Rendiconti del Seminario Matematico della Università di Padova

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Asymptotics for Meixner Polynomials.

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SUMMARY - Asymptotic expansion for Meixner polynomials $m_n(x/(1-c); \alpha+1, c)$ in terms of Laguerre polynomials $L_n^{(\alpha)}(x)$, $c \to 1$, is derived. Asymptotic approximation and limit relations of the zeros of these polynomials are also given.

1. Introduction and results.

The Meixner polynomials

$$m_n(x; \beta, c) = (\beta)_n \, {}_2F_1\left(-n, \; -x; \beta; 1 - \frac{1}{c}\right),$$

$$c \neq 0, \qquad \beta \neq 0, \; -1, \; -2, \; \ldots,$$

satisfying, for $0 < |c| < 1$, the orthogonality relation

$$\sum_{k=0}^{\infty} m_n(k; \beta, c) \, m_p(k; \beta, c) \, \frac{c^k (\beta)_k}{k!} = (1-c)^{-\beta} c^{-n} n! (\beta)_n \, \delta_{np},$$

are a discrete analogue of the Laguerre polynomials $L_n^{(\beta-1)}(x)$.

These polynomials are of importance for the description of certain Markov processes and in the numerical evaluation of the Hankel transform [3]. Their main properties can be found in Chihara [1], including

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the limit relation

$$\lim_{c \to 1} m_n\left(\frac{x}{1-c}; \beta, c\right) = n!\, L_n^{(\beta-1)}(x), \tag{1.1}$$

and the recurrence formula

$$cm_{n+1}(x) = \tag{1.2}$$

$$= [(c-1)x + (1+c)n + c\beta]\, m_n(x) - n(n+\beta-1)\, m_{n-1}(x), \qquad n \geq 0.$$

Now, let $L_n^{(\alpha)}(x)$ be Laguerre polynomials and let $Q_n^{(\alpha)}(x; \lambda)$ be defined by

$$Q_n^{(\alpha)}(x; \lambda) = \frac{1}{n!}\left(\frac{\lambda+1}{\lambda-1}\right)^{n/2} m_n\left(-\frac{\lambda x}{2} - \frac{\alpha+1}{2}; \alpha+1, \frac{\lambda+1}{\lambda-1}\right). \tag{1.3}$$

We notice that, for the transformation formula

$${}_2F_1(a, b; c; z) = (1-z)^{-a}{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right),$$

the function $Q_n^{(\alpha)}(x; \lambda)$ is even with respect to λ.

We shall assume throughout this paper that the parameter α satisfies the condition $\alpha > -1$.

By putting

$$\frac{x}{1-c} = -\frac{\lambda x}{2} - \frac{\alpha+1}{2} = -\frac{\lambda-1}{2}\, x\{1 + O(\lambda^{-1})\}, \qquad \lambda \to \infty,$$

from the (1.1) and (1.3) it follows that

$$\lim_{\lambda \to \infty} Q_n^{(\alpha)}(x; \lambda) = L_n^{(\alpha)}(x). \tag{1.4}$$

If $l_{n,k}^{(\alpha)}$ and $x_{n,k}^{(\alpha,\lambda)}$, $k = 1, 2, \ldots, n$, are the zeros of Laguerre and Meixner polynomials (1.3) respectively, that is

$$L_n^{(\alpha)}(l_{n,k}^{(\alpha)}) = 0, \tag{1.5}$$

$$Q_n^{(\alpha)}(x_{n,k}^{(\alpha,\lambda)}; \lambda) = 0, \tag{1.6}$$

then from (1.4) it will be

$$\lim_{\lambda \to \infty} x_{n,k}^{(\alpha,\lambda)} = l_{n,k}^{(\alpha)}, \qquad k = 1, 2, \ldots, n. \tag{1.7}$$

In this note we show that the limits (1.4) and (1.7) can be replaced by accurate asymptotic estimates. More precisely, we can prove

THEOREM 1.1. The polynomial $Q_n^{(\alpha)}(x;\lambda)$ defined in (1.3) has the asymptotic representation

$$(1.8)\qquad Q_n^{(\alpha)}(x;\lambda) = L_n^{(\alpha)}(x) + \frac{1}{\lambda^2} Q_{n,1}^{(\alpha)}(x) + O\left(\frac{1}{\lambda^4}\right), \qquad \lambda \to \infty,$$

where $L_n^{(\alpha)}(x)$ is the Laguerre polynomial and

$$(1.9)\qquad Q_{n,1}^{(\alpha)}(x) = \frac{x}{3} L_n^{(\alpha)}(x) + $$

$$- \frac{x+\alpha+1}{2} L_n^{(\alpha+1)}(x) + \frac{x+\alpha+1}{2} L_n^{(\alpha+2)}(x) - \frac{x}{3} L_n^{(\alpha+3)}(x).$$

THEOREM 1.2. Let $x_{n,k}^{(\alpha,\lambda)}$ and $l_{n,k}^{(\alpha)}$ be the zeros of $Q_n^{(\alpha)}(x;\lambda)$ and $L_n^{(\alpha)}(x)$ respectively. Then

$$(1.10)\qquad x_{n,k}^{(\alpha,\lambda)} = l_{n,k}^{(\alpha)} - \frac{1}{6\lambda^2}\left[2 l_{n,k}^{(\alpha)} - (2n+\alpha+1) + \frac{1-\alpha^2}{l_{n,k}^{(\alpha)}}\right] + O\left(\frac{1}{\lambda^4}\right),$$

$$k = 1, 2, \ldots, n, \qquad \lambda \to \infty.$$

THEOREM 1.3. For the zeros $x_{n,k}^{(\alpha,\lambda)}$ the following limit relations hold

$$(1.11)\qquad \lim_{\lambda\to\infty} \lambda^3 \frac{\partial x_{n,k}^{(\alpha,\lambda)}}{\partial\lambda} = \frac{1}{3}\left[2 l_{n,k}^{(\alpha)} - (2n+\alpha+1) + \frac{1-\alpha^2}{l_{n,k}^{(\alpha)}}\right],$$

$$(1.12)\qquad \lim_{\lambda\to\infty} \lambda^4 \frac{\partial^2 x_{n,k}^{(\alpha,\lambda)}}{\partial\lambda^2} = -\left[2 l_{n,k}^{(\alpha)} - (2n+\alpha+1) + \frac{1-\alpha^2}{l_{n,k}^{(\alpha)}}\right].$$

The proofs of the above Theorems are given in the next Sections.

The outlined procedure shows to be efficient in deriving higher order terms in the asymptotic formulas. The required task is, unfortunately, highly manipulative.

2. Proof of Theorem 1.1.

The polynomials $Q_n^{(\alpha)}(x;\lambda)$ satisfy, for $\lambda > 1$, the recurrence formula

$$(2.1)\qquad (n+1)\,Q_{n+1}^{(\alpha)}(x;\lambda) = \\ = \frac{\lambda}{\sqrt{\lambda^2-1}}\,(2n+\alpha+1-x)\,Q_n^{(\alpha)}(x;\lambda) - (n+\alpha)\,Q_{n-1}^{(\alpha)}(x;\lambda),$$

with the initial values

$$(2.2)\qquad Q_{-1}^{(\alpha)}(x;\lambda) = 0, \qquad Q_0^{(\alpha)}(x;\lambda) = 1\,.$$

This is an immediate consequence of the Meixner polynomials recurrence formula (1.2). Assume

$$(2.3)\qquad Q_n^{(\alpha)}(x;\lambda) = Q_{n,0}^{(\alpha)}(x) + \frac{1}{\lambda^2}\,Q_{n,1}^{(\alpha)}(x) + \frac{1}{\lambda^4}\,Q_{n,2}^{(\alpha)}(x) + \dots,$$

and note

$$(2.4)\qquad \frac{\lambda}{\sqrt{\lambda^2-1}} = \sum_{k=0}^{\infty}\binom{k-1/2}{k}\frac{1}{\lambda^{2k}}, \qquad \lambda > 1\,.$$

Substituting (2.3) and (2.4) into (2.1) and equating the coefficients of the same powers of λ, we find

$$(2.5)\qquad (n+1)\,Q_{n+1,0}^{(\alpha)}(x) = (2n+\alpha+1-x)\,Q_{n,0}^{(\alpha)}(x) - (n+\alpha)\,Q_{n-1,0}^{(\alpha)}(x),$$

and

$$(2.6)\qquad (n+1)\,Q_{n+1j}^{(\alpha)}(x) - (2n+\alpha+1-x)\,Q_{nj}^{(\alpha)}(x) + (n+\alpha)\,Q_{n-1j}^{(\alpha)}(x) = \\ = \sum_{k=1}^{j}\binom{k-1/2}{k}(2n+\alpha+1-x)\,Q_{n,j-k}^{(\alpha)}(x),$$

where $j = 1, 2, \dots$.

Moreover we have

$$(2.7)\qquad Q_{-1,0}^{(\alpha)}(x) = 0, \qquad Q_{0,0}^{(\alpha)}(x) = 1\,,$$

and

$$(2.8)\qquad Q_{-1,j}^{(\alpha)}(x) = Q_{0,j}^{(\alpha)}(x) = 0, \qquad j = 1, 2, \dots.$$

From equations (2.5) and (2.7) one recognizes

$$Q_{n,0}^{(\alpha)}(x) = L_n^{(\alpha)}(x), \qquad n = 0, 1, \ldots .$$

The explicit expression of $Q_{n,j}^{(\alpha)}(x), j = 1, 2, \ldots$, can be derived by means of the method of generating function [4, p. 35].

In order to determine $Q_{n,1}^{(\alpha)}(x)$, we observe that, from (2.6) and (2.8) with $j = 1$, we have

$$\text{(2.9)} \quad \sum_{n=0}^{\infty} (n+1) Q_{n+1,1}^{(\alpha)}(x) z^n - \sum_{n=0}^{\infty} (2n + \alpha + 1 - x) Q_{n,1}^{(\alpha)}(x) z^n + \\ + \sum_{n=0}^{\infty} (n + \alpha) Q_{n-1,1}^{(\alpha)}(x) z^n = \frac{1}{2} \sum_{n=0}^{\infty} (2n + \alpha + 1 - x) Q_{n,0}^{(\alpha)}(x) z^n .$$

Now, setting

$$\text{(2.10)} \qquad G_1(z) = \sum_{n=0}^{\infty} Q_{n,1}^{(\alpha)}(x) z^n ,$$

and

$$\text{(2.11)} \qquad H_0(z) = \frac{1}{2} \sum_{n=0}^{\infty} (2n + \alpha + 1 - x) Q_{n,0}^{(\alpha)}(x) z^n ,$$

by straightforward calculations the (2.9) can be rewritten in the form

$$\text{(2.12)} \qquad (1-z)^2 G_1'(z) - [(\alpha+1)(1-z) - x] G_1(z) = H_0(z) ,$$

with

$$\text{(2.13)} \qquad G_1(0) = 0 ,$$

where the prime denotes the derivative respect to z.

Using [2, p. 189]

$$\text{(2.14)} \qquad \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) z^n = (1-z)^{-\alpha-1} \exp\left(\frac{xz}{z-1}\right),$$

and

$$\sum_{n=1}^{\infty} n L_n^{(\alpha)}(x) z^{n-1} = \frac{(\alpha+1)(1-z) - x}{(1-z)^{\alpha+3}} \exp\left(\frac{xz}{z-1}\right),$$

in equation (2.11), we easily obtain

$$H_0(z) = \frac{\alpha + 1 - x - z^2(\alpha + 1 + x)}{2(1 - z)^{\alpha + 3}} \exp\left(\frac{xz}{z - 1}\right).$$

Thus, from (2.12) and (2.13), it follows

$$G_1(z) = \exp\left(\frac{xz}{z - 1}\right) \cdot$$

$$\cdot \left[\frac{x}{3(1 - z)^{\alpha + 1}} - \frac{\alpha + 1 + x}{2(1 - z)^{\alpha + 2}} + \frac{\alpha + 1 + x}{2(1 - z)^{\alpha + 3}} - \frac{x}{3(1 - z)^{\alpha + 4}}\right].$$

Then, from (2.10), taking into account (2.14), we find that $Q_{n,1}^{(\alpha)}(x)$ satisfies (1.9). So the expansion (2.3) prove part of estimate (1.8).

To complete the proof, we observe merely that $G_2(z)$, defined by

$$G_2(z) = \sum_{n=0}^{\infty} Q_{n,2}^{(\alpha)}(x)\, z^n,$$

is analitic near the origin.

In fact, with a procedure perfectly analogous to the one previously used, we can observe that $G_2(z)$ satisfies the non-homogeneous equation

$$(1 - z)^2 G_2'(z) - [(\alpha + 1)(1 - z) - x]\, G_2(x) = H_1(z),$$

with $G_2(0) = 0$ and

$$H_1(z) = \frac{1}{2} \sum_{n=0}^{\infty} (2n + \alpha + 1 - x)\left[\frac{3}{4}\, Q_{n,0}^{(\alpha)}(x) + Q_{n,1}^{(\alpha)}(x)\right] z^n.$$

3. Proof of Theorem 1.2.

Suggested by (1.8), we assume that the zeros $x_{n,k}^{(\alpha,\lambda)}$, $k = 1, 2, \ldots, n$, of $Q_n^{(\alpha)}(x; \lambda)$ look as

$$x_{n,k}^{(\alpha,\lambda)} = l_{n,k}^{(\alpha)} + \frac{1}{\lambda^2}\, c_1 + \frac{1}{\lambda^4}\, c_2 + \ldots, \tag{3.1}$$

where $l_{n,k}^{(\alpha)}$ satisfy (1.5).

Putting (3.1) in expansion (1.8) and (1.9) we get

$$c_1 = \frac{Q_{n,1}^{(\alpha)}(l_{n,k}^{(\alpha)})}{D_n^{(\alpha)}}, \tag{3.2}$$

where

$$D_n^{(\alpha)} = -\frac{d}{dx} L_n^{(\alpha)}(x)\bigg|_{x = l_{n,k}^{(\alpha)}}.$$

The coefficient c_1 may be written in easy form by the repeated use in (1.9) of the following functional relations of Laguerre polynomials, [2, p. 190]

$$L_n^{(\alpha+1)}(x) = \frac{1}{x}\left[(x-n)L_n^{(\alpha)}(x) + (n+\alpha)L_{n-1}^{(\alpha)}(x)\right], \tag{3.3}$$

and [2, p. 189]

$$x\frac{d}{dx} L_n^{(\alpha)}(x) = nL_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x). \tag{3.4}$$

So, we obtain

$$Q_{n,1}^{(\alpha)}(l_{n,k}^{(\alpha)}) = \tag{3.5}$$

$$= -\frac{n+\alpha}{6(l_{n,k}^{(\alpha)})^2} L_{n-1}^{(\alpha)}(l_{n,k}^{(\alpha)})\left[2(l_{n,k}^{(\alpha)})^2 - (2n+\alpha+1)l_{n,k}^{(\alpha)} + 1 - \alpha^2\right].$$

Moreover, since (3.4) and (1.5), we have

$$D_n^{(\alpha)} = \frac{n+\alpha}{l_{n,k}^{(\alpha)}} L_{n-1}^{(\alpha)}(l_{n,k}^{(\alpha)}). \tag{3.6}$$

Finally, by means of (3.5), (3.6) and (3.2), Theorem 1.2 is immediately established.

4. Proof of Theorem 1.3.

To prove the first limit relation of the Theorem 1.3, let us set

$$R_n^{(\alpha)}(x) = \lim_{\lambda \to \infty} \frac{\partial Q_n^{(\alpha)}(x;\lambda)}{\partial x}, \tag{4.1}$$

and

$$S_n^{(a)}(x) = \lim_{\lambda \to \infty} \lambda^3 \frac{\partial Q_n^{(a)}(x; \lambda)}{\partial \lambda}. \tag{4.2}$$

Thus it will be

$$\lim_{\lambda \to \infty} \lambda^3 \frac{\partial x_{n,k}^{(a,\lambda)}}{\partial \lambda} = -\frac{S_n^{(a)}(l_{n,k}^{(a)})}{R_n^{(a)}(l_{n,k}^{(a)})}. \tag{4.3}$$

To estimate this ratio we establish recurrence formulas for $R_n^{(a)}(x)$ and $S_n^{(a)}(x)$. By differentiating (2.1) and (2.2) with respect to x, we see that $R_n^{(a)}(x)$ satisfies

$$(n+1)R_{n+1}^{(a)}(x) - \\ -(2n+a+1-x)R_n^{(a)}(x) + (n+a)R_{n-1}^{(a)}(x) = -L_n^{(a)}(x),$$

with

$$R_{-1}^{(a)}(x) = R_0^{(a)}(x) = 0.$$

In similar manner, by differentiating (2.1) and (2.2) with respect to λ, we find

$$(n+1)S_{n+1}^{(a)}(x) - (2n+a+1-x)S_n^{(a)}(x) + (n+a)S_{n-1}^{(a)}(x) \\ = -(2n+a+1-x)L_n^{(a)}(x),$$

with

$$S_{-1}^{(a)}(x) = S_0^{(a)}(x) = 0.$$

By using the method outlined in Section 2 for the (1.9), we can derive that

$$R_n^{(a)}(x) = L_n^{(a)}(x) - L_n^{(a+1)}(x) \tag{4.4}$$

and

$$S_n^{(a)}(x) = -2Q_{n,1}^{(a)}(x), \tag{4.5}$$

where $Q_{n,1}^{(a)}(x)$ is given by (1.9).

Since (1.5) and (3.3), it follows

$$R_n^{(a)}(l_{n,k}^{(a)}) = -\frac{1}{l_{n,k}^{(a)}}(a+n)L_{n-1}^{(a)}(l_{n,k}^{(a)}). \tag{4.6}$$

So, taking into account (3.5), the limit relation (1.11) can be immediately proved by means of (4.5) and (4.6).

Now, to complete the proof of Theorem 1.3, we first consider that

$$\lim_{\lambda\to\infty} \lambda^4 \frac{\partial^2 x_{n,k}^{(\alpha,\lambda)}}{\partial\lambda^2} =$$

$$= -\frac{T_n^{(\alpha)}(l_{n,k}^{(\alpha)})R_n^{(\alpha)}(l_{n,k}^{(\alpha)}) - S_n^{(\alpha)}(l_{n,k}^{(\alpha)})V_n^{(\alpha)}(l_{n,k}^{(\alpha)})}{[R_n^{(\alpha)}(l_{n,k}^{(\alpha)})]^2} +$$

$$-\frac{V_n^{(\alpha)}(l_{n,k}^{(\alpha)})R_n^{(\alpha)}(l_{n,k}^{(\alpha)}) - W_n^{(\alpha)}(l_{n,k}^{(\alpha)})Z_n^{(\alpha)}(l_{n,k}^{(\alpha)})}{[R_n^{(\alpha)}(l_{n,k}^{(\alpha)})]^2} \lim_{\lambda\to\infty} \lambda^3 \frac{\partial x_{n,k}^{(\alpha,\lambda)}}{\partial\lambda},$$

where $R_n^{(\alpha)}(x)$ and $S_n^{(\alpha)}(x)$ are defined respectively in (4.1) and (4.2) and

$$T_n^{(\alpha)}(x) = \lim_{\lambda\to\infty} \lambda^4 \frac{\partial^2}{\partial\lambda^2} Q_n^{(\alpha)}(x;\lambda),$$

$$V_n^{(\alpha)}(x) = \lim_{\lambda\to\infty} \lambda \frac{\partial^2}{\partial x\,\partial\lambda} Q_n^{(\alpha)}(x;\lambda),$$

$$W_n^{(\alpha)}(x) = \lim_{\lambda\to\infty} \lambda \frac{\partial}{\partial\lambda} Q_n^{(\alpha)}(x;\lambda),$$

$$Z_n^{(\alpha)}(x) = \lim_{\lambda\to\infty} \frac{\partial^2}{\partial x^2} Q_n^{(\alpha)}(x;\lambda).$$

By reiterating the previous procedure, we obtain the following results

$$T_n^{(\alpha)}(x) = 6Q_{n,1}^{(\alpha)}(x),$$

$$V_n^{(\alpha)}(x) = W_n^{(\alpha)}(x) = L_n^{(\alpha)}(x),$$

$$Z_n^{(\alpha)}(x) = L_n^{(\alpha+2)}(x) - 2L_n^{(\alpha+1)}(x) + L_n^{(\alpha)}(x),$$

which, remembering (1.5), (3.3), (3.5), (4.3), (4.4) and (4.5), prove the (1.12).

REMARK. In Table 1 we compare the approximations $\bar{x}_{n,k}^{(\alpha,\lambda)}$ obtained by omitting the O-term in the asymptotic representation (1.10) with the exact values $x_{n,k}^{(\alpha,\lambda)}$, for $n = 3$, $\alpha = 2$, and $\lambda = 20, 100$.

TABLE 1.

k	$x_{3,k}^{(2,20)}$	$\bar{x}_{3,k}^{(2,20)}$	$x_{3,k}^{(2,100)}$	$\bar{x}_{3,k}^{(2,100)}$
1	1.5206974	1.5206964	1.5175194542	1.5175194524
2	4.3120309	4.3120301	4.3116010123	4.3116010109
3	9.1672716	9.1672735	9.1708795334	9.1708795366

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