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Functional Differential Equations of Mixed Type in Banach Spaces.

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1. Introduction.

In the last few year the fundamental theory of functional differential equations with a single delay in a Banach space has undergone intensive development (see [1, 2]). The fundamental theory of functional differential equations of mixed type, however, is still in a initial stage of development [3-8]. In this paper, we consider the functional differential equation of mixed type of the form

$$x'(t) = f(t, x(t), x(t+\tau(t)), x(t-\tau(t))) \quad \text{a.e. in } (\alpha, \beta), \tag{1}$$

in a Banach space B.

Let $B = (B, \|\cdot\|)$ be a Banach space, $r \geq 0$, $s > 0$ are constants and $-\infty < \alpha < \beta \leq +\infty$. Assume that $f: [\alpha, \beta] \times B \times B \times B \to B$ satisfies the conditions:

(a) $f(\cdot, x, y, z)$ is strongly measurable for all fixed $x, y, z \in B$ and there exist $x_0, y_0, z_0 \in B$ such that $\int_\alpha^\beta \|f(t, x_0, y_0, z_0)\| dt < \infty$;

(b) there exist nonnegative constants a, b and c such that

$$\|f(t, x, y, z) - f(t, \bar{x}, \bar{y}, \bar{z})\| \leq a\|x - \bar{x}\| + b\|y - \bar{y}\| + c\|z - \bar{z}\|,$$

for all $x, y, z, \bar{x}, \bar{y}, \bar{z} \in B$ and $t \in (\alpha, \beta)$; and

(c) $\tau(t)$ is continuous and $r \leq \tau(t) \leq s$.

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We consider the equation (1) with two boundary conditions:

$$(2)\qquad \begin{cases} x(t) = \phi(t), & t \in [\alpha - s, \alpha], \\ x(t) = \psi(t), & t \in [\beta, \beta + s), \qquad \text{if } \beta < \infty, \end{cases}$$

where

$$\phi \in \mathcal{D} = \{\phi\colon [\alpha - s, \alpha] \to B \mid \phi \text{ is Bocher integrable function}\},$$

$$\psi \in \mathfrak{J} = \{\psi\colon [\beta, \beta + s) \to B \mid \psi \text{ is Bocher integrable function}\},$$

DEFINITION. $x\colon [\alpha - s, \beta + s) \to B$ is a solution of the Problem (1) (2), if $x(t)$ is continuous on $[\alpha, \beta)$ and satisfies (1) and (2).

Let

$$\mathcal{C} = \{x\colon [\alpha - s, \beta + s) \to B \mid x(t) \text{ is continuous on } [\alpha, \beta) \text{ and satisfies (2)}\},$$

$$\mathcal{C}_\lambda = \{x \in \mathcal{C} \mid \sup_{\alpha \leqslant t < \beta} \exp[-\lambda t]\|x(t)\| < \infty\} \qquad \text{for each } \lambda \geqslant 0.$$

It is easy to see that x is a solution of the problem (1), (2) if and only if x is a fixed point of the operato T, defined on $\mathcal{C}$ by

$$(3)\qquad Tx(t) = \begin{cases} \phi(t), & t \in [\alpha - s, \alpha], \\ \psi(t), & t \in [\beta, \beta + s), \qquad \text{if } \beta < \infty, \\ \phi(\alpha) + \int_\alpha^t f(u, x(u), x(u + \tau(u)), x(u - \tau(u)))\, du, & t \in (\alpha, \beta). \end{cases}$$

2. Existence and uniqueness.

THEOREM 1 (Existence and uniqueness). Let $(a), (b)$ and (c) hold and $s \leqslant l = \beta - \alpha$. If there exists a $\lambda \geqslant 0$ such that

$$(4)\qquad D(a, b, c, \lambda) = a\int_0^l e^{-\lambda u}\, du + b\int_{2r-l}^{s} e^{\lambda u}\, du + c\int_{2r-s}^{l} e^{-\lambda u}\, du < 1,$$

then, for each $x_0 \in \mathcal{C}_\lambda$ the iterates $T^n x_0$, where $T\colon \mathcal{C}_\lambda \to \mathcal{C}_\lambda$ is defined by

(3), converge in the metric ϱ_λ of $\mathcal{C}_\lambda$ defined by

(5) $$\varrho_\lambda(x, \bar{x}) = \sup_{\alpha \leqslant t < \beta} \{e^{-\lambda t} \|x(t) - \bar{x}(t)\|\}$$

to a solution of the problem (1) (2), which is unique in $\mathcal{C}_\lambda$.

PROOF. Let $x \in \mathcal{C}_\lambda$, then $\sup_{\alpha \leqslant t < \beta} e^{-\lambda t} \|x(t)\| < \infty$ and $x(t)$ is continuous on $[\alpha, \beta)$ and satisfies (2). Hence $Tx(t)$ is continuous on $[\alpha, \beta)$ and satisfies (2).

From (3), for each $\lambda > 0$ and any $t \in [\alpha, \beta)$, we have that

$$e^{-\lambda t} \|Tx(t)\| \leqslant$$

$$\leqslant e^{-\lambda t} \left\| Tx(t) - \int_\alpha^\beta f(t, x_0, y_0, z_0)\, dt \right\| + e^{-\lambda t} \left\| \int_\alpha^\beta f(t, x_0, y_0, z_0)\, dt \right\| \leqslant$$

$$\leqslant e^{-\lambda t} \|\phi(\alpha)\| + e^{-\lambda t} \left\| \int_\alpha^t f(u, x(u), x(u + \tau(u)), x(u - \tau(u)))\, du - \right.$$

$$\left. - \int_\alpha^t f(u, x_0, y_0, z_0)\, du \right\| +$$

$$+ e^{-\lambda t} \left\| \int_t^\beta f(u, x_0, y_0, z_0)\, du \right\| + e^{-\lambda t} \left\| \int_\alpha^\beta f(t, x_0, y_0, z_0)\, dt \right\| \leqslant e^{-\lambda t} \|\phi(\alpha)\| +$$

$$+ e^{-\lambda t} \left[a \int_\alpha^t \|x(u)\| du + b \int_\alpha^t \|x(u + \tau(u))\|\, du + c \int_\alpha^t \|x(u - \tau(u)\|\, du \right] +$$

$$+ e^{-\lambda t}(t - \alpha)(a\|x_0\| + b\|y_0\| + c\|z_0\|) + 2e^{-\lambda t} \left\| \int_\alpha^\beta f(t, x_0, y_0, z_0)\, dt \right\|.$$

Note that $x \in \mathcal{C}_\lambda$, ϕ and ψ are Bocher integrable functions, so we can obtain that

$$\sup_{\alpha \leqslant t < \beta} e^{-\lambda t} \|Tx(t)\| < \infty .$$

For $\lambda = 0$, using (a) and $x \in \mathcal{C}_0$, i.e. $\sup_{\alpha \leqslant t < \beta} \|Tx(t)\| < \infty$, we have $Tx \in \mathcal{C}_0$. So $T(\mathcal{C}_\lambda) \subset \mathcal{C}_\lambda$ for each $\lambda \geqslant 0$.

Now we define

$$\mathcal{B} = \{y\colon [\alpha - s, \beta + s) \to (0, \infty) \,|\, y(t) \text{ is continuous in } [\alpha, \beta),$$
$$y(t) = 0 \text{ for } t \in [\alpha - s, \alpha) \text{ and } t \in [\beta, \beta + s)\}$$

and

$$Wy(t) = 0 \quad \text{for } t \in [\alpha - s, \alpha) \text{ and } t \in [\beta, \beta + s),$$

$$Wy(t) =$$
$$= a\int_\alpha^t y(u)\,du + b\int_\alpha^t y(u + \tau(u))\,du + c\int_\alpha^t y(u - \tau(u))\,du \text{ for } t \in [\alpha, \beta),$$

then the operator W maps the set $\mathcal{B}$ into itself. We can prove that

$$\|Wy\|_\lambda < D(a, b, c, \lambda)\|y\|_\lambda \quad \text{for any } y \in \mathcal{B} \text{ and each } \lambda \geqslant 0,$$

where

$$\|y\|_\lambda = \sup_{\alpha \leqslant t < \beta} \{e^{-\lambda t} y(t)\} \quad \text{for any } y \in \mathcal{B},$$

and

$$\|Tx - T\overline{x}\| \leqslant W(\|x(t) - \overline{x}(t)\|) \quad \text{for any } x, \overline{x} \in \mathcal{A}_\lambda \text{ and } t \in [\alpha, \beta).$$

Thus from (5) we have that

$$\varrho_\lambda(Tx, T\overline{x}) \leqslant D(a, b, c, \lambda)\varrho_\lambda(x, \overline{x}), \qquad x, \overline{x} \in \mathcal{A}_\lambda.$$

Hence T is contractive with respect to ϱ_λ by (4). Since $\mathcal{A}_\lambda$ is complete in this metric, the Banach's contractive mapping theorem implies that the iterates $T^n x_0$, $x_0 \in \mathcal{A}_\lambda$, converge in $(\mathcal{A}_\lambda, \varrho_\lambda)$ to a unique fixed point of T, i.e. to a unique solution of the problem (1), (2) in $\mathcal{A}_\lambda$.

COROLLARY 1. Let (a) and (b) hold, and $\tau(t) = r = \text{const} > 0$. If there exists $\lambda \geqslant 0$ such that

$$a\int_0^l e^{-\lambda u}\,du + b\int_{2r-l}^r e^{\lambda u}\,du + c\int_r^l e^{-\lambda u}\,du < 1, \tag{6}$$

then the problem (1), (2) has an unique solution in $\mathcal{A}_\lambda$.

EXAMPLE. Consider the Lecornu's equations [4, 6]

$$x'(t) = ax(t + 1) + bx(t - 1) \tag{7}$$

where a and b are constants and $\tau(t) = 1$. The conditions (a) and (b) are verified. By the Corollary 1 and (6), if there exists a $\lambda \geqslant 0$ such that $|a| \int_{2-l}^{1} e^{\lambda u}\, du + |b| \int_{1}^{l} e^{-\lambda u}\, du < 1$, then the problem (7), (2) has a unique solution in $\mathcal{C}_\lambda$ for $r = s = 1$.

COROLLARY 2. Let the hypotheses of the Theorem 1 hold and $\beta < < \infty$, then the problem (1), (2) has a bounded solution on $[\alpha, \beta)$.

PROOF. Since $\beta < \infty$, then $\mathcal{C}_\lambda = \mathcal{C}_0$ for all $\lambda \geqslant 0$. Hence the problem (1), (2) has a unique solution in $\mathcal{C}_0$ by the Theorem 1, and the solution is bounded.

3. Dependence on the boundary equations.

THEOREM 2 (Dependence). Let all conditions of the Theorem 1 hold. Suppose that $\{\phi_n\} \subset \mathcal{D}$, $\{\psi_n\} \subset \mathfrak{J}$,

$$|||\phi_n - \phi||| \overset{\text{def}}{=} \int_{\alpha - s}^{\alpha} \|\phi_n(t) - \phi(t)\|\, dt \to 0, \qquad n \to \infty,$$

$$|||\psi_n - \psi||| \overset{\text{def}}{=} \int_{\beta}^{\beta + s} \|\psi_n - \psi(t)\|\, dt \to 0, \qquad n \to \infty,$$

and $\phi_n(\alpha) \to \phi(\alpha)$ as $n \to \infty$. Let $x_n(t)$ be the solution of the equation (1) with the boundary conditions

$$(8)_n \qquad \begin{cases} x_n(t) = \phi_n(t), & t \in [\alpha - s, \alpha], \\ x_n(t) = \psi_n(t), & t \in [\beta, \beta + s), \quad \text{if } \beta < \infty, \end{cases}$$

$n = 1, 2, \ldots$. Then the sequence $\{x_n(t)\}$ of the solutions of the problem (1), $(8)_n$ ($n = 1, 2, \ldots$) has the following properties:

(i) if $\beta < \infty$, then $x_n(t) \to x(t)$ as $n \to \infty$ uniformly on $[\alpha, \beta)$;

(ii) if $\beta = \infty$, then $x_n(t) \to x(t)$ as $n \to \infty$ uniformly on the compact intervals of $[\alpha, \beta)$.

PROOF. Let

$$\mathcal{A}_n = \{x\colon [\alpha - s, \beta + s) \to B \,|\, x(t) \text{ is continuous on } [\alpha, \beta) \text{ and satisfies } (8)_n\},$$

$$\mathcal{A}_{n\lambda} \equiv \{x \in \mathcal{A}_n \,|\, \sup e^{-\lambda t}\|x(t)\| < \infty\} \quad \text{for each } \lambda \geqslant 0,$$

$n = 1, 2, \ldots$, and we define the operators T_n:

$$(9)_n \qquad T_n x_n(t) = \begin{cases} \phi_n(t), & t \in [\alpha - s, \alpha], \\ \psi_n(t), & t \in [\beta, \beta + s), \quad \text{if } \beta < \infty, \\ \phi_n(\alpha) + \int_\alpha^t (u.\ x_n(u), x_n(u + \tau(u)), x_n(u - \tau)))\, du, & t \in [\alpha, \beta), \end{cases}$$

for $n = 1, 2, \ldots$. Similarly to the proof of the Theorem 1, we can prove that $T_n(\mathcal{A}_{n\lambda}) \subset \mathcal{A}_{n\lambda}$ and the operator T_n has a unique fixed point for each n. Hence for each n the problem (1), $(6)_n$ has a unique solution in $\mathcal{A}_{n\lambda}$.

Let $x \in \mathcal{A}_\lambda$ be the fixed point of T and $x_n \in \mathcal{A}_{n\lambda}$ be the fixed point of T_n for each n, then we have that

$$\varrho_\lambda(x, x_n) \leqslant M_n e^{-\lambda\alpha}/(1 - D(a, b, c, \lambda)) \quad \text{for all } n,$$

where $M_n = \text{Const.} \geqslant 0$ and $\lim_{n\to\infty} M_n = 0$. Thus from $\lim_{n\to\infty} \varrho_\lambda(x, x_n) = 0$, it is easy to prove our results.

4. Remarks.

REMARK 1.. The above results can be estended naturally to the problem involving several arguments

$$(10) \qquad x'(t) = f(t, x(t), x(t + \tau_1(t)), x(t + \tau_2(t)), \ldots, x(t + \tau_m(t)), x(t - \tau_1(t)), x(t - \tau_2(t)), \ldots, x(t - \tau_m(t))) \quad \text{a.e. in } (\alpha,\beta),$$

where $\tau_1(t) > 0$ $(i = 1, 2, \ldots, m)$ are continuous, and the exist nonnegative constants r_i and s_i $(i = 1, 2, \ldots, m)$ such that $0 \leqslant r_i \leqslant \tau_i(t) \leqslant s_i$ and also to functional differential equation of mixed type of the form

$$x'(t) = f(t, x(t), x(t+\tau_1(t)), x(t+\tau_2(t)), \ldots, x(t+\tau_p(t)),$$
$$x(t-h_1(t)), x(t-h_2(t)), \ldots, x(t-h_q(t))) \quad \text{a.e. in } (\alpha, \beta),$$

where p and q are positive numbers, $\tau_i(t)$ and $h_j(t)$ are nonnegative continuous functions.

REMARK 2. The equations

$$x'(t) = f(t, x(t), x(g(t)t), x(h(t)t)) \quad \text{a.e. in } (\alpha, \infty), \tag{11}$$

and

$$x'(t) = f(t, x(t), x(g_1(t)t), x(g_2(t)t), \ldots, x(g_p(t)t),$$
$$x(h_1(t)t), x(h_2(t)t), \ldots, x(h_q(t)t)) \quad \text{a.e. in } (\alpha,\infty),$$

where g, h, g_i $(i = 1, 2, \ldots, p)$ and h_j $(j = 1, 2, \ldots, q)$ are continuous, and $g > 1, g_i > 1, 0 < h < 1, 0 < h_j < 1$ for each i and j, are of type (10) too.

In fact, let $t = e^y$, $X(y) = x(e^y)$, then we have that from (11)

$$X'(y) = x'(e^y)e^y = e^y f(e^y, x(e^y), x(e^{y+\log g(e^y)}), x(e^{y+\log h(e^y)})) \stackrel{\text{def}}{=}$$
$$\stackrel{\text{def}}{=} F(y, X(y), X(y+\log g(e^y)), X(y+\log h(e^y))) \text{ a.e.}$$

where $\log(e^y) > 0$, $\log h(e^y) < 0$.

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