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# On the Rank-One-Convexity Domain of the Saint Venant-Kirchhoff Stored Energy Function.

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ABSTRACT - The goal of this paper is to give a characterization of the rank-one-convexity domain of the Saint Venant-Kirchhoff stored energy function.

## 1. Introduction.

For a homogeneous, isotropic, elastic material, whose reference configuration is a natural state, one can show that the response function associated to the second Piola-Kirchhoff stress tensor is of the form

$$\Sigma(F) = \lambda(\operatorname{tr} E) I + 2\mu E + o(E) \tag{1.1}$$

for a matrix $F \in \mathbb{M}_+^3$ neighbouring of the identity matrix $I \in \mathbb{M}^3$ and where

$$E = E(F) = \frac{1}{2}(F^T F - I). \tag{1.2}$$

For $n \in \mathbb{N}^*$, we denote by $\mathbb{M}^n$ the set of all $n \times n$ real matrices and by $\mathbb{M}_+^n$ the subset of $\mathbb{M}^n$ of the matrices $A$ verifying $\det A > 0$. For $A \in \mathbb{M}^n$, $A^T$ is the transpose of the matrix $A$ and $\operatorname{tr} A$ the trace of $A$.

The *positive* constants $\lambda$ and $\mu$ are called the *Lamé constants* of the material under consideration (see [Ci.$_1$] or [Ci.$_2$] for a complete point of view about these notions).

Now, we call *Saint Venant-Kirchhoff* material, a homogeneous, isotropic, elastic material, whose response function $\widehat{\Sigma}$ associated to the second Piola-Kirchhoff stress tensor is defined by (1.1) where we have

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neglected the term $o(E)$, that is to say such that

$$\widehat{\Sigma}(F) = \lambda(\operatorname{tr} E) I + 2\mu E\,.$$

One can prove that such a material is a hyperelastic material and that its stored energy function $\widehat{W}$ is given by

$$\widehat{W}(F) = \frac{\lambda}{2}(\operatorname{tr} E)^2 + \mu \operatorname{tr}(E^2) \tag{1.3}$$

(see [Ci.$_1$] Th. 4.4-3, p. 155 or [Ci.$_2$] Th. 1.4-7, p. 76).

«... *Since Saint Venant-Kirchhoff materials are the simplest among the nonlinear models (in the sense that they are the simplest that are compatible with* (1.1)), *they are quite popular in actual computations ... but the relative simplicity of their practical implementation is more than compensated by various shortcomings.*

... *their associated stored energy function is not polyconvex...*» (P. G. Ciarlet - 1988).

See [B.], [Ci.$_1$], [Ci.$_2$] or [D.] for details about polyconvexity, and [Ra.] for the non-polyconvexity of the stored energy function of a Saint Venant-Kirchhoff material.

On the other hand, if $W\colon \mathbb{M}^3_+ \to \mathbb{R}$ is the stored energy function associated to some hyperelastic material, it is suitable for physical reasons, that one has

$$\lim_{\det F \to 0^+} W(F) = +\infty \tag{1.4}$$

(see [Ci.$_1$] or [Ci.$_2$]). In this case, it is usual to extend $W$ to a continuous function defined of $\mathbb{M}^3$ into $\mathbb{R} \cup \{+\infty\}$ by

$$W(F) = +\infty \quad \text{if } \det F \leqslant 0\,.$$

The Saint Venant-Kirchhoff stored energy function does not satisfy (1.4) and thus we can not obtain an extension as above.

«... *At their best, Saint Venant-Kirchhoff materials can be only expected to be useful in a narrow range of 'small' strains* $E$, *as indeed they should be from their very definition ; this is why such materials are often referred to as 'large displacement-small strain' models. In spite of these various inadequacies, Saint Venant-Kirchhoff materials can be nevertheless expected to perform better than the linearized models that are so often used...*» (P. G. Ciarlet - 1988).

It turns out that we can consider the equality (1.3) as a definition of $\widehat{W}$ which is not only valid on a neighbourhood of $I$, but on $\mathbb{M}^3_+$ and even on the whole set $\mathbb{M}^3$.

As we have said above, the function $\widehat{W}$ is not polyconvex on $\mathbb{M}^3_+$ ; in fact, we know more since the Saint Venant-Kirchhoff stored energy function is not rank-one-convex on $\mathbb{M}^3_+$ (see [A.] or [Br.]). In this paper, we are interested in *the rank-one convexity domain* of $\widehat{W}$, in other terms in the greatest subset of $\mathbb{M}^n$ on which $\widehat{W}$ is rank-one-convex, and we are going to characterize exactly this domain.

So, from now on, we will denote by $\widehat{W}$ the function defined on $\mathbb{M}^n$ (with $n \geqslant 2$) by the equality (1.3).

## 2. The «Legendre-Hadamard» condition for $\widehat{W}$.

First, let us recall the definition of rank-one-convexity on a subset $U$ of $\mathbb{M}^n$. A function $W: \mathbb{M}^n \to \mathbb{R}$ is said *rank-1-convex on $U$* if

$$\forall F, G \in U \quad \text{such that } [F, G] \subset U \quad \text{and } \operatorname{rank}(F - G) = 1$$

one has

$$\forall \lambda \in [0, 1], \qquad W(\lambda F + (1 - \lambda) G) \leqslant \lambda W(F) + (1 - \lambda) W(G).$$

It is well known that, if $W$ is twice continuously differentiable, then $W$ is rank-1-convex on $U$ if and only if one has

$$\inf_{a,\, b \in \mathbb{R}^n} \sum_{i, j, k, l} \frac{\partial^2 W(F)}{\partial F_{ij} \partial F_{kl}} a_i b_j a_k b_l \geqslant 0, \qquad \forall F \in U. \tag{2.1}$$

(*Legendre-Hadamard* condition on $U$) (see [B.] Th. 3.3, p. 352).

Now, let us introduce some notation; for $F, G \in \mathbb{M}^n$ we set $F: G = \operatorname{tr}(F^T G)$ the matrix inner product in $\mathbb{M}^n$ and $|F| = (F: F)^{1/2}$ the corresponding norm.

Next, if $a$, $b \in \mathbb{R}^n$ let us denote by $a \otimes b$ the $n \times n$ matrix defined by

$$(a \otimes b)_{ij} = a_i b_j .$$

Therefore, if the vectors $a$ and $b$ are column matrices one has

$$a \otimes b = ab^T \tag{2.2}$$

and the euclidean inner product in $\mathbb{R}^n$ (respectivly the euclidean norm in $\mathbb{R}^n$) can be written

$$a.b = a^T b \quad \text{(respectivly } |a| = (a^T a)^{1/2}\text{)}. \tag{2.3}$$

Using (2.2) et (2.3) it is easy to verify that, for $F \in \mathbb{M}^n$ and $a, b, c, d \in \mathbb{R}^n$, the following identities hold:

$$(a \otimes b)^T = b \otimes a, \tag{2.4}$$

$$F(a \otimes b) = (Fa) \otimes b, \tag{2.5}$$

$$(a \otimes b)F = a \otimes (F^T b), \tag{2.6}$$

$$(a \otimes b)(c \otimes d) = (b.c)(a \otimes d), \tag{2.7}$$

$$\operatorname{tr}(a \otimes b) = a.b, \tag{2.8}$$

$$F : (a \otimes b) = Fb.a = F^T a.b, \tag{2.9}$$

$$(a \otimes b) : (c \otimes d) = (a.c)(b.d), \tag{2.10}$$

$$|a \otimes b|^2 = |a|^2 |b|^2. \tag{2.11}$$

Since it is clear that the Saint Venant-Kirchhoff stored energy function $\widehat{W}$ is indefinitly differentiable we can compute $\dfrac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}}$ and use the criterion (2.1) to study the rank-1-convexity of $\widehat{W}$. We have the following result:

LEMMA 2.1. *For $F \in \mathbb{M}^n$ and $a, b \in \mathbb{R}^n$ one has*

$$\sum_{i,j,k,l} \frac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}} a_i b_j a_k b_l = \tag{2.12}$$

$$= \left( \frac{\lambda}{2} (|F|^2 - n) |b|^2 + \mu(|Fb|^2 - |b|^2) \right) |a|^2 (\lambda + \mu)(F^T a.b)^2 + \mu |F^T a|^2 |b|^2.$$

PROOF. Let us consider $F \in \mathbb{M}^n$ and $a$, $b \in \mathbb{R}^n$. First, since $E = E^T$ (where $E$ is defined by (1.2)), we can write

$$\widehat{W}(F) = \frac{\lambda}{2}(\mathrm{tr}E)^2 + \mu|E|^2.$$

Next, for a matrix $H \in \mathbb{M}^n$ the quantity

$$\frac{1}{2}\sum_{i,j,k,l}\frac{\partial^2 \widehat{W}(F)}{\partial F_{ij}\,\partial F_{kl}} H_{ij}H_{kl} \overset{def}{=} \frac{1}{2}\varphi_F(H)$$

is the second-order term of the Taylor expansion in $H$ of $\widehat{W}(F+H)$.
We have

$$E(F+H) = \frac{1}{2}(F^TF - I + F^TH + H^TF + H^TH) \tag{2.13}$$

thus

$$\mathrm{tr}(E(F+H)) = \frac{1}{2}(|F|^2 - n + 2F\colon H + |H|^2)$$

then the second-order term in $H$ of $(\mathrm{tr}(E(F+H)))^2$ is

$$\frac{1}{4}((2F\colon H)^2 + 2(|F|^2 - n)|H|^2).$$

On the other hand, by using (2.13) and the equality $|A+B|^2 = |A|^2 + |B|^2 + 2A\colon B$ we see that the second-order term in $H$ of $|E(F+H)|^2$ is

$$\frac{1}{4}(|F^TH + H^TF|^2 + 2(F^TF - I)\colon H^TH)$$

therefore we deduce that for any matrix $H \in \mathbb{M}^n$, one has

$$\varphi_F(H) = \frac{\lambda}{2}(2(F\colon H)^2 + (|F|^2 - n)|H|^2) + \frac{\mu}{2}(|F^TH + H^TF|^2 + 2(F^TF - I)\colon H^TH). \tag{2.14}$$

Now, for $a, b \in \mathbb{R}^n$ and $H = a \otimes b$, (2.14) can be written

$$\varphi_F(a \otimes b) = \frac{\lambda}{2}\left(2(F : (a \otimes b))^2 + (|F|^2 - n)\,|a \otimes b|^2\right) +$$

$$+\frac{\mu}{2}\left(|F^T(a \otimes b) + (a \otimes b)^T F|^2 + 2(F^T F - I) : (a \otimes b)^T(a \otimes b)\right)$$

which becomes, by using the equalities (2.4)-(2.11)

$$\varphi_F(a \otimes b) = \frac{\lambda}{2}\left(2(F^T a.b)^2 + (|F|^2 - n)\,|a|^2\,|b|^2\right) + \tag{2.15}$$

$$+\frac{\mu}{2}\left(|(F^T a) \otimes b + b \otimes (F^T a)|^2 + 2(F^T F - I) : \,|a|^2(b \otimes b)\right).$$

But, one has

$$|(F^T a) \otimes b + b \otimes (F^T a)|^2 = |(F^T a) \otimes b|^2 + \,|b \otimes (F^T a)|^2 +$$

$$+2((F^T a) \otimes b) : (b \otimes (F^T a)) = 2\,|F^T a|^2\;|b|^2 + 2(F^T a.\,b)^2$$

and

$$(F^T F - I) : \,|a|^2(b \otimes b) = \,|a|^2(F^T F : (b \otimes b) - I : (b \otimes b)) =$$

$$= \,|a|^2(\,|Fb|^2 - \,|b|^2)$$

in such way that (2.15) gives

$$\varphi_F(a \otimes b) = \left(\frac{\lambda}{2}(\,|F|^2 - n)\,|b|^2 + \mu(\,|Fb|^2 - \,|b|^2)\right)|a|^2 +$$

$$+(\lambda + \mu)(F^T a.b)^2 + \mu\,|F^T a|^2\,|b|^2\,.$$

The proof is now complete. ■

Before to go on, let us recall what are the singular values of a matrix. Let $F \in \mathrm{M}^n$; if we denote by $\alpha_1, \ldots, \alpha_n$ the eigenvalues of the symmetric matrix $F^T.F$, one has $\forall i = 1, \ldots, n$, $\alpha_i \in \mathbb{R}_+$ and the $n$ non-negative numbers $v_i = \sqrt{\alpha_i}$ are called the *singular values* of $F$. Then, for some orthogonal matrices $Q_1$ and $Q_2$, we have

$$F = Q_1 \begin{bmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_n \end{bmatrix} Q_2 . \tag{2.16}$$

Next, we set $S^{n-1} = \{x \in \mathbb{R}^n ;\ |x| = 1\}$.

PROPOSITION 2.1. *Let $F \in \mathrm{M}^n$ and $v_1, \ldots, v_n$ the singular values of $F$. Then*

$$\inf_{a,\, b \in \mathbb{R}^n} \sum_{i, j, k, l} \frac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}} a_i b_j a_k b_l \geqslant 0 \Leftrightarrow \inf_{a,\, b \in S^{n-1}} f_{v_1, \ldots, v_n}(a, b) \geqslant 0 \tag{2.17}$$

*where $f_{v_1, \ldots, v_n}(a, b)$ is given by*

$$f_{v_1, \ldots, v_n}(a, b) = \frac{\lambda}{2}(v_1^2 + \ldots + v_n^2) + \mu(v_1^2(a_1^2 + b_1^2) + \ldots + v_n^2(a_n^2 + b_n^2)) + \\ + (\lambda + \mu)(v_1 a_1 b_1 + \ldots + v_n a_n b_n)^2 - \left(\frac{n\lambda}{2} + \mu\right).$$

PROOF. Since the functions

$$a \mapsto \sum_{i, j, k, l} \frac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}} a_i b_j a_k b_l \quad \text{and} \quad b \mapsto \sum_{i, j, k, l} \frac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}} a_i b_j a_k b_l$$

are homogeneous polynomials of degree 2, we can write

$$\inf_{a,\, b \in \mathbb{R}^n} \sum_{i, j, k, l} \frac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}} a_i b_j a_k b_l \geqslant 0 \Leftrightarrow \tag{2.18}$$

$$\Leftrightarrow \inf_{a,\, b \in S^{n-1}} \sum_{i, j, k, l} \frac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}} a_i b_j a_k b_l \geqslant 0 \Leftrightarrow$$

$$\Leftrightarrow \inf_{a,\, b \in S^{n-1}} \left[ \frac{\lambda}{2}(|F|^2 - n) + \mu(|F^T a|^2 + |Fb|^2 - 1) + (\lambda + \mu)(F^T a . b)^2 \right] \geqslant 0$$

by using, for $a, b \in S^{n-1}$, the identity (2.12) of the previous lemma.

Now, if we use (2.16) and denote by $V$ the matrix

$$\begin{bmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_n \end{bmatrix},$$

we get

$$(2.18) \Leftrightarrow \inf_{a,\, b \in S^{n-1}} \left[ \frac{\lambda}{2} (|Q_1 V Q_2|^2 - n) + \mu(|Q_2^T V Q_1^T a|^2 + |Q_1 V Q_2 b|^2 - 1) + \right.$$

$$\left. + (\lambda + \mu)(Q_2^T V Q_1^T a.b)^2 \right] \geqslant 0$$

$$\Leftrightarrow \inf_{a,\, b \in S^{n-1}} \left[ \frac{\lambda}{2} (|V|^2 - n) + \mu(|V(Q_1^T a)|^2 + |V(Q_2 b)|^2 - 1) + \right.$$

$$\left. + (\lambda + \mu)(V(Q_1^T a).(Q_2 b))^2 \right] \geqslant 0$$

$$\Leftrightarrow \inf_{a,\, b \in S^{n-1}} \left( \frac{\lambda}{2} (|V|^2 - n) + \mu(|Va|^2 + |Vb|^2 - 1) + (\lambda + \mu)(Va.b)^2 \right] \geqslant 0$$

$$\Leftrightarrow \inf_{a,\, b \in S^{n-1}} f_{v_1, \dots, v_n}(a, b) \geqslant 0 .$$

So, we have (2.17). ■

Now, we are able to obtain some results about the rank-1-convexity domain of $\widehat{W}$.

## 3. The rank-one convexity domain of $\widehat{W}$.

Let us denote by $\widehat{\Omega}$ the rank-1-convexity domain of $\widehat{W}$. The Legendre-Hadamard condition (2.1) implies that

$$\widehat{\Omega} = \left\{ F \in \mathrm{M}^n ;\ \inf_{a,\, b \in \mathrm{R}^n} \sum_{i,j,k,l} \frac{\partial^2 \widehat{W}(F)}{\partial F_{ij} \partial F_{kl}} a_i b_j a_k b_l \geqslant 0 \right\}$$

which can be written, by using proposition 2.1

$$\hat{\Omega} = \left\{F \in \mathbb{M}^n;\ \inf_{a,\, b \in S^{n-1}} f_{v_1, \dots, v_n}(a, b) \geq 0\right\} \tag{3.1}$$

where $v_1, \dots, v_n$ are the singular values of $F$.

For a matrix $F \in \mathbb{M}^n$ we would like to obtain necessary and sufficient conditions, in terms of its singular values, in order to have $F \in \hat{\Omega}$.

Our main result will rely on the following lemma:

Lemma 3.1. *Let $v_1, \dots, v_n$ be some real numbers such that $0 \leq v_1 \leq \dots \leq v_n$, and $g$ be the function defined on $\mathbb{R}^n \times \mathbb{R}^n$ by*

$$g(a, b) = \mu(v_1^2(a_1^2 + b_1^2) + \dots + v_n^2(a_n^2 + b_n^2)) + \\ +(\lambda + \mu)(v_1 a_1 b_1 + \dots + v_n a_n b_n)^2 .$$

*Next, let us denote by $K$ the following set*

$$K = \{(i, j) \in \mathbb{N}^2 \ ;\ 1 \leq i < j \leq n \text{ and } (\lambda + 2\mu) v_i \geq \mu v_j\}.$$

*If $\mho_g = \inf_{x,\, y \in S^{n-1}} g(x, y)$ then*

$$\mho_g = \begin{cases} (\lambda + 3\mu) v_1^2 & \text{if } (1, 2) \notin K, \\ \min_{(i, j) \in K} \dfrac{\mu}{\lambda + \mu}(-\mu v_i^2 + 2(\lambda + 2\mu) v_i v_j - \mu v_j^2) & \text{if } (1, 2) \in K. \end{cases}$$

Proof. First of all, let us denote by $e^1, \dots, e^n$ the canonical basis of $\mathbb{R}^n$, and recall that the constantes $\lambda$, $\mu$ are positive.

If $v_1 = 0$, then $g(e^1, e^1) = 0$ and since $g \geq 0$, we get $\mho_g = 0$, in such way that the lemma is proved in this case.

So, from now on, we will assume $v_1 > 0$, and we will denote by $I_1, \dots, I_p$ the subsets of $\{1, \dots, n\}$ defined by $I_1 \cup \dots \cup I_p = \{1, \dots, n\}$ and

$$\begin{cases} i, j \in I_k \Rightarrow v_i = v_j , \\ i \in I_k,\ j \in I_l \text{ and } k < l \Rightarrow v_i < v_j . \end{cases}$$

Since $\mathcal{S}^{n-1} \times \mathcal{S}^{n-1}$ is a compact set,

$$\mathfrak{I}_g = g(a, b) \tag{3.2}$$

for some $(a, b) \in \mathcal{S}^{n-1} \times \mathcal{S}^{n-1}$.

Now, let us divide the rest of the proof in two steps.

*Step* 1. Here we are going to give necessary conditions in order to have (3.2). So, let $(a, b) \in \mathcal{S}^{n-1} \times \mathcal{S}^{n-1}$ satisfying (3.2); then, there exist Lagrange multipliers $\alpha$ and $\beta$ such that $\forall i = 1, \ldots, n$ one has

$$\frac{\partial}{\partial a_i}(g(a, b) + \alpha(|a|^2 - 1) + \beta(|b|^2 - 1)) =$$

$$= \frac{\partial}{\partial b_i}(g(a, b) + \alpha(|a|^2 - 1) + \beta(|b|^2 - 1)) = 0.$$

Therefore, $\forall i = 1, \ldots, n$ one has

$$\mu v_i^2 a_i + (\lambda + \mu) c v_i b_i + \alpha a_i = 0 \tag{3.3$_i$}$$

and

$$\mu v_i^2 b_i + (\lambda + \mu) c v_i a_i + \beta b_i = 0 \tag{3.4$_i$}$$

where we have set $c = \sum_{j=1}^{n} v_j a_j b_j$.

Now, multiplying (3.3$_i$) by $b_i / v_i$ and adding for $i = 1, \ldots, n$ we get

$$(\lambda + 2\mu) c + \alpha \sum_{i=1}^{n} \frac{a_i b_i}{v_i} = 0 . \tag{3.5}$$

From (3.4$_i$) we obtain, by the same way

$$(\lambda + 2\mu) c + \beta \sum_{i=1}^{n} \frac{a_i b_i}{v_i} = 0 . \tag{3.6}$$

So, (3.5) and (3.6) imply that

$$(\alpha - \beta) \sum_{i=1}^{n} \frac{a_i b_i}{v_i} = 0 .$$

— In a first case, let us assume that one has $\sum_{i=1}^{n} \frac{a_i b_i}{v_i} = 0$; then by (3.5), $c = 0$. Therefore, ($3.3_i$) and ($3.4_i$) can be written

$$(\mu v_i^2 + \alpha) a_i = 0 \quad \text{and } (\mu v_i^2 + \beta) b_i = 0$$

and thus, since this equalities are valid for all $i = 1, \ldots, n$, and that $\alpha$ (respectivly $\beta$) does not depend on $i$, we get necessarily, for some $k$, $l$

$$a = \sum_{i \in I_k} a_i e^i \quad \text{and } b = \sum_{i \in I_l} b_i e^i .$$

Consequently, since $c = 0$ and $a, b \in S^{n-1}$, we easily deduce

$$g(a, b) = \mu(v_i^2 + v_j^2) \quad \text{for some } i, j. \tag{3.7}$$

— Now, let us assume that $\alpha = \beta$; then we can suppose $c \neq 0$ (indeed, if $c = 0$, then thanks to ($3.3_i$), we obtain $\alpha \neq 0$ and by (3.5), $\sum_{i=1}^{n} \frac{a_i b_i}{v_i} = 0$ which brought us back to the previous case). Next, if we multiply ($3.3_i$) by $b_i$ and ($3.4_i$) by $a_i$, we get by substraction

$$(\lambda + \mu) c v_i (b_i^2 - a_i^2) = 0 \quad \text{implying that } b_i^2 = a_i^2 \ \forall i . \tag{3.8}$$

Let $i \in \{1, \ldots, n\}$; using (3.8), one has, either

$$\begin{cases} a_i = b_i = 0 \\ \text{or} \\ a_i = b_i \neq 0 \text{ and thus } c = -\dfrac{\alpha + \mu v_i^2}{(\lambda + \mu) v_i} \text{ by } (3.\ 3_i) \\ \text{or} \\ a_i = -b_i \neq 0 \text{ and thus } c = \dfrac{\alpha + \mu v_i^2}{(\lambda + \mu) v_i} \text{ by } (3.\ 3_i). \end{cases} \tag{3.9}$$

Let us remark that the equality

$$\frac{\alpha + \mu v_i^2}{(\lambda + \mu) v_i} = \frac{\alpha + \mu v_j^2}{(\lambda + \mu) v_j}$$

implies $(v_j - v_i)(\alpha - \mu v_i v_j) = 0$ and $v_i = v_j$ (since, if we multiply $(3.3_i)$ by $a_i$, and add for $i = 1, \dots, n$, we get $\mu \sum_{i=1}^{n} v_i^2 a_i^2 + (\lambda + \mu) c^2 + \alpha = 0$ and $\alpha \leqslant 0$).

Consequently, if we denote by $A$ and $B$ the following subsets of $\{1, \dots, n\}$

$$A = \{i \; ; \; a_i = b_i \neq 0\} \quad \text{and} \quad B = \{i \; ; \; a_i = -b_i \neq 0\}$$

we get for some $k, l \in \{1, \dots, p\}$

$$A \subset I_k \quad \text{and} \quad B \subset I_l \, .$$

(3.10) $\rightarrow$ If $A = \emptyset$ then $g(a, b) = (\lambda + 3\mu) v_j^2$ for $j \in I_l$.

(3.11) $\rightarrow$ If $B = \emptyset$ then $g(a, b) = (\lambda + 3\mu) v_j^2$ for $j \in I_k$.

$\rightarrow$ If $A \neq \emptyset$ and $B \neq \emptyset$ then $k \neq l$ (indeed, if $k = l$, then (3.9) implies $c = 0$, but we have supposed the opposite). So, $a$ and $b$ are of the form

$$a = \sum_{i \in I_k} a_i e^i + \sum_{i \in I_l} a_i e^i \quad \text{and} \quad b = \sum_{i \in I_k} a_i e^i - \sum_{i \in I_l} a_i e^i \tag{3.12}$$

and thus, if $i \in I_k$ and $j \in I_l$, we have $i \neq j$ and

$$g(a, b) = 2\mu(v_i^2 x^2 + v_j^2 y^2) + (\lambda + \mu) c^2 \tag{3.13}$$

where $x^2 = \sum_{s \in I_k} a_s^2$ and $y^2 = \sum_{s \in I_l} a_s^2$.

But, using (3.9), we can write

$$c = -\frac{\alpha + \mu v_i^2}{(\lambda + \mu) v_i} = \frac{\alpha + \mu v_j^2}{(\lambda + \mu) v_j}$$

which gives easily

$$\alpha = -\mu v_i v_j \quad \text{and} \quad c = \frac{\mu}{\lambda + \mu} (v_j - v_i) \, . \tag{3.14}$$

On the other hand, by definition of $c$ and since $y^2 = 1 - x^2$, we have

$$c = v_i x^2 - v_j y^2 = (v_i + v_j) x^2 - v_j$$

implying that

$$(3.15) \qquad x^2 = \frac{-\mu v_i + (\lambda + 2\mu) v_j}{(\lambda + \mu)(v_i + v_j)} \quad \text{and} \quad y^2 = \frac{(\lambda + 2\mu) v_i - \mu v_j}{(\lambda + \mu)(v_i + v_j)} .$$

By using (3.13), (3.14) and (3.15) we get

$$(3.16) \qquad g(a, b) = \frac{\mu}{\lambda + \mu} ( - \mu v_i^2 + 2(\lambda + 2\mu) v_i v_j - \mu v_j^2 ) .$$

So, we have proved that, if $\mathfrak{J}_g = g(a, b)$ then necessarily, $g(a, b)$ is given by (3.7), (3.10), (3.11) or (3.16).

*Step* 2. Now, we can compute $\mathfrak{J}_g$ according to the values of $v_1 / v_2$.

First, let us denote by $h$ the function

$$h(x, y) = \frac{\mu}{\lambda + \mu}( - \mu x^2 + 2(\lambda + 2\mu) xy - \mu y^2 ) .$$

Now, let us remark that, if $K = \emptyset$, then (3.15) is not possible for $i \neq j$ and thus (3.12) does not hold. Consequently, thanks to (3.7), (3.10), (3.11) and (3.16) we get

$$(3.17) \quad \mathfrak{J}_g = \begin{cases} \min [(\lambda + 3\mu) v_1^2 ;\ \mu(v_1^2 + v_2^2)] & \textit{if } K = \emptyset , \\ \min [(\lambda + 3\mu) v_1^2 ;\ \mu(v_1^2 + v_2^2);\ \min_{(i, j) \in K} h(v_i, v_j)] & \textit{if } K \neq \emptyset . \end{cases}$$

— If $(1, 2) \in K$ then

$$\min_{(i, j) \in K} h(v_i, v_j) \leqslant h(v_1, v_2) \leqslant \min [(\lambda + 3\mu) v_1^2 ;\ \mu(v_1^2 + v_2^2)]$$

the last inequality arising from the following identities

$$\begin{cases} h(v_1, v_2) - \mu(v_1^2 + v_2^2) = - \dfrac{\mu(\lambda + 2\mu)}{\lambda + \mu} (v_1 - v_2)^2 , \\ h(v_1, v_2) - (\lambda + 3\mu) v_1^2 = - \dfrac{1}{\lambda + \mu} (\mu v_1 - (\lambda + 2\mu) v_2)^2 . \end{cases}$$

So, by using (3.17) we obtain

$$(1,2) \in K \Rightarrow \mathfrak{I}_g = \min_{(i,j)\in K} h(v_i, v_j).$$

— If $(1,2) \notin K$ then

$$(\lambda + 3\mu) v_1^2 - \mu(v_1^2 + v_2^2) = (\lambda + 2\mu) v_1^2 - \mu v_2^2 < 0$$

$\left(\text{indeed, } \frac{v_1}{v_2} < \frac{\mu}{\lambda + 2\mu} < 1 \text{ and thus } \frac{v_1^2}{v_2^2} < \frac{\mu}{\lambda + 2\mu} \text{ which implies that } (\lambda + 2\mu) v_1^2 < \mu v_2^2\right)$.

Therefore, taking into account of (3.17), we can write when $(1,2) \notin K$

$$\mathfrak{I}_g = \begin{cases} (\lambda + 3\mu) v_1^2 & \text{if } K = \emptyset, \\ \min[(\lambda + 3\mu) v_1^2;\ \min_{(i,j)\in K} h(v_i, v_j)] & \text{if } K \neq \emptyset. \end{cases}$$

Finally, to conclude, it remains to show that

$$\forall (i,j) \in K, \quad h(v_i, v_j) - (\lambda + 3\mu) v_1^2 \geq 0 \tag{3.18}$$

holds when $K \neq \emptyset$ and $(1,2) \notin K$.

Let $(i,j) \in K$; then

$$(\lambda + 2\mu) v_i - \mu v_j \geq 0. \tag{3.19}$$

Next, $(1,2) \notin K$ and thus

$$(\lambda + 2\mu) v_1 \leq \mu v_2 \leq \mu v_j. \tag{3.20}$$

By using (3.19) and (3.20) we get

$$((\lambda + 2\mu) v_1 - \mu v_j)((\lambda + 2\mu) v_i - \mu v_j) \leq 0$$

that we can write

$$\mu^2 v_j^2 - \mu(\lambda + 2\mu) v_i v_j \leq -(\lambda + 2\mu)^2 v_1 v_i + \mu(\lambda + 2\mu) v_1 v_j.$$

Consequently,

$$h(v_i, v_j) - (\lambda + 3\mu) v_1^2 =$$

$$= -\frac{1}{\lambda + \mu}(\mu^2 v_i^2 - 2\mu(\lambda + 2\mu) v_i v_j + \mu^2 v_j^2 + (\lambda + 3\mu)(\lambda + \mu) v_1^2) \geq$$

$$\geqslant -\frac{1}{\lambda+\mu}(\mu^2 v_i^2 - \mu(\lambda+2\mu)v_i v_j + (\lambda+3\mu)(\lambda+\mu)v_1^2 - $$
$$-(\lambda+2\mu)^2 v_1 v_i + \mu(\lambda+2\mu)v_1 v_j) =$$

$$= -\frac{1}{\lambda+\mu}(\mu^2 v_i^2 - \mu(\lambda+2\mu)v_i v_j + (\lambda+2\mu)^2 v_1^2 - \mu^2 v_1^2 - $$
$$-(\lambda+2\mu)^2 v_1 v_i + \mu(\lambda+2\mu)v_1 v_j) =$$

$$= -\frac{1}{\lambda+\mu}(\mu^2(v_i - v_1)(v_i + v_1) - \mu(\lambda+2\mu)v_j(v_i - v_1) - $$
$$-(\lambda+2\mu)^2 v_1(v_i - v_1)) =$$

$$= \frac{v_i - v_1}{\lambda+\mu}(\mu(\lambda+2\mu)v_j + (\lambda+2\mu)^2 v_1 - \mu^2(v_i + v_1)) =$$

$$= \frac{v_i - v_1}{\lambda+\mu}((\lambda+3\mu)(\lambda+\mu)v_1 + \mu(\lambda+2\mu)v_j - \mu^2 v_i) \geqslant 0$$

since $v_j \geqslant v_i$ and $v_i \geqslant v_1$.

So, (3.18) holds, and the proof is complete. ■

The following theorem gives the requested characterization of $\hat{\mathfrak{Q}}$.

THEOREM 3.1. *Let* $F \in \mathbb{M}^n$ *and* $v_1 \leqslant \ldots \leqslant v_n$ *be the singular values of* $F$. *Let us denote by* $K(F)$ *the following set*

$$K(F) = \{(i,j) \in \mathbb{N}^2;\ 1 \leqslant i < j \leqslant n \text{ and } (\lambda+2\mu)v_i \geqslant \mu v_j\}.$$

(i) *If* $(1,2) \notin K(F)$ *then,* $F \in \hat{\mathfrak{Q}}$ *if and only if*

$$(\lambda+3\mu)v_1^2 \geqslant \left(\frac{n\lambda}{2}+\mu\right) - \frac{\lambda}{2}(v_1^2 + \ldots + v_n^2).$$

(ii) *If* $(1,2) \in K(F)$ *then,* $F \in \hat{\mathfrak{Q}}$ *if and only if*

$$\min_{(i,j)\in K(F)} \frac{\mu}{\lambda+\mu}(-\mu v_i^2 + 2(\lambda+2\mu)v_i v_j - \mu v_j^2) \geqslant$$
$$\geqslant \left(\frac{n\lambda}{2}+\mu\right) - \frac{\lambda}{2}(v_1^2 + \ldots + v_n^2).$$

PROOF. This follows immediatly from (3.1), Proposition 2.1 and Lemma 3.1. ■

REMARK 3.1. Clearly, one can see that $\widehat{\mathfrak{Q}}$ is a neighbourhood of the identity matrix of $\mathbb{M}^n$.

## 4. Characterization of $\widehat{\mathfrak{Q}}$ in the two-dimensional case.

In this last section, we will assume that $n = 2$. We would like to precise the results of the previous part, and to give a representation in $\mathbb{R}_+ \times \mathbb{R}_+$ of the following set

$$\widehat{\mathfrak{V}} = \left\{ (v_1, v_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \ ; \ \forall Q_1, Q_2 \in \mathrm{O}(2), \ Q_1 \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} Q_2 \in \widehat{\mathfrak{Q}} \right\}$$

(where $\mathrm{O}(2)$ is the set of all the $2 \times 2$ orthogonal matrices).

The first theorem is the two-dimensional version of Theorem 3.1.

THEOREM 4.1. *Let $F \in \mathbb{M}^2$ and $v_1, v_2$ be the singular values of $F$.*

(i) *If* $\dfrac{v_1}{v_2} \in \left[0, \dfrac{\mu}{\lambda + 2\mu}\right]$ *then,*

$$F \in \widehat{\mathfrak{Q}} \Leftrightarrow 3\left(\frac{\lambda}{2} + \mu\right) v_1^2 + \frac{\lambda}{2} v_2^2 \geqslant \lambda + \mu .$$

(ii) *If* $\dfrac{v_1}{v_2} \in \left[\dfrac{\mu}{\lambda + 2\mu}, \dfrac{\lambda + 2\mu}{\mu}\right]$ *then,*

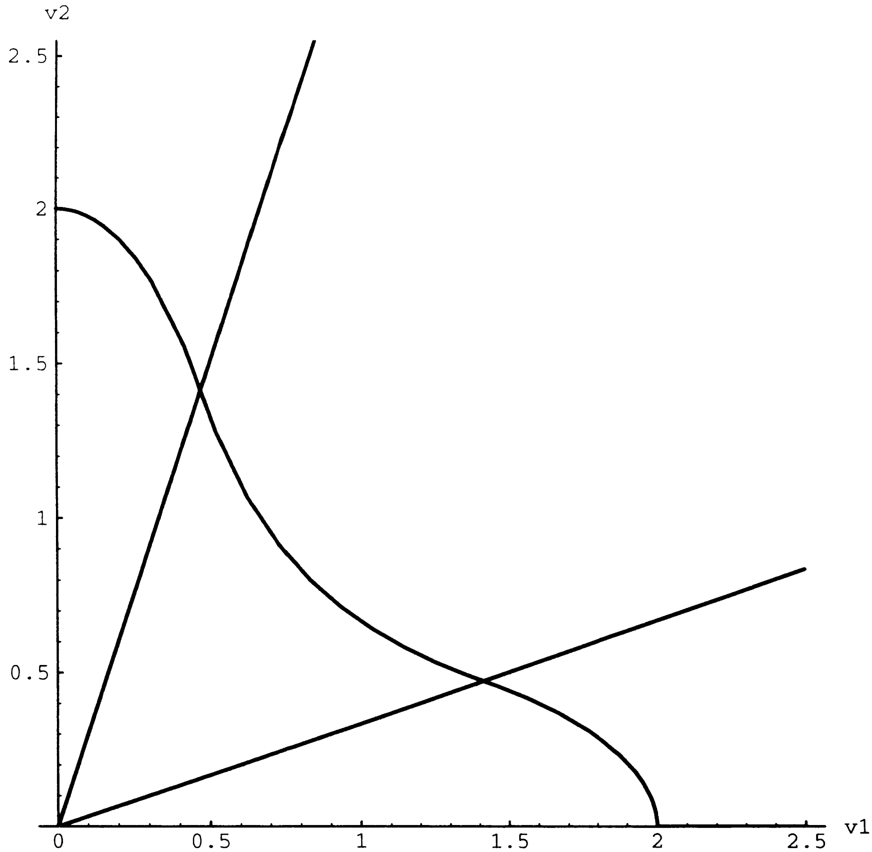
$$F \in \widehat{\mathfrak{Q}} \Leftrightarrow \frac{\lambda - \mu}{2} (v_1^2 + v_2^2) + 2\mu v_1 v_2 \geqslant \frac{(\lambda + \mu)^2}{\lambda + 2\mu} .$$

(iii) *If* $\dfrac{v_1}{v_2} \in \left[\dfrac{\lambda + 2\mu}{\mu}, +\infty\right]$ *then,*

$$F \in \widehat{\mathfrak{Q}} \Leftrightarrow \frac{\lambda}{2} v_1^2 + 3\left(\frac{\lambda}{2} + \mu\right) v_2^2 \geqslant \lambda + \mu.$$

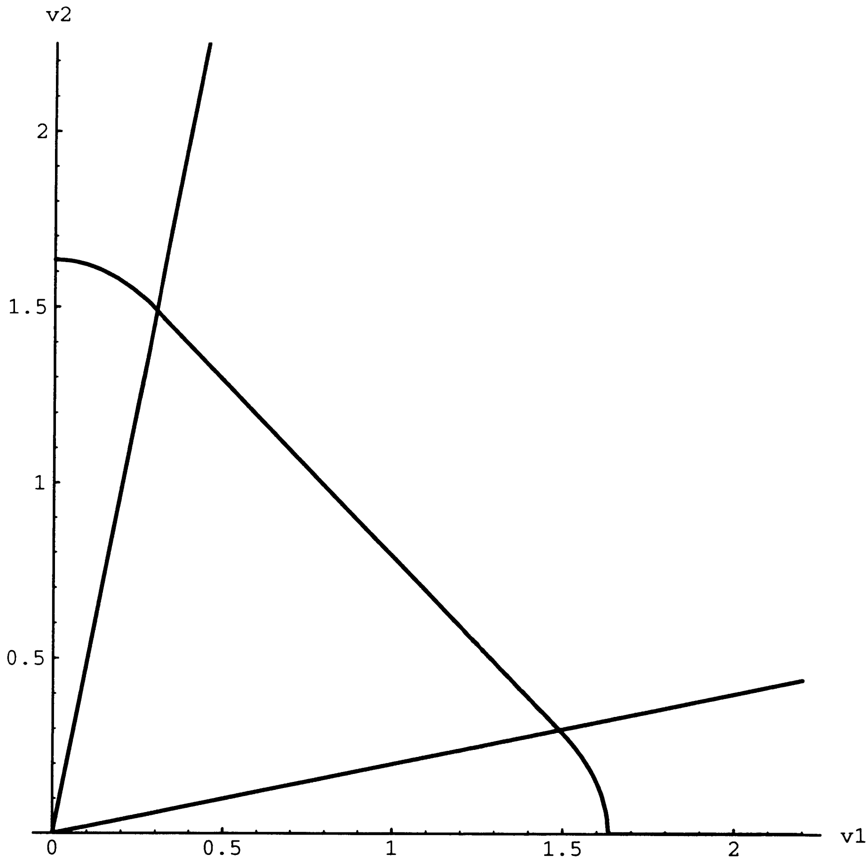
PROOF. This follows immediatly from Theorem 3.1. Let us note that we have not assumed $v_1 \leqslant v_2$, which explains the three cases (i), (ii) and (iii). ∎

On each of the following figures, the region located above the curve represent the set $\widehat{\mathfrak{V}}$ for some values of the quotient $\lambda/\mu$.



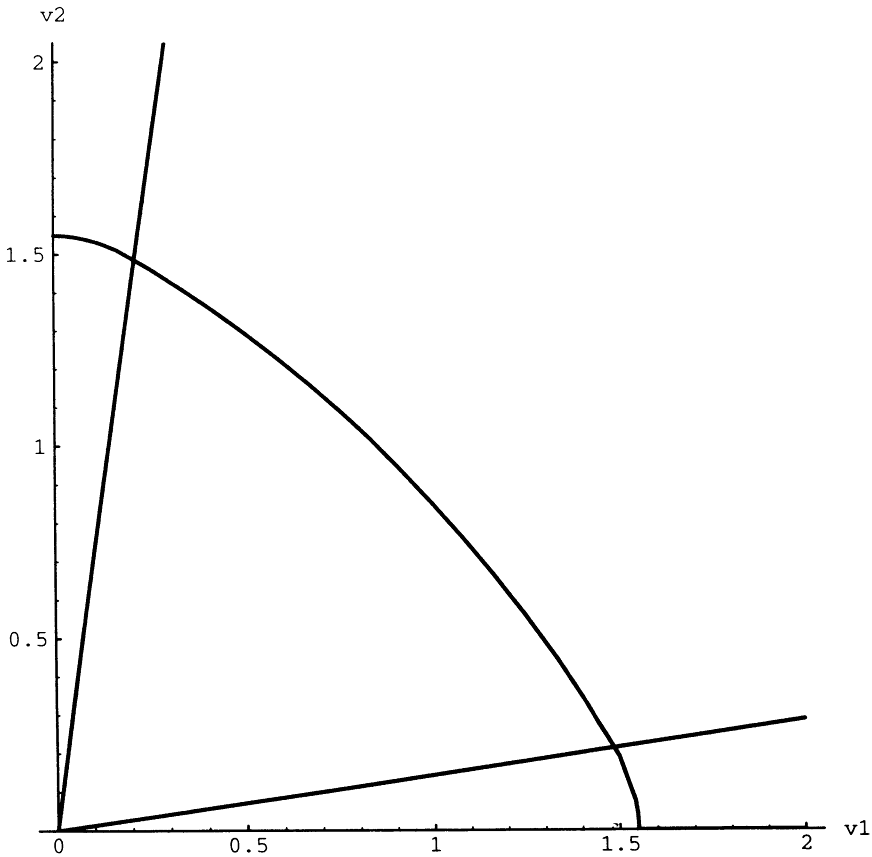

$$(v_1, v_2) \in \widehat{\mathcal{V}} \Leftrightarrow \begin{cases} 9v_1^2 + v_2^2 \geqslant 4 & \text{when} \quad 0 \leqslant \dfrac{v_1}{v_2} \leqslant \dfrac{1}{3}, \\ v_1 v_2 \geqslant \dfrac{2}{3} & \text{when} \quad \dfrac{1}{3} \leqslant \dfrac{v_1}{v_2} \leqslant 3, \\ v_1^2 + 9v_2^2 \geqslant 4 & \text{when} \quad 3 \leqslant \dfrac{v_1}{v_2} \leqslant +\infty. \end{cases}$$

Figure 1. – $\dfrac{\lambda}{\mu} = 1.$




$$(v_1, v_2) \in \widehat{\mathcal{V}} \Leftrightarrow \begin{cases} 15v_1^2 + 3v_2^2 \geqslant 8 & \text{when} \quad 0 \leqslant \dfrac{v_1}{v_2} \leqslant \dfrac{1}{5}, \\ v_1 + v_2 \geqslant \dfrac{4}{\sqrt{5}} & \text{when} \quad \dfrac{1}{5} \leqslant \dfrac{v_1}{v_2} \leqslant 5, \\ 3v_1^2 + 15v_2^2 \geqslant 8 & \text{when} \quad 5 \leqslant \dfrac{v_1}{v_2} \leqslant +\infty. \end{cases}$$

Figure 2. - $\dfrac{\lambda}{\mu} = 3$.




$$(v_1, v_2) \in \widehat{\mathcal{V}} \Leftrightarrow \begin{cases} 21v_1^2 + 5v_2^2 \geqslant 12 & \text{when} \quad 0 \leqslant \dfrac{v_1}{v_2} \leqslant \dfrac{1}{7}, \\ v_1^2 + v_2^2 + v_1 v_2 \geqslant \dfrac{18}{17} & \text{when} \quad \dfrac{1}{7} \leqslant \dfrac{v_1}{v_2} \leqslant 7, \\ 5v_1^2 + 21v_2^2 \geqslant 12 & \text{when} \quad 7 \leqslant \dfrac{v_1}{v_2} \leqslant +\infty. \end{cases}$$

Figure 3. – $\dfrac{\lambda}{\mu} = 5.$

Now, for $F \in M^2$ and $v_1 \leqslant v_2$ its singular values, one has

$$v_1^2 + v_2^2 = |F|^2 \quad \text{and} \quad v_1 v_2 = \det F. \tag{4.1}$$

Moreover, if we denote by $\|\cdot\|$ the matrix norm defined on $M^2$ by

$$\|F\|^2 = \sup_{|x|=1} |Fx|^2$$

where $|\cdot|$ denotes the euclidean norm (see (2.3)). It is well known that $\|F\| = v_2$ and thus

$$(\lambda + 2\mu) v_1 \geqslant \mu v_2 \Leftrightarrow (\lambda + 2\mu) \det F \geqslant \mu \|F\|^2$$

in such way that Theorem 4.1 leads to:

THEOREM 4.2. *Let* $F \in M^2 \setminus \{0\}$.

(i) *If* $(\lambda + 2\mu) \det F \leqslant \mu \|F\|^2$, *then* $F \in \hat{\Omega}$ *if and only if*

$$\frac{\lambda}{2} \|F\|^4 - (\lambda + \mu) \|F\|^2 + 3\left(\frac{\lambda}{2} + \mu\right)(\det F)^2 \geqslant 0.$$

(ii) *If* $(\lambda + 2\mu) \det F \geqslant \mu \|F\|^2$, *then* $F \in \hat{\Omega}$ *if and only if*

$$\frac{\lambda - \mu}{2} |F|^2 + 2\mu \det F \geqslant \frac{(\lambda + \mu)^2}{\lambda + 2\mu}.$$

PROOF. It is immediate since

$$v_1 = \frac{\det F}{\|F\|} \quad (\text{if } F \neq 0) \quad \text{and } v_2 = \|F\|. \qquad \blacksquare$$

REMARK 4.1. This can be express also without using the norm $\|\cdot\|$; indeed (4.1) implies that $v_1^2$ and $v_2^2$ are the solutions of the following equations

$$X^2 - |F|^2 X + (\det F)^2 = 0$$

and thus

$$v_i^2 = \frac{1}{2}\left(|F|^2 \pm \sqrt{|F|^4 - 4(\det F)^2}\right).$$

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