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Edge-of-the-wedge theorem for elliptic systems

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Edge-of-the-Wedge Theorem for Elliptic Systems.

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ABSTRACT - Let M be a real analytic manifold, X a complexification of M, $N \subset M$ a submanifold, and $Y \subset X$ a complexification of N. One denotes by $\mathcal{A}_M$ the sheaf of real analytic functions on M, and by $\mathcal{B}_M$ the sheaf of Sato hyperfunctions. Let $\mathcal{M}$ be an elliptic system of linear differential operators on M for which Y is non-characteristic. Using the language of the microlocal study of sheaves of [K-S] we give a new proof of a result of Kashiwara-Kawai [K-K] which asserts that

$$(\ddagger) \quad H^j \mu_N(\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, *)) = 0 \qquad \text{for } * = \mathcal{A}_M,\ \mathcal{B}_M,\ j < \operatorname{cod}_M N,$$

where μ_N denotes the Sato microlocalization functor. For $\operatorname{cod}_M N = 1$, the previous result reduces to the Holmgren's theorem for hyperfunctions, and of course in this case the ellipticity assumption is not necessary. For $\operatorname{cod}_M N > 1$, this implies that the sheaf of analytic (resp. hyperfunction) solutions to $\mathcal{M}$ satisfies the edge-of-the-wedge theorem for two wedges in M with edge N. Dropping the ellipticity hypothesis in this higher codimensional case, we then show how $(\ddagger)$ no longer holds for $* = \mathcal{A}_M$. In the frame of tempered distributions, Liess [L] gives an example of constant coefficient system for which the edge-of-the-wedge theorem is not true. We don't know whether $(\ddagger)$ holds or not for $* = \mathcal{B}_M$ in the non-elliptic case.

1. Notations and statement of the result.

1.1. Let X be a real analytic manifold and $N \subset M \subset X$ real analytic submanifolds. One denotes by $\pi\colon T^*X \to X$ the cotangent bundle to X, and by $T^*_N X$ the conormal bundle to N in X. The embedding $f\colon M \to X$ induces a smooth morphism ${}^t f'_N\colon T^*_N X \to T^*_N M$.

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Let $\gamma \subset T_N X$ be an open convex cone of the normal bundle $T_N X$. We denote by γ^a its antipodal, and by $\gamma^0 \subset T_N^* X$ its polar. For a subset $U \subset X$ one denotes by $C_N(U) \subset T_N X$ its normal Whitney cone.

DEFINITION 1.1. An open connected set $U \subset X$ is called a wedge in X with profile γ if $C_N(X \backslash U) \cap \gamma = \emptyset$. The submanifold N is called the edge of U. We denote by $\mathcal{W}_\gamma$ the family of wedges with profile γ.

1.2. Let us recall some notions from [K-S]. Let $D^b(X)$ denote the derived category of the category of bounded complexes of sheaves of $\boldsymbol{C}$-vector spaces on X. For F an object of $D^b(X)$, one denotes by SS(F) its micro-support, a closed, conic, involutive subset of T^*X. One says that M is non-characteristic for F if $SS(F) \cap T_M^* X \subset M \times_X T_X^* X$. Recall that in this case, one has $f^! F \simeq F|_M \otimes or_{M/X}[-\operatorname{cod}_X M]$, where $or_{M/X}$ denotes the relative orientation sheaf of M in X.

Denote by $\mu_N(F)$ the Sato microlocalization of F along N, an object of $D^b(T_N^* X)$.

PROPOSITION 1.2. (cf. [K-S, Theorem 4.3.2])

(i) $R\Gamma_N(\mu_N(F)) \simeq F|_N \otimes or_{N/X}[-\operatorname{cod}_X N]$,

(ii) *for $\gamma \subset T_N X$ an open proper convex cone, there is an isomorphism for all $j \in \boldsymbol{Z}$:*

$$H^j_{\gamma^{0a}}(T_N^* X; \mu_N(F) \otimes or_{N/X}) \simeq \varinjlim_{U \in \mathcal{W}_\gamma} H^{j-\operatorname{cod}_X N}(U; F),$$

The main tool of this paper will be the following result on commutation for microlocalization and inverse image due to Kashiwara-Schapira.

THEOREM 1.3. (cf. [K-S, Corollary 6.7.3]) *Assume that M is non-characteristic for F. Then the natural morphism:*

$$\mu_N(f^! F) \to R\,{}^t f'_{N*}\mu_N(F)$$

is an isomorphism.

1.3. We will consider the following geometrical frame.

Let M be a real analytic manifold of dimension n, and let $N \subset M$ be a real analytic submanifold of codimension d. Let X be a complexification of M, $Y \subset X$ a complexification of N, and consider the embeddings.

$$\begin{array}{ccc} M & \xrightarrow{f} & X \\ {\scriptstyle j}\uparrow & & {\scriptstyle g}\uparrow \\ N & \xrightarrow{i} & Y \end{array}$$

One denotes by $\mathcal{O}_X$ the sheaf of germs of holomorphic functions on X, and by $\mathcal{D}_X$ the sheaf of rings of linear holomorphic differential operators on X. The sheaf $\mathcal{O}_X|_M$ of real analytic functions is denoted by $\mathcal{A}_M$. Moreover, one considers the sheaves:

$$\mathcal{B}_M = \mathrm{R}\Gamma_M(\mathcal{O}_X) \otimes or_{M/X}[n] \simeq f^! \mathcal{O}_X \otimes or_{M/X}[n],$$

$$\mathcal{C}_M = \mu_M(\mathcal{O}_X) \otimes or_{M/X}[n].$$

These are the sheaves of Sato's hyperfunctions and microfunctions respectively.

Let $\mathcal{M}$ be a left coherent $\mathcal{D}_X$-module. One says that $\mathcal{M}$ is *non-characteristic* for Y if $\mathrm{char}(\mathcal{M}) \cap T_Y^* X \subset Y \times_X T^* X$ (here $\mathrm{char}(\mathcal{M}) \subset T^* X$ denotes the characteristic variety of $\mathcal{M}$), and one denotes by $\mathcal{M}_Y$ the induced system on Y, a left coherent $\mathcal{D}_Y$-module. One says that $\mathcal{M}$ is *elliptic* if $\mathcal{M}$ is non-characteristic for M. Recall that in this case, by the fundamental theorem of Sato, one has:

$$\mathrm{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M) \simeq \mathrm{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M). \tag{1.1}$$

1.4. In the next section we will give a new proof of the following theorem of Kashiwara and Kawai:

THEOREM 1.4. (cf. [K-K]). *Let $\mathcal{M}$ be a left coherent elliptic $\mathcal{D}_X$-module, non-characteristic for Y. Then*

$$H^j \mu_N(\mathrm{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, *)) = 0 \qquad \textit{for } * = \mathcal{A}_M,\ \mathcal{B}_M,\ j < d. \tag{1.2}$$

Let us discuss here some corollaries of this result.

1.4.1. Let X be a complex analytic manifold. One denotes by $\overline{X}$ the complex conjugate of X, and by $X^{\boldsymbol{R}}$ the underlying real analytic manifold to X. Identifying $X^{\boldsymbol{R}}$ to the diagonal of $X \times \overline{X}$, the complex manifold $X \times \overline{X}$ is a natural complexification of $X^{\boldsymbol{R}}$.

Let $S \subset X^{\boldsymbol{R}}$ be a real analytic submanifold (identified to a subset of X), and let $S^{\boldsymbol{C}} \subset X \times \overline{X}$ be a complexification of S. Denoting by $\bar{\partial}$ the

Cauchy-Riemann system (i.e. $\overline{\partial} = \mathcal{D}_X \underline{\boxtimes} \mathcal{O}_{\overline{X}}$), one has an obvious isomorphism

$$\mu_S(\mathcal{O}_X) \simeq \mu_S(\mathrm{R}\ \mathcal{H}om_{\mathcal{D}_{X\times\overline{X}}}(\overline{\partial}, \mathcal{B}_{X^R})). \tag{1.3}$$

Assume S is generic, i.e. $TS +_S iTS = S \times_X TX$. Then the embedding $g\colon S^C \to X \times \overline{X}$ is non-characteristic for $\overline{\partial}$ (which is, of course, elliptic) and hence, combining (1.3) with Theorem 1.4, one recovers the well known result:

COROLLARY 1.5 *Let $S \subset X$ be a generic submanifold with* $\mathrm{cod}_X S = d$. *Then*

$$H^j \mu_S(\mathcal{O}_X) = 0 \qquad \text{for } j < d.$$

1.4.2 Let us go back to the notations of 1.3, and assume that N is a hypersurface of M defined by the equation $\phi(x) = 0$ with $d\phi \neq 0$. Assume that $M \backslash N$ has the two open connected components $M^{\pm} = \{x;\ \pm\phi(x) > 0\}$. Let $\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X P$ for an elliptic differential operator P non-characteristic for Y.

By Theorem 1.4 we then recover the classical Holmgren's theorem for hyperfunctions:

COROLLARY 1.6. *Let $u \in \mathcal{B}_M$ be a solution of $Pu = 0$ such that $u|_{M^+} = 0$. Then $u = 0$.*

As it is well known, this result remains true even for non elliptic operators.

1.4.3 Assume now $\mathrm{cod}_M N = d > 1$, and let $\mathcal{M}$ be a left coherent elliptic $\mathcal{D}_X$-module which is non-characteristic for Y. Let γ be an open convex proper cone of the normal bundle $T_N M$, and let $U \in \mathcal{W}_\gamma$ be a wedge with profile γ.

By Proposition 1.2 and Theorem 1.4, one has

$$\varinjlim_{U \in \mathcal{W}_\gamma} \Gamma(U; \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)) \simeq \tag{1.4}$$

$$\simeq H^d \mathrm{R}\Gamma_{\gamma^{0a}}(T_N^* M; \mu_N(\mathrm{R}\ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \otimes or_{N/X})) \simeq$$

$$\simeq \Gamma_{\gamma^{0a}}(T_N^* M; H^d \mu_N(\mathrm{R}\ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \otimes or_{N/X})).$$

The induced morphism

$$b_\gamma\colon \Gamma(U;\ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\ \mathcal{B}_M)) \to$$

$$\to \Gamma_{\gamma^{0a}}(T_N^* M;\ H^d \mu_N(\mathrm{R}\ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\ \mathcal{B}_M) \otimes or_{N/X}))$$

is called «boundary value morphism» (cf. [S]). Using (1.1), one easily sees that b_γ is injective by analytic continuation.

Let γ_1, γ_2 be open convex proper cones of $T_N M$ and denote by $\langle \gamma_1, \gamma_2 \rangle$ their convex envelope. One deduces the following edge-of-the-wedge theorem.

COROLLARY 1.7. *Let* $U_i \in \mathcal{W}_{\gamma_i}$ *(for* $i = 1, 2$*), and let* $u_i \in \Gamma(U_i;\ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\ \mathcal{B}_M))$ *with* $b_{\gamma_1}(u_1) = b_{\gamma_2}(u_2)$. *Then there exist a wedge* $U \in \mathcal{W}_{\langle \gamma_1, \gamma_2 \rangle}$ *with* $U \supset U_1 \cup U_2$, *and a section* $u \in \Gamma(U;\ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\ \mathcal{B}_M))$ *such that* $u|_{U_1} = u_1$, $u|_{U_2} = u_2$.

PROOF. We will neglect orientation sheaves for simplicity. Notice that $\tilde{u} = b_{\gamma_1}(u_1) = b_{\gamma_2}(u_2)$ is a section of $H^d \mu_N(\mathrm{R}\ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\ \mathcal{B}_M))$ whose support is contained in $\gamma_1^{0a} \cap \gamma_2^{0a}$. If $\gamma_1^{0a} \cap \gamma_2^{0a} = \{0\}$, the result follows by (i) of Proposition 1.2. If $\gamma_1^{0a} \cap \gamma_2^{0a} \neq \{0\}$, one remarks that $\mathrm{Int}\,(\gamma_1^{0a} \cap \gamma_2^{0a})^{0a}$ is precisely the convex envelope of γ_1 and γ_2, and hence by (1.4) there exists a wedge $U' \in \mathcal{W}_{\langle \gamma_1, \gamma_2 \rangle}$ and a section $u \in \Gamma(U';\ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\ \mathcal{B}_M))$ with $b_{\langle \gamma_1, \gamma_2 \rangle}(u) = \tilde{u}$. Again by analytic continuation, one checks that u extends to an open set $U \in \mathcal{W}_{\langle \gamma_1, \gamma_2 \rangle}$ with $U \supset U_1 \cup U_2$. Q.E.D.

Notice that in the case where one replaces N by M, M by $X^{\mathbf{R}}$, X by $X \times \overline{X}$, and $\mathcal{M}$ by $\bar{\partial}$, the boundary value morphism considered above is the classical:

$$b_\gamma\colon \Gamma(U;\ \mathcal{O}_X) \to \Gamma_{\gamma^{0a}}(T_M^* X;\ \mathcal{C}_M)$$

2. Proof of Theorem 1.4.

Set $F = \mathrm{R}\ \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\ \mathcal{O}_X)$, the complex of holomorphic solutions to $\mathcal{M}$, and consider the natural projections

$$T_N^* Y \xrightarrow{{}^t g_N'} T_N^* X \xrightarrow{{}^t f_N'} T_N^* M.$$

We shall reduce the proof of Theorem 1.4 to the two following isomorphisms:

(2.1) if $\mathcal{M}$ is an elliptic left coherent $\mathcal{D}_X$-module, one has:

$$\mu_N(\mathrm{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)) \simeq \mathrm{R}\,{}^t f'_{N*}\mu_N(F) \otimes or_{M/X}[n],$$

(2.2) if $\mathcal{M}$ is a left coherent $\mathcal{D}_X$-module non-characteristic for Y, one has:

$$\mathrm{R}\,{}^t g'_{N*}\mu_N(F) \otimes or_{N/X}[n] \simeq \mathrm{R}\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N) \otimes or_{Y/X}[-d].$$

In fact, since the restriction of ${}^t g'_N$ to $\mathrm{char}(\mathcal{M}) \cap T^*_N X$ is finite, it follows from (2.2) that $H^j\mu_N(F) = 0$ for $j < n + d$. The conclusion of Theorem 1.4 then follows by formula (2.1).

Let us prove (2.1). By [K-S], Theorem 11.3.3 one has the equality

$$\mathrm{SS}(F) = \mathrm{char}(\mathcal{M}). \tag{2.3}$$

According to (2.3), $\mathcal{M}$ is elliptic if and only if M is non-characteristic for F. One then has the following chain of isomorphisms:

$$\mu_N(\mathrm{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)) \simeq \mu_N(f^! F) \otimes or_{M/X}[n] \simeq \mathrm{R}\,{}^t f'_{N*}\mu_N(F) \otimes or_{M/X}[n],$$

where the second isomorphism follows from Theorem 1.3. This proves (2.1).

Let us prove (2.2). According to (2.3), Y is non-characteristic for $\mathcal{M}$ if and only if Y is non-characteristic for F. One then has the following chain of isomorphisms:

$$\begin{aligned}\mathrm{R}\,{}^t g'_{N*}\mu_N(F) \otimes or_{N/X}[n] &\simeq \mu_N(g^! F) \otimes or_{N/X}[n] \simeq \\ &\simeq \mu_N(F|_Y) \otimes or_{N/Y}[n-2d] \simeq \\ &\simeq \mu_N(\mathrm{R}\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)) \otimes or_{N/Y}[n-2d] \simeq \\ &\simeq \mathrm{R}\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N) \otimes or_{Y/X}[-d],\end{aligned}$$

where the second isomorphism follows from Theorem 1.3, and the third from the Cauchy-Kowalevski-Kashiwara theorem which asserts that, $\mathcal{M}$ being non-characteristic for Y, $\mathrm{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y \simeq \simeq \mathrm{R}\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)$.

3. Remarks for non-elliptic systems.

As already pointed out, for $\mathrm{cod}_M N = 1$ Theorem 1.4 reduces to the Holmgren theorem, and to prove the latter the ellipticity assumption is not necessary.

For $\operatorname{cod}_M N > 1$, one may then wonder whether Theorem 1.4 holds or not if the ellipticity hypothesis is dropped out.

3.1. In the frame of tempered distribution, Liess [L] gives an example of a differential system with constant coefficients for which the corresponding Corollary 1.7 does not hold.

3.2. In order to deal with the real analytic case (i.e. $* = \mathcal{A}_M$ in (1.2)), consider $M \simeq \boldsymbol{R}^3$ with coordinates (t, x_1, x_2), let N be defined by $x_1 = x_2 = 0$, and set $X = \boldsymbol{C}^3$, $Y = \boldsymbol{C} \times \{0\}$. Let $\mathcal{M}$ be the (non-elliptic) module associated to the system

$$D_{x_1} + i\, x_1 D_t\,, \qquad D_{x_2} + i\, x_2 D_t\,,$$

which is non-characteristic for Y.

In this case, one has $H^1 \mu_N \mathrm{R}\, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M) \neq 0$ as implied by the following:

PROPOSITION 3.1. *One has*

$$H^1 \mathrm{R}\Gamma_N \mathrm{R}\, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M) \neq 0\,.$$

PROOF. By a change of holomorphic coordinates, $\mathcal{M}$ is associated to a system of constant coefficient differential equations on X, and hence $H^j \mathrm{R}\, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M) = 0$ for $j \neq 0$. It is then enough to find a solution $f \in \mathcal{A}_M(M \backslash N)$ for $\mathcal{M}$ which does not extend analytically to M. This is the case for

$$f = \frac{1}{2t + i(x_1^2 + x_2^2)}\,. \tag{3.1}$$

Q.E.D.

Of course, the function f in (3.1) extends to M as a hyperfunction, since its domain of holomorphy in X contains a wedge with edge N.

3.3. We don't know whether Theorem 1.4 holds or not in the frame of hyperfunctions without the ellipticity assumption. However, note that $H^j \mathrm{R}\Gamma_N \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) = 0$ for $j < \operatorname{cod}_M N$, as implied by the following division theorem (cf. [S-K-K]) of which we give here a sheaf theoretical proof.

LEMMA 3.2. *Assume that Y is non-characteristic for $\mathcal{M}$. Then there is an isomorphism*:

$$\mathrm{R}\, \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_N \mathcal{B}_M) \simeq \mathrm{R}\, \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N) \otimes or_{N/M}[-d]\,.$$

PROOF. We will neglect orientation sheaves for simplicity. Setting $F = \mathrm{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$, one has the isomorphisms:

$$\mathrm{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_N \mathcal{B}_M) \simeq \mathrm{R}\Gamma_N(F)[n] \simeq i^! g^! F[n] \simeq \mathrm{R}\Gamma_N(F|_Y)[n-2d].$$

By the Cauchy-Kowalevski-Kashiwara theorem, $F|_Y \simeq \mathrm{R}\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)$, and one concludes.

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