) follows from (1.17). However, the convergence of v to w is still an open problem except if ν and μ vanish identically, i.e., if $\nu_0 = \mu_0 = 0$ (see [BV6]). In the part II of this work, see [BV7], we prove the convergence of v to w in the $C(0, T; H^k)$ norm whenever $(v_0, \lambda g_0, \lambda, \nu)$ converges to $(w_0, 0, \infty, \bar{\nu})$ and μ remains bounded. If, moreover, $\bar{\nu} \neq 0$, then we also show convergence in the $L^2(0, T; H^{k+1})$ norm. On the other hand, if $\mu \to \bar{\mu} \neq 0$, then $\nabla \cdot v \to \nabla \cdot w$ in $L^2(0, T; H^k)$. We also prove that $\lambda \|\nabla g\|_{k-1, T}$ converges to 0 (the sharper convergence result under the natural hypotheses made here). More precisely, in reference [BV7] we prove the following result:

THEOREM 1.6 (1). *Under the assumptions of Theorem* 1.5 *one has*

$$\lim_{(v_0, \lambda g_0, \lambda, \nu) \to (w_0, 0, \infty, \bar{\nu})} (\|v - w\|_{k,T}^2 + \lambda^2 \|\nabla g\|_{k-1,T}^2 + \bar{\nu}[v - w]_{k+1,T}^2) = 0 . \tag{1.19}$$

If, in addition, $\mu \to \bar{\mu} \neq 0$ *then* $[\nabla \cdot (v - w)]_{k,T} \to 0$.

We point out that in spite of the uniform boundedness of the v_t's in $C(0, T; H^l)$ it is false (in general) that $v_t \to w_t$ in the $C(O, T; H^l)$ norm. Analogously, if $\bar{\nu} > 0$, then the v_t's are uniformly bounded in $L^2(0, T; H^l)$. However, it is false (in general) that $v_t \to w_t$ in the $L^2(0, T; H^l)$ norm. The convergence of v_t to w_t in the strong norms can be proved if additional assumptions are done on the initial data. However, these additional assumptions look quite artificial.

(1) Here, we assume that $\phi' \in C^{k+1}(\mathbb{R}; \mathbb{R}^+)$ (i.e. $p \in C^{k+2}(\mathbb{R}^+; \mathbb{R})$).

Finally, we note that the main points in the proof of Theorem 1.6 can be easily extended in order to cover the general class of problems treated in reference [KM1] (in this direction, see [BV6], Theorem 2.2).

Proofs.

We will prove the a priori estimates claimed in the previous section. The proof of the existence of the solution follows then well known devices: Linearization of equations (1.4), proof of the fundamental a priori estimates (obtained just as done below) and construction of a fixed point.

In the sequel we denote integrals $\int_\Omega f(x)\,dx$ simply by $\int f(x)$ or even by $\int f$. If D^α denotes partial differentiation, $\alpha = (\alpha_1, \dots, \alpha_n)$, we set

$$\widetilde{D}^\alpha \{fg\} = D^\alpha (fg) - f(D^\alpha g).$$

As usual, $|D^m f|^2 = \sum_{|\alpha| = m} |D^\alpha f|^2$. For brevity, we set

$$\widetilde{e}(f) \equiv e^f .$$

We start by applying the operator D^α to equation $(1.4)_1$, by multiplying by $\lambda^2 \phi'(g) D^\alpha g$ the equation obtained (this corresponds to symmetrization; see [KM1]) and by integrating over Ω. This shows that

$$\begin{aligned}(2.1)\quad & \frac{\lambda^2}{2}\frac{d}{dt}\int \phi'(g)(D^\alpha g)^2 + \frac{\lambda^2}{2}\int \phi''(g)(v\cdot\nabla g + \nabla\cdot v)(D^\alpha g)^2 - \\ & -\frac{\lambda^2}{2}\int \nabla\cdot(\phi'(g)v)(D^\alpha g)^2 + \lambda^2\int \widetilde{D}^\alpha\{\phi'(g)v\cdot\nabla g\}D^\alpha g - \\ & -\lambda^2\int \phi'(g)D^\alpha v\cdot\nabla(D^\alpha g) - \lambda^2\int \phi''(g)\nabla g\cdot D^\alpha v D^\alpha g = 0 .\end{aligned}$$

Next, we apply the operator D^α to the equation $(1.4)_2$, we multiply it by $D^\alpha v$ and integrate over Ω. This shows that

$$\begin{aligned}(2.2)\quad & \frac{1}{2}\frac{d}{dt}\|D^\alpha v\|^2 + \lambda^2\int \phi'(g)\nabla D^\alpha g\cdot D^\alpha v + \lambda^2\int \widetilde{D}^\alpha\{\phi'(g)\nabla g\}\cdot D^\alpha v - \\ & -\frac{1}{2}\int(\nabla\cdot v)|D^\alpha v|^2 + \int \widetilde{D}^\alpha\{(v\cdot\nabla)v\}\cdot D^\alpha v = \\ & = -\nu\int \widetilde{e}(-g)|\nabla D^\alpha v|^2 + \nu\int \widetilde{e}(-g)\nabla g\cdot\nabla D^\alpha v\cdot D^\alpha v +\end{aligned}$$

$$+\nu\int\widetilde{D}^{\alpha}\{\widetilde{e}(-g)\Delta v\}\cdot D^{\alpha}v-$$

$$-\mu\int\widetilde{e}(-g)|D^{\alpha}(\nabla\cdot v)|^{2}+\mu\int\widetilde{e}(-g)\nabla g\cdot(D^{\alpha}\nabla\cdot v)D^{\alpha}v+$$

$$+\mu\int\widetilde{D}^{\alpha}\{\widetilde{e}(-g)\nabla(\nabla\cdot v)\}\cdot D^{\alpha}v.$$

Now, we add, side by side, the equations (2.1) for $0\leqslant|\alpha|\leqslant m$ to the equations (2.2) for $0\leqslant|\alpha|\leqslant m$. Then, we estimate the L^2-norms of the single terms according to the following devices: Estimate the $\widetilde{D}^{\alpha}$ terms by using (3.5). Take into account that $|\cdot|_{\infty}\leqslant c\|\cdot\|_{k_0}$. Take into account (see (3.7)) that $\|D^{\alpha}\phi'(g)\|\leqslant\beta(\|g\|_{k_0})\|\nabla g\|_{m-1}$. Use the Cauchy-Schwarz inequality in order to split the ν and the μ terms according to a standard and well kknown device. This shows that

$$\frac{1}{2}\frac{d}{dt}E_{m}^{2}(t)+\frac{1}{2}\widetilde{e}(-|g|_{\infty})(\nu\|\nabla v\|_{m}^{2}+\mu\|\nabla\cdot v\|_{m}^{2})\leqslant \tag{2.3}$$

$$\leqslant\lambda^{2}\beta(\|v\|_{k_0}\|\nabla g\|_{k_0}+\|v\|_{k_0+1})\|g\|_{m}^{2}+\lambda^{2}\beta\|\nabla g\|_{k_0}\|v\|_{m}\|g\|_{m}+c\|v\|_{k_0+1}\|v\|_{m}^{2}+$$

$$+\nu\beta(\|v\|_{m}^{2}+\|\nabla v\|_{k_0+1}\|v\|_{m}\|g\|_{m})+\mu\beta(\|v\|_{m}^{2}+\|\nabla\cdot v\|_{k_0+1}\|v\|_{m}\|g\|_{m}),$$

where, by definition,

$$E_{m}^{2}(t)\equiv\lambda^{2}\int\phi'(g)\sum_{j=0}^{m}|D^{j}g|^{2}+\|v\|_{m}^{2}, \tag{2.4}$$

and $\beta=\beta(\|g\|_{k_0})$. Here and in the sequel we denote by $\beta(\cdot)$ increasing functions that belong to the class $C^{\infty}(\mathbb{R}_0^+;\mathbb{R}^+)$. The same symbol β is used to denote distinct functions. These functions depend only on ϕ', on n and on k (we will use (2.3) only for $m\leqslant k$).

The estimate (2.3) is not sufficient to furnish an a priori estimate for $E_k^2(t)$, except if one assumes that $\phi'(\tau)\geqslant c>0$ for each $\tau\in\mathbb{R}$, a too strong hpothesis (for instance, if $p(\varrho)=\varrho^{\gamma}$ this hypothesis is satisfied only when $\gamma=1$).

Let us turn back to equation (1.4). Apply to this equation the operator D^{α}, for each α such that $|\alpha|\leqslant k_0$, multiply by $D^{\alpha}g$ the equation obtained, integrate over Ω, and use the estimate (3.4) with k and l given by k_0+1 and k_0, respectively. After same manipulations one shows that

$$\frac{1}{2}\frac{d}{dt}\|g\|_{k_0}^{2}\leqslant c\|v\|_{k_0+1}(\|g\|_{k_0}^{2}+\|g\|_{k_0}). \tag{2.5}$$

On the other hand, we also note (for further use) that, since ϕ' is a continuous, strictly positive function, there are two β functions such that

$$\beta^{-1}(\|g\|_{k_0}) \leq \phi'(g(x)) \leq \beta(\|g\|_{k_0}), \qquad \forall x \in \Omega. \tag{2.6}$$

Note that $|g|_\infty \leq c\|g\|_{k_0}$. In particular, $\lambda^2\|g\|_m^2 \leq \phi(\|g\|_{k_0})E_m^2(t)$ for each $m \geq k_0$.

PROOF OF LEMMA 1.1. Set

$$G^2(t) \equiv E_{k_0+1}^2(t) + \|g(t)\|_{k_0}^2 .$$

From (2.3) and (2.5) it readily follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}G^2(t) + \frac{1}{4}\tilde{e}(-|g|_\infty)(\nu\|\nabla v\|_{k_0+1}^2 + \mu\|\nabla\cdot v\|_{k_0+1}^2) \leq \\ \leq \lambda^2\beta(\|v\|_{k_0}\|g\|_{k_0+1}^3 + \|v\|_{k_0+1}\|g\|_{k_0+1}^2) + \\ + c\|v\|_{k_0+1}^3 + \beta\|v\|_{k_0+1}^2 + (\nu_0+\mu_0)\beta\|v\|_{k_0+1}^2\|g\|_{k_0+1}^2 ,\end{aligned} \tag{2.7}$$

where $\beta = \beta(\|g\|_{k_0})$. Equation (2.7) shows that

$$\frac{1}{2}\frac{d}{dt}G^2(t) \leq \Phi(G(t)) \tag{2.8}$$

where

$$\Phi(\tau) = \beta(\tau)(\tau^4 + \tau^3) + c\tau^3 + \beta(\tau)\tau^2 + (\nu_0+\mu_0)\beta(\tau)\tau^4$$

belongs to $C^\infty(\mathbb{R}_0^+;\mathbb{R}^+)$. Since

$$G^2(0) = \|g_0\|_{k_0}^2 + \lambda^2\int \phi'(g_0)\sum_{j=0}^{k_0+1}|D^jg_0|^2 + \|v_0\|_{k_0+1}^2$$

is bounded by $c_1^2 + \beta(c_1)c_1^2 + c_1^2$, for a suitable β, it readily follows from (2.8) that $G^2(t) \leq C_1$ for each $t \in [0, T]$, where C_1 and T depend only on c_1 (dependence on n, ν_0, μ_0 is not taken into account). Finally, integration of (2.7) on $[0, T]$ completes the proof of (1.7). ∎

PROOF OF LEMMA 1.2. By (2.3) and (1.7) one shows that

$$\frac{d}{dt}E_k^2(t) \leq \beta(C_1)(1 + \nu\|\nabla v\|_{k_0+1} + \mu\|\nabla\cdot v\|_{k_0+1})E_k^2(t). \tag{2.9}$$

Moreover, $\nu[\nabla v]_{k_0+1,T}^2 + \mu[\nabla\cdot v]_{k_0+1,T}^2 \leq C_1$. ∎

PROOF OF LEMMA 1.3. From (1.4) it follows that

$$
(2.10)\begin{cases} g_{tt} + v\cdot\nabla g_t + \nabla\cdot v_t + v_t\cdot\nabla g = 0\,, \\ v_{tt} + \lambda^2\phi'(g)\nabla g_t + \lambda^2\phi''(g)g_t\nabla g + (v\cdot\nabla)v_t + (v_t\cdot\nabla)v = \\ \qquad = \tilde{e}(-g)[\nu\Delta v_t + \mu\nabla(\nabla\cdot v_t)] - \tilde{e}(-g)g_t[\nu\Delta v + \mu\nabla(\nabla\cdot v)]\,. \end{cases}
$$

Next we apply the operator D^α, $0 \leqslant |\alpha| \leqslant l \leqslant k-1$, to equation $(2.10)_1$, we multiply both sides by $\lambda^2\phi'(g)D^\alpha g_t$ and integrate over Ω. This yields

$$
\frac{\lambda^2}{2}\frac{d}{dt}\int \phi'(g)(D^\alpha g_t)^2 - \frac{\lambda^2}{2}\int |\phi''(g)|\ |v\cdot\nabla g + \nabla\cdot v|(D^\alpha g_t)^2 -
$$

$$
-\frac{\lambda^2}{2}\int |\nabla\cdot(\phi'(g)v)|(D^\alpha g_t)^2 + \lambda^2\int \phi'(g)\tilde{D}^\alpha\{v\cdot\nabla g_t\}D^\alpha g_t -
$$

$$
-\lambda^2\int \phi'(g)(D^\alpha v_t)\cdot\nabla D^\alpha g_t - \lambda^2\int |\phi''(g)|\ |\nabla g|\ |D^\alpha v_t|\ |D^\alpha g_t| -
$$

$$
-\lambda^2\int |D^\alpha(v_t\cdot\nabla g)|\ |D^\alpha g_t| \leqslant 0\,.
$$

By using (1.10), (3.2), (3.4) and (3.7) one shows that

$$
(2.11)\qquad \frac{\lambda^2}{2}\frac{d}{dt}\int \phi'(g)(D^\alpha g_t)^2 - \lambda^2 C_2\|g_t\|_l\|D^\alpha g_t\| -
$$

$$
-\lambda^2\int \phi'(g)(D^\alpha v_t)\cdot\nabla D^\alpha g_t - \lambda C_2\|v_t\|_l\|D^\alpha g_t\| \leqslant 0
$$

where the constants C_2 depend at most on c_1 and c_2.

Next, we apply the operator D^α to the equation $(2.10)_2$, we multiply by $D^\alpha v_t$ and integrate over Ω. This gives

$$
\frac{1}{2}\frac{d}{dt}\|D^\alpha v_t\|^2 + \lambda^2\int \phi'(g)(\nabla D^\alpha g_t)\cdot D^\alpha v_t +
$$

$$
+\lambda^2\int \tilde{D}^\alpha\{\phi'(g)\nabla g_t\}\cdot D^\alpha v_t +
$$

$$
+\lambda^2\int D^\alpha[\phi''(g)g_t\nabla g]\cdot D^\alpha v_t - \frac{1}{2}\int (\nabla\cdot v)|D^\alpha v_t|^2 +
$$

$$+\int \widetilde{D}^\alpha\{(v\cdot\nabla)v_t\}\cdot D^\alpha v_t + \int D^\alpha[(v_t\cdot\nabla)v]\cdot D^\alpha v_t =$$

$$= -\nu\int \widetilde{e}(-g)|\nabla D^\alpha v_t|^2 - \nu\int \nabla\widetilde{e}(-g)\cdot(\nabla D^\alpha v_t)\cdot(D^\alpha v_t) +$$

$$+\nu\int \widetilde{D}^\alpha\{\widetilde{e}(-g)\Delta v_t\}\cdot D^\alpha v_t - \nu\int D^\alpha[\widetilde{e}(-g)g_t\Delta v]\cdot D^\alpha v_t -$$

$$-\mu\int \widetilde{e}(-g)|D^\alpha(\nabla\cdot v_t)|^2 - \mu\int \widetilde{e}(-g)(\nabla\cdot D^\alpha v_t)\cdot(D^\alpha v_t) +$$

$$+\mu\int \widetilde{D}^\alpha\{\widetilde{e}(-g)\nabla(\nabla\cdot v_t)\}\cdot D^\alpha v_t - \mu\int D^\alpha[\widetilde{e}(-g)v_t\nabla(\nabla\cdot v)]\cdot D^\alpha v_t\,.$$

By using (1.10), (3.1), (3.2), (3.4) and (3.7) we show, after addition with respect to α, that

$$\frac{1}{2}\frac{d}{dt}\|v_t\|_l^2 + \lambda^2\int \phi'(g)\sum_{|\alpha|\leqslant l}(\nabla D^\alpha g_t)\cdot(D^\alpha v_t) + \tag{2.12}$$

$$+\frac{1}{2}\widetilde{e}(-|g|_\infty)(\nu\|\nabla v_t\|_l^2 + \mu\|\nabla\cdot v_t\|_l^2) \leqslant$$

$$\leqslant C_2(\lambda\|g_t\|_l\|v_t\|_l + \|v_t\|_l^2 + (\nu+\mu)\|v_t\|_l^2) +$$

$$+C_2(\nu\|\nabla v\|_k + \mu\|\nabla\cdot v\|_k)\|g_t\|_l\|v_t\|_l\,.$$

Set

$$F_l^2(t) \equiv \lambda^2\int \phi'(g)\sum_{|\alpha|\leqslant l}|D^\alpha g_t|^2 + \|v_t\|_l^2\,.$$

By adding, side by side, the equation (2.11) for each α, $0\leqslant|\alpha|\leqslant l$, and equation (2.12) one obtains

$$\frac{d}{dt}F_l^2(t) + \frac{1}{2}\widetilde{e}(-|g|_\infty)(\nu\|\nabla v_t\|_l^2 + \mu\|\nabla\cdot v_t\|_l^2) \leqslant \tag{2.13}$$

$$\leqslant C_2(1+\nu\|\nabla v\|_k + \mu\|\nabla\cdot v\|_k)F_l^2(t)\,.$$

By taking into account that $\nu[\nabla v]_{k,T}^2 + \mu[\nabla\cdot v]_{k,T}^2 \leqslant C_2$ one proves that $\|F_l(t)\|_{0,T}^2 \leqslant C_2\|F_l(0)\|^2$. Straightforward manipulations show (1.11). ∎

Appendix.

For the reader's convenience we state here some useful results. Here, Ω is the n-dimensional thorus, an open bounded regular subset of $\mathbb{R}^n$, $\mathbb{R}^n$ itself, or $\mathbb{R}^n_+ \equiv \{x : x_n > 0\}$.

LEMMA 3.1. *Let be $r > n/2$. If $0 \leq s \leq r$, then*

$$\|fg\| \leq c\|f\|_{r-s}\|g\|_s . \tag{3.1}$$

If $0 \leq l$ and $0 \leq s \leq r - l$, then

$$\|fg\|_l \leq c\|f\|_{r-s}\|g\|_{l+s} . \tag{3.2}$$

For the proof, see [BV3], Appendix A, Lemma A.1 and Proposition A.1.

LEMMA 3.2. *Let be $r > n/2$, $0 \leq l \leq r$, $l \leq l_i \leq r$ for $i = 1, \ldots, m$, and $l_1 + \ldots + l_m = l + (m-1)r$. Then*

$$\|f_1 \cdots f_m\|_l \leq c\|f_1\|_{l_1} \cdots \|f_m\|_{l_m} . \tag{3.3}$$

PROOF. By induction on m. For $m = 2$ the result holds, by (3.2). Assume it for some $m \geq 2$. Since the result holds for $m = 2$ one has

$$\|f_1 \cdots f_{m+1}\|_l \leq c\,\|f_1 \cdots f_m\|_{l+r-l_{m+1}}\|f_{m+1}\|_{l_{m+1}} .$$

Let us show that

$$\|f_1 \cdots f_m\|_{l+r-l_{m+1}} \leq c\|f_1\|_{l_1} \cdots \|f_m\|_{l_m} ,$$

where $l_1 + \ldots + l_m = l + mr$. In fact, by setting $\bar{l} = l + r - l_{m+1}$ one has $0 \leq \bar{l} \leq r$, $l_1 + \ldots + l_m = \bar{l} + (m-1)r$, and $\bar{l} \leq l_1 \leq r$ for each $i = 1, \ldots, m$. Note that if it was $l_i < \bar{l}$ for some i (say, for $i = m$) then $l + mr = (l_1 + \ldots + l_{m-1}) + l_m + l_{m+1} < (m-1)r + l + r$. ■

LEMMA 3.3. *Let be $k > 1 + n/2$ and $1 \leq l \leq k$. If $|\alpha| \leq l$ then*

$$\|\widetilde{D}^\alpha\{fg\}\| \leq c\|Df\|_{k-1}\|g\|_{l-1} . \tag{3.4}$$

For the proof see [BV3], appendix A, Corollary A.3.

LEMMA 3.4. *Let be $|\alpha| \leq l$. Then*

$$\|\widetilde{D}^\alpha\{fg\}\| \leq c(|Df|_\infty\|g\|_{l-1} + |g|_\infty\|Df\|_{l-1}) . \tag{3.5}$$

See [Mo] or [K.M] Lemma A.1. Use the Gagliardo-Nirenberg inequalities ([Ga], [Ni]) in the form

$$(3.6) \qquad |D^j g|_{2r/j} \leq c |g|_\infty^{1-j/r} \|Dg\|_{r-1}^{j/2} ,$$

where $0 \leq j \leq r$.

LEMMA 3.5. *Let $\psi \in C(\mathbb{R}; \mathbb{R})$, $r \geq 1$. Then, there are increasing functions $\beta_1 \in C^\infty(\mathbb{R}_0^+; \mathbb{R}^+)$ and $\beta_2 \in C^\infty(\mathbb{R}_0^+ \times \mathbb{R}_0^+; \mathbb{R}^+)$ such that*

$$(3.7) \qquad \|D^\alpha \psi(g)\|^2 \leq \beta_1(|g|_\infty) \|g\|_r^2 ,$$

and

$$(3.8) \qquad \|D^\alpha \psi(g) - D^\alpha \psi(f)\|^2 \leq \beta_2(|g|_\infty, |f|_\infty) \|g - f\|_r^2 ,$$

for each α, $1 \leq |\alpha| \leq r$.

For the reader's convenience we show here the proof (due to Moser ([Mo]). Without loss of generality assume that $|\alpha| = r$. With abbreviate but clear notation one has

$$(3.9) \qquad D^r \psi(g) = \sum c\, \psi^{(\varrho)}(g) (Dg)^{s_1} (D^2 g)^{s_2} \dots (D^r g)^{s_r} ,$$

where $s_1 + \dots + s_r = \varrho$, $s_1 + 2s_2 + \dots + rs_r = r$, and $1 \leq \varrho \leq r$. The s_j are nonegative integers. Set $p_j = r/j\, s_j$. Note that it could be $p_j = \infty$. Since $(p_1)^{-1} + \dots + (p_r)^{-1} = 1$ it follows that

$$I \equiv \int |\psi^{(\varrho)}(g)(Dg)^{s_1} \dots (D^r g)^{s_r}|^2 \leq |\psi^{(\varrho)}(g)|_\infty^2 \prod_{j=1}^{r} \left(\int |D^j g|^{2r/j} \right)^{js_j/r} .$$

Hence, by (3.6),

$$I \leq c |\psi^{(\varrho)}(g)|_\infty^2 \prod_{j=1}^{r} |g|_\infty^{2s_j(1-j/r)} \|Dg\|_{r-1}^{2js_j/r} .$$

This shows that each term I is bounded by $\tilde{\beta}(|g|_\infty) \|Dg\|_{r-1}^2$, for a suitable increasing function $\tilde{\beta}$. Clearly, $\tilde{\beta}$ is bounded from above by an increasing function $\beta_1 \in C^\infty(\mathbb{R}_0^+; \mathbb{R}^+)$ obtained, for instance, by applying to $\tilde{\beta}$ a translation to the left and a mollification.

Next, by using the decomposition (3.9) for $D^r \psi(g)$ and for $D^r \psi(f)$, by taking the difference between each pair of homologous terms and by splitting each one of these differences into a summation of products (according to a standard device), an obvious extension of the above argument shows (3.8).

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