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The Formation Generated by a Finite Group.

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ABSTRACT - It is proved that the formation generated by a finite group which is an extension of a soluble group by a non-abelian simple group contains only finitely many subformations. This extends the work of Bryant, Bryce and Hartley.

1. Introduction.

In [1] Bryant, Bryce and Hartley showed that the formation generated by a finite soluble group contains only finitely many subformations. In this paper we show that the same result is true for a finite group which is an extension of a soluble group by a non-abelian simple group. We refer the reader to [1] and [2] for notation and definitions relating to formations. If Σ is a class of finite groups we write Form (Σ) for the formation generated by Σ and note that Form $(\Sigma) = \mathrm{QR}_0(\Sigma)$. If G is a finite group then every group in Form (G) is isomorphic to a quotient of a subdirect subgroup of the direct power G^n for some positive integer n.

THEOREM 1.1. *Let G be a finite group. Suppose G is an extension of a soluble group by a non-abelian simple group T. Then* Form (G) *contains only finitely many subformations.*

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A finite group A is called *formation critical* if the formation generated by those proper factors of A which lie in Form (A) does not contain A. Every formation is generated by those formation critical groups contained within it (see [1]). As in [1], 1.1 will be proved by showing that the formation generated by a finite soluble-by-simple group contains only finitely many formation critical groups.

THEOREM 1.1.1. *Let G be a finite group. Suppose G is an extension of a soluble group by a non-abelian simple group T. Then* Form (G) *contains only finitely many formation critical groups.*

As previously remarked, 1.1.1 implies 1.1. It will be shown that the following structure theorem for formation critical groups implies 1.1.1.

THEOREM 1.2. *Let G be a finite group which is an extension of a soluble group by a non-abelian simple group T. Let A be a formation critical group in* Form (G), *and let $R(A)$ denote the soluble radical of A. Then either A is soluble or $A/R(A)$ is isomorphic to either T or $T \times T$.*

It is perhaps worth remarking that there exists a group G which is formation critical and is such that $G/R(G)$ is isomorphic to the direct product of two copies of a non-abelian simple group. For example take G to be the central product of two copies of the special linear group SL (2, 5). See the last part of [1] for details.

Section 2 will contain some necessary preliminaries and section 3 the proofs of 1.1.1 and 1.2. It would be of considerable interest to know whether Theorem 1.1 holds for an arbitrary finite group G. But even very special cases seem to be difficult. We have unsuccessfully tried to extend 1.1 to the case where G is a finite extension of a soluble group by the direct product of two or more copies of T or to the case where G is a finite group possessing precisely one non-abelian composition factor.

2. Preliminaries.

Consider the direct power G^n where G is a finite group and n is a positive integer. Let G_i denote the subgroup of G^n consisting of all elements with the identity in the j-th position for all $j \neq i$, and let $\pi_i : G^n \to G_i$ be the i-th projection homomorphism. Identify G_i with G $(1 \leq i \leq n)$. Let D be a subgroup of G^n. Then we say that D is a diagonal subgroup if $\pi_i(D) = G$ and $\ker(\pi_i|_D) = 1$ $(1 \leq i \leq$ $\leq n)$. Note that, when this holds, $D \cong G$.

Suppose D is a diagonal subgroup of G^n. Then each $\pi_i|_D$ is an isomorphism from D to G. Hence there are automorphisms $\phi_1, \phi_2, \ldots, \phi_n$ of G such that

$$D = \{(\phi_1(g), \phi_2(g), \ldots, \phi_n(g)) \mid g \in G\}.$$

Conversely, if $\phi_1, \ldots, \phi_n$ are automorphisms of G and D is the subgroup of G^n given by the preceding equation then D is a diagonal subgroup.

Let $I = \{1, 2, \ldots, n\}$. For each subset J of I we write G^J for the subgroup of G^n consisting of those elements w of G^n such that $\pi_i(w) = 1$ for all $i \notin J$. Thus $G^J \cong G^m$ where $m = |J|$.

We shall consider carefully the case where G is a non-abelian simple group.

LEMMA 2.1. *Let G be a finite non-abelian simple group, n a positive integer and $I = \{1, 2, \ldots, n\}$. Let H be a subdirect subgroup of G^n. Then there exist pairwise disjoint non-empty subsets $I_1, \ldots, I_r$ of I with $I = I_1 \cup \ldots \cup I_r$ such that $H = H_1 \times \ldots \times H_r$ and H_j is a diagonal subgroup of G^{I_j} for $j = 1, \ldots, r$. Furthermore there exists a subgroup H_0 of H such that H_0 is a diagonal subgroup of G^n.*

PROOF. The proof of the first part is well known and may be done, for example, by induction on $|I|$. Since the statement of the second part is vital for this work we shall give a proof.

After a suitable renumbering we may assume without loss of generality that

$$I_1 = \{1, \ldots, n_1\}, I_2 = \{n_1 + 1, \ldots, n_2\}, \ldots, I_r = \{n_{r-1} + 1, \ldots, n\},$$

with $n_1 < n_2 < \ldots < n_{r-1} < n$. Then there exist automorphisms ϕ_i of G $(1 \leq i \leq n)$ such that

$$H_1 = \{(\phi_1(g), \ldots, \phi_{n_1}(g), 1, \ldots, 1) \mid g \in G\},$$

$$H_2 = \{(1, \ldots, 1, \phi_{n_1+1}(g), \ldots, \phi_{n_2}(g), 1, \ldots, 1) \mid g \in G\},$$

and so on. Let H_0 be the subgroup of $H = H_1 \times \ldots \times H_r$ defined by

$$H_0 = \{(\phi_1(g), \phi_2(g), \ldots, \phi_n(g)) \mid g \in G\}.$$

Then it is easy to see that H_0 is a diagonal subgroup of G^n. This completes the proof of the second part of the lemma.

LEMMA 2.2. *Let G be a finite non-abelian simple group. Let n be a positive integer and let $G_1, \ldots, G_n$ be isomorphic copies of G. Suppose N is a normal subgroup of $G_1 \times \ldots \times G_n$. Then there exists a subset $\{i_1, \ldots, i_c\}$ of $\{1, \ldots, n\}$ such that $N = G_{i_1} \times \ldots \times G_{i_c}$.*

PROOF. This is well known.

The following result is the key to the development.

LEMMA 2.3. *Let A be a finite group which is generated by the subgroup H together with normal subgroups $N_1, N_2, \ldots, N_n$. Suppose $[N_{\pi(1)}, \ldots, N_{\pi(n)}] = 1$ holds for every permutation π of $I = \{1, 2, \ldots, n\}$. For each subset J of I let $A_J = \left(\prod_{i \in J} N_i\right) H$ (taking $A_\emptyset = H$). Then*

$$A_J \in \mathrm{QR}_0 \{A_\Gamma \mid \Gamma \subseteq I, \Gamma \neq J\}.$$

PROOF. See [3], Theorem α.19, page 843.

We shall also need some notions from the theory of varieties. The necessary elementary notation and results can be found in chapter 1 of [4]. In particular we shall need the following facts.

(i) The free groups of finite rank of the variety generated by a finite group are finite ([4], 15.71).

(ii) If m is a positive integer, any m-generator group in a variety is isomorphic to a quotient of the free group of rank m of the variety ([4], 14.23).

Thus the order of any m-generator group in the variety $\mathrm{Var}(G)$ generated by a finite group G is bounded by the (finite) order of the free group of rank m of $\mathrm{Var}(G)$.

3. The main result.

We shall now embark upon the proof of 1.2. The proof is long and proceeds via several lemmas. Let G denote a finite group which is an extension of a soluble group by a non-abelian simple group T. Let A be a formation critical group in $\mathrm{Form}(G)$. Throughout the proof, for any finite group H, $R(H)$ denotes the soluble radical of H.

Since $A \in \mathrm{Form}(G)$, $A \cong S/L$ where S is a subdirect subgroup of G^n for some positive integer n and L is a normal subgroup of S. Without loss of generality we assume that $A = S/L$. Furthermore, let G_i and π_i $(1 \leqslant i \leqslant n)$ be as defined in section 2.

Let Var $(T) = \text{QSC}\,(T)$, the variety generated by T. Let V be any set of words defining Var (T); for example, the set of all laws of T. Then, for any group H, $H \in \text{Var}\,(T)$ if and only if $V(H) = 1$. (Here $V(H)$ denotes the verbal subgroup of H corresponding to V.)

LEMMA 3.1. *$G/V(G) = \hat{T} \times (R(G)/V(G))$ where $\hat{T} \cong T$.*

PROOF. Let $\hat{G} = G/V(G)$. Since $G/R(G) \cong T$ we have $V(G) \leqslant R(G)$. Hence $R(\hat{G}) = R(G)/V(G)$ and $\hat{G}/R(\hat{G}) \cong T$. Since $\hat{G} \in \text{Var}\,(T)$ it follows by 53.56 of [4] that $R(\hat{G})$ does not contain the socle of $\hat{G}$. Hence there is a minimal normal subgroup $\hat{T}$ of $\hat{G}$ such that $\hat{T} \nleqq R(\hat{G})$. Since $\hat{G}/R(\hat{G}) \cong T$ it follows that $\hat{G} = \hat{T} \times R(\hat{G})$ and $\hat{T} \cong T$. This proves the result.

Let $\hat{G} = G/V(G)$. Then, by 3.1, $\hat{G} = \hat{T} \times \hat{W}$. Here $\hat{T}$ is a subgroup of $\hat{G}$ isomorphic to T and $\hat{W}$ is a soluble normal subgroup of $\hat{G}$.

Define a homomorphism ϕ from S to $(G/V(G))^n$ by

$$\phi(s) = (\pi_1(s)V(G), \ldots, \pi_n(s)V(G))$$

for all $s \in S$. Then $\ker(\phi) = S \cap (V(G))^n = M$ say. Since S is a subdirect subgroup of G^n it is easy to see that $\phi(S) = \hat{S}$ is a subdirect subgroup of $(\hat{G})^n$. The next lemma gives the structure of $\hat{S}$.

LEMMA 3.2. *$\hat{S} = \hat{X} \times \hat{Y}$, where $\hat{X}$ is a subdirect subgroup of $(\hat{T})^n$ and $\hat{Y}$ is a subdirect subgroup of $(\hat{W})^n$.*

PROOF. We have $\hat{S} \leqslant (\hat{G})^n = (\hat{T} \times \hat{W})^n \cong (\hat{T})^n \times (\hat{W})^n$ and there are natural projections $\lambda_1 : (\hat{G})^n \to (\hat{T})^n$ and $\lambda_2 : (\hat{G})^n \to (\hat{W})^n$. Let $\mu_1 : \hat{S} \to (\hat{T})^n$ and $\mu_2 : \hat{S} \to (\hat{W})^n$ be the restrictions to $\hat{S}$ of λ_1 and λ_2, respectively. Then im (μ_1) (the image of μ_1) is a subdirect subgroup of $(\hat{T})^n$ and im (μ_2) is a subdirect subgroup of $(\hat{W})^n$.

Set $\hat{X} = \ker(\mu_2) = \hat{S} \cap (\hat{T})^n$ and $\hat{Y} = \ker(\mu_1) = \hat{S} \cap (\hat{W})^n$. Clearly $\hat{X}\hat{Y} = \hat{X} \times \hat{Y}$. We will show that $\hat{S} = \hat{X}\hat{Y}$. Now $\hat{S}/\hat{X}\hat{Y} \cong (\hat{S}/\hat{X})/(\hat{X}\hat{Y}/\hat{X})$, and $\hat{S}/\hat{X} \cong$ im (μ_2), which is soluble. Thus $\hat{S}/\hat{X}\hat{Y}$ is soluble. But also $\hat{S}/\hat{X}\hat{Y} \cong (\hat{S}/\hat{Y})/(\hat{X}\hat{Y}/\hat{Y})$, and $\hat{S}/\hat{Y} \cong$ im (μ_1), which is a subdirect subgroup of $(\hat{T})^n$. By 2.1 and 2.2 it follows that $\hat{S}/\hat{X}\hat{Y}$ is either 1 or a direct product of copies of the simple group T. This situation can only occur if $\hat{S} = \hat{X}\hat{Y} = \hat{X} \times \hat{Y}$.

From the subdirect nature of $\hat{S}$ it follows that $\hat{X}$ and $\hat{Y}$ are indeed subdirect on their respective factors. This completes the proof of the lemma.

Let $\widehat{T}_i (1 \leqslant i \leqslant n)$ denote the subgroup of $(\widehat{T})^n$ consisting of those elements of $(\widehat{T})^n$ with the identity in the j-th position for all $j \neq i$. Let $\widehat{X}_0$ be a diagonal subgroup of $(\widehat{T})^n$ such that $\widehat{X}_0 \leqslant \widehat{X}$, as given by 2.1. Hence $\widehat{X}_0 \cong T$ and $\widehat{X}_0$ projects onto each $\widehat{T}_i$ $(1 \leqslant i \leqslant n)$. Furthermore $\widehat{X}_0 \times \widehat{Y} \leqslant \leqslant \widehat{X} \times \widehat{Y}$.

Now $\phi(S) = \widehat{S} = \widehat{X} \times \widehat{Y}$ and $\ker(\phi) = M$. Let X and Y be the normal subgroups of S containing M such that $\phi(X) = \widehat{X}$ and $\phi(Y) = \widehat{Y}$. Also, let X_0 be the subgroup of X containing M such that $\phi(X_0) = \widehat{X}_0$. Thus $X/M \cong \widehat{X}$, $Y/M \cong \widehat{Y}$ and $X_0/M \cong \widehat{X}_0 \cong T$. Since $\widehat{Y}$ and M are soluble it follows that Y is soluble. Set $S_0 = X_0 Y$. Thus $\phi(S_0) = \widehat{X}_0 \times \widehat{Y}$, which is a subdirect subgroup of $(\widehat{G})^n$. Also, $S_0/M \cong X_0/M \times Y/M$. Thus $S_0/Y \cong \cong T$.

We need a lemma which gives an important property of S_0.

LEMMA 3.3. *S_0 is a subdirect subgroup of G^n.*

PROOF. Let $i \in \{1, \ldots, n\}$ and let g be an arbitrary element of G_i. We need $s_0 \in S_0$ such that $\pi_i(s_0) = g$. Let $\widehat{\pi}_i$ denote the natural projection homomorphism of $(G/V(G))^n$ onto the i-th factor.

Since $\phi(S_0)$ is subdirect there exists $p_0 \in S_0$ such that

$$\widehat{\pi}_i(\phi(p_0)) = \pi_i(p_0)V(G_i) = gV(G_i).$$

Thus $g = \pi_i(p_0)v$ for some $v \in V(G_i)$. Now $V(S)$ is a subdirect subgroup of $(V(G))^n$ and $V(S) \leqslant S \cap (V(G))^n \leqslant S_0$. Thus $V(S) \leqslant S_0 \cap (V(G))^n$ and so $S_0 \cap (V(G))^n$ is a subdirect subgroup of $(V(G))^n$. Hence there exists $r_0 \in S_0 \cap (V(G))^n$ such that $\pi_i(r_0) = v$. Set $s_0 = p_0 r_0$. Then $\pi_i(s_0) = g$ as desired.

We now consider the group $A = S/L$ and its subgroup $A_0 = = S_0 L/L$.

LEMMA 3.4. $|A_0 R(A)/R(A)| \leqslant |T|$.

PROOF. Since $Y \leqslant S_0$ and Y is soluble, $YL/L \leqslant A_0 \cap R(A)$. Thus

$$|A_0 R(A)/R(A)| = |A_0/A_0 \cap R(A)| \leqslant |A_0/(YL/L)|.$$

But

$$|A_0/(YL/L)| = |S_0L/YL| \leqslant |S_0/Y| = |T|.$$

The result follows.

LEMMA 3.5. *Let H be a subgroup of A containing A_0. Then $H \in \text{Form}(A)$.*

PROOF. Set $M_i = S \cap \ker(\pi_i)$ $(1 \leqslant i \leqslant n)$. Then each M_iL/L is a normal subgroup of A.

Since S_0 is a subdirect subgroup of G^n it follows easily that $S = S_0M_i$. Therefore $H(M_iL/L) = A$ $(1 \leqslant i \leqslant n)$. If Γ is a non-empty subset of $\{1, \dots, n\}$ we set $A_\Gamma = H\left(\prod_{i \in \Gamma} M_iL/L\right) = A$, and we set $A_\emptyset = H$.

Since $\bigcap_{i=1}^{n} M_i = 1$ it follows that $[M_{\pi(1)}, \dots, M_{\pi(n)}] = 1$ for all permutations π of $\{1, \dots, n\}$. Thus $[M_{\pi(1)}L/L, \dots, M_{\pi(n)}L/L] = 1$ for all π, and the hypotheses of 2.3 are satisfied. Therefore

$$A_\emptyset \in \text{Form}\{A_\Gamma \mid \emptyset \neq \Gamma \subseteq \{1, \dots, n\}\}.$$

Hence $H \in \text{Form}(A)$, as required.

LEMMA 3.6. *$A/R(A)$ is isomorphic to a direct product of copies of T.*

PROOF. Let $R(A) = R(S/L) = K/L$, where $L \leqslant K \leqslant S$. Then $R(S)L \leqslant K$ as $R(S)L/L$ is a soluble normal subgroup of S/L. We have

$$A/R(A) = (S/L)/(K/L) \cong S/K \cong (S/R(S)L)/(K/R(S)L).$$

The lemma will then follow from 2.2 if it can be shown that $S/R(S)L$ is isomorphic to a direct product of copies of T.

Now $R(G)^n$ is a soluble normal subgroup of G^n. Hence $S \cap R(G)^n$ is a soluble normal subgroup of S. Therefore $S \cap R(G)^n \leqslant R(S)$. Hence $S/R(S)L$ is isomorphic to a quotient of $S/S \cap R(G)^n$. But it is easy to see that $S/S \cap R(G)^n$ is isomorphic to a subdirect subgroup of $(G/R(G))^n$. By 2.1, $S/S \cap R(G)^n$ is thus isomorphic to a direct product of copies of T. Hence so is $S/R(S)L$.

PROOF OF 1.2. Let d denote the derived length of $R(G)$. For $i = 1, \dots, n$, let $N_i = R(G_i) \times \prod_{j \neq i} G_j$, $S_i = S \cap N_i$ and $\overline{S}_i = S_iL/L$.

Then $S/S_i \cong SN_i/N_i$. Since S is a subdirect subgroup of G^n it is easy to see that SN_i equals G^n. Therefore SN_i/N_i is isomorphic to G^n/N_i which is isomorphic to T. Hence $S/S_i \cong T$ for all i.

Case 1. $\bar{S}_i$ is a subgroup of $R(A)$ for some i $(1 \leqslant i \leqslant n)$.

Then $A/R(A)$ is isomorphic to a quotient of $A/\bar{S}_i$. But $A/\bar{S}_i$ is isomorphic to a quotient of S/S_i and S/S_i is isomorphic to T. Hence in this case, either $A/R(A) = 1$ and A is soluble, or $A/R(A) \cong T$.

Case 2. For all i, $\bar{S}_i \nleqq R(A)$.

Then $\bar{S}_i^{(d)}$ (the d-th term of the derived series of $\bar{S}_i$) is not a subgroup of $R(A)$, for otherwise $\bar{S}_i$ would be a soluble normal subgroup of A and, by assumption, this is not so.

Since $S_i^{(d)} \leqslant N_i^{(d)} \leqslant \prod_{j \neq i} G_j$ we have $[S_{\pi(1)}^{(d)}, \dots, S_{\pi(n)}^{(d)}] = 1$ for all permutations π. Write $A_i = \bar{S}_i^{(d)}$. Then $[A_{\pi(1)}, \dots, A_{\pi(n)}] = 1$ for all π.

Now, as in case 1, $A/\bar{S}_i$ is isomorphic to a quotient of T. Now a comparison of the composition series of A/A_i which pass through $\bar{S}_i/A_i$ and $A_iR(A)/A_i$ respectively shows, by the Jordan-Hölder theorem, that $A/A_iR(A)$ possesses at most one composition factor isomorphic to T. By 3.6, $A/A_iR(A)$ is isomorphic to a quotient of a direct product of copies of T. Hence, by 2.2, $A/A_iR(A)$ is isomorphic to 1 or T $(1 \leqslant i \leqslant n)$.

From 3.6 it follows that there exists a positive integer e and normal subgroups T_i of A containing $R(A)$ such that $T_i/R(A)$ is isomorphic to T $(1 \leqslant i \leqslant e)$ and $A/R(A) = T_1/R(A) \times \dots \times T_e/R(A)$.

Now $A_iR(A)/R(A)$ is a normal subgroup of $A/R(A)$ with quotient isomorphic to 1 or T. Hence $A_iR(A)/R(A)$ contains at least $e-1$ of the factors $T_1/R(A), \dots, T_e/R(A)$.

Suppose (for a contradiction) that $e \geqslant 3$.

Let $U = (S_0L/L)R(A) = A_0R(A)$. Then, by 3.4, $|U/R(A)| \leqslant |T|$. By 3.5 every subgroup of A containing U belongs to Form(A). Clearly

$$A/R(A) = (U/R(A))(T_1/R(A) \times \dots \times T_e/R(A)).$$

Choose $i_1, \dots, i_c \in \{1, \dots, e\}$ with c minimal subject to

$$A/R(A) = (U/R(A))(T_{i_1}/R(A) \times \dots \times T_{i_c}/R(A)).$$

Then, by a suitable renumbering of $T_1, \dots, T_e$, we may take

$$A/R(A) = (U/R(A))(T_1/R(A) \times \dots \times T_c/R(A)).$$

Since $|A/R(A)| = |T|^e$ and $|U/R(A)| \leqslant |T|$ we must have $c \geqslant e - 1 \geqslant \geqslant 2$. Let B be the subgroup of A containing $R(A)$ such that

$$B/R(A) = (U/R(A))(T_3/R(A))\dots(T_c/R(A)).$$

Therefore

$$A/R(A) = (B/R(A))(T_1/R(A))(T_2/R(A)).$$

By the minimality of c, $(B/R(A))(T_1/R(A))$ and $(B/R(A))(T_2/R(A))$ are proper subgroups of $A/R(A)$.

Since every $A_iR(A)/R(A)$ contains at least $e-1$ of the $T_j/R(A)$, every $A_iR(A)/R(A)$ contains either $T_1/R(A)$ or $T_2/R(A)$.

For $i = 1, \dots, n$ we define a normal subgroup X_i of A by $X_i = A_i \cap T_1$ if $A_iR(A)/R(A)$ contains $T_1/R(A)$ and $X_i = A_i \cap T_2$ otherwise. Thus, by Dedekind's rule, $X_iR(A)/R(A)$ is either $T_1/R(A)$ or $T_2/R(A)$. Since $X_i \leqslant A_i$ we have $[X_{\pi(1)}, \dots, X_{\pi(n)}] = 1$ for all permutations π. Hence

$$[X_{\pi(1)}R(A)/R(A), \dots, X_{\pi(n)}R(A)/R(A)] = 1,$$

for all π. Since $T_1/R(A)$ is non-nilpotent there exists at least one value of i for which $X_iR(A)/R(A) = T_2/R(A)$. Similarly there exists at least one value of i for which $X_iR(A)/R(A) = T_1/R(A)$.

Suppose after renumbering that $X_1R(A)/R(A), \dots, X_pR(A)/R(A)$ equal $T_1/R(A)$ and that $X_{p+1}R(A)/R(A), \dots, X_nR(A)/R(A)$ equal $T_2/R(A)$ where p is a positive integer and $p < n$. By standard commutator identities,

$$[[X_1, \dots, X_p], [X_{p+1}, \dots, X_n]] \leqslant \prod [X_{\pi(1)}, \dots, X_{\pi(n)}],$$

where the product is over all permutations π of $\{1, \dots, n\}$. Thus

$$[[X_1, \dots, X_p], [X_{p+1}, \dots, X_n]] = 1.$$

Let $J_1 = [X_1, \dots, X_p]$ and $J_2 = [X_{p+1}, \dots, X_n]$. Then J_1 and J_2 are normal subgroups of A and $[J_1, J_2] = 1$. Since $T_1/R(A)$ and $T_2/R(A)$ are both perfect groups we have

$$J_1R(A)/R(A) = T_1/R(A)$$

and

$$J_2R(A)/R(A) = T_2/R(A).$$

Hence $A = BJ_1J_2$ where $[J_1, J_2] = 1$ and B, BJ_1 and BJ_2 are proper subgroups of A. These subgroups contain U and so they all belong to Form (A). The hypotheses of 2.3 are satisfied (take $n = 2$, $H = B$ and $N_i = J_i$ $(i = 1, 2)$ in the statement of 2.3). Thus $BJ_1J_2 \in \in \mathrm{Form}\{B, BJ_1, BJ_2\}$. This contradicts the fact that A is formation critical.

Hence $e \leqslant 2$ and $A/R(A)$ is isomorphic to T or $T \times T$. This completes the proof of 1.2.

PROOF OF 1.1.1. We shall show that the order of any formation critical group A in Form (G) is less than some constant which depends only upon the group G. This is what we shall mean when speaking of bounding $|A|$.

Let F be the Fitting subgroup of A, Φ the Frattini subgroup of A, and R the soluble radical of A. The arguments used in [1], section 1, show that $|F/\Phi|$ is bounded: these arguments do not require A to be soluble.

Let $C = C_A(F/\Phi) = \{a \in A : [a, f] \in \Phi \text{ for all } f \in F\}$. We claim that $F = C \cap R$. Now F/Φ is abelian. Therefore $F \leqslant C$. Since $F \leqslant R$ we have $F \leqslant C \cap R$.

For the reverse inclusion note that $F(R)$, the Fitting subgroup of R, is a normal nilpotent subgroup of A. Hence $F(R) \leqslant F$. Certainly $F \leqslant \leqslant F(R)$ as $F \leqslant R$. Thus $F(R) = F$. Similarly, $F(R/\Phi) = F(A/\Phi)$. But $F(A/\Phi) = F/\Phi$. Thus $F/\Phi = F(R/\Phi)$. Also, $C \cap R = C_R(F/\Phi)$. Thus

$$(C \cap R)/\Phi = C_R(F/\Phi)/\Phi = C_{R/\Phi}(F/\Phi) = C_{R/\Phi}(F(R/\Phi)).$$

Since R/Φ is a soluble group, 7.67 of [5] yields that $C_{R/\Phi}(F(R/\Phi)) \leqslant \leqslant F(R/\Phi)$. Hence $(C \cap R)/\Phi \leqslant F(R/\Phi) = F/\Phi$, and so $C \cap R \leqslant F$. Therefore $F = C \cap R$ as required.

Each element a of A induces the automorphism $\alpha_a : f\Phi \mapsto (afa^{-1})\Phi$ of F/Φ. This gives rise to a homomorphism $a \mapsto \alpha_a$ of A into Aut (F/Φ). But C is the kernel of this homomorphism and so A/C can be embedded in Aut (F/Φ). Since $|F/\Phi|$ is bounded it follows that $|A/C|$ is bounded. Also

$$|A/C| = |A/CR||CR/C| = |A/CR||R/F|,$$

since $F = C \cap R$. Hence $|R/F|$ is bounded.

Also, by 1.2, $|A/R|$ is bounded. But

$$|A/\Phi| = |A/R||R/F||F/\Phi| .$$

Thus $|A/\Phi|$ is bounded. Therefore the number of generators of A is bounded. It follows from the remarks at the end of section 2 that $|A|$ is bounded.

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