Rendiconti
del
Seminario Matematico
della
Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 94 (1995), p. 165-213

<http://www.numdam.org/item?id=RSMUP_1995__94__165_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 94 (1995)

Some Results of Gevrey and Analytic Regularity for Semilinear Weakly Hyperbolic Equations of Oleinik Type.

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1. Introduction.

Consider the *quasi-linear* hyperbolic equation

$$u_{tt} - \sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(t, x, u, \nabla_x u)\, \partial_{x_i} u) = f(t, x, u, \nabla_x u), \tag{1.1}$$

where the coefficients a_{ij} and the nonlinear term f are real analytic functions of all their arguments and assume that the strict hyperbolicity condition

$$\lambda_0 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j} \xi_i \xi_j \leq \lambda |\xi|^2 \qquad (\xi \in \boldsymbol{R}^n), \tag{2.1}$$

is satisfied for some $0 < \lambda_0 \leq \lambda$. Thanks to results obtained by S. Alinhac and G. Métivier ([AM]) we know that every solution $u(t, x)$, with analytic initial data $u(0, x)$, $u_t(0, x)$, is also analytic as soon as it belongs to some Sobolev space $\boldsymbol{H}^k$ for $k > k(n)$ (see also [J1] for a more accurate estimate of the regularity bound $k(n)$; see [M3] for the *linear* case). Later, it was proved by S. Spagnolo [S1], [S2] that a similar result is also valid in the *weakly hyperbolic* case (that is, $\lambda_0 = 0$ in (2.1)), at least if we restrict ourselves to the subclass of the *semilinear* equations of the form

$$u_{tt} - \sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(t, x)\, \partial_{x_i} u) = f(t, x, u), \tag{3.1}$$

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where

$$a_{ij}(t, x) = a_{ji}(t, x) \tag{4.1}$$

and *one* of the following additional conditions is fulfilled:

i) the coefficients $a_{ij}(t, x)$ have the special form

$$a_{ij}(t, x) = b(t) \cdot \tilde{a}_{ij}(x); \tag{5.1}$$

ii) the solution $u(t, x)$ is assumed to belong *a priori* to a *Gevrey* class of order less than two.

We note that the Cauchy problem for a *linear* weakly hyperbolic equation is well-posed in the space $\mathcal{A}(\boldsymbol{R}_x^n)$ of real analytic functions, provided the coefficients of the equations are analytic; but, for $\lambda_0 = 0$, the linearized of (3.1) at a $\boldsymbol{C}^\infty$ solution, is a *weakly hyperbolic* equation with $\boldsymbol{C}^\infty$ coefficients, which could present the phenomena of non-existence or non-uniqueness. An example of a Cauchy problem (in one space dimension) of the form

$$\begin{cases} u_{tt} = a(t) u_{xx} , \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \end{cases} \tag{6.1}$$

with $a(t) \geqslant 0$, $a(t) \in \boldsymbol{C}^\infty$, $u_0(x)$, $u_1(x) \in \boldsymbol{C}^\infty$, without local solution, was constructed in [CS] (see also [CJS1]). A similar problem arises if we add a *lower order term* to the linear part of eq. (3.1) (see [N3]). Thus, in the *weakly hyperbolic* case, it is likely that some other assumptions are necessary in order to prove the *analytic regularity* of the solutions.

Furthermore, the Cauchy problem for a second order *weakly hyperbolic* linear equation (with smooth coefficients) is well-posed in the *Gevrey* class $\gamma^{(s)}$ for $s < 2$ (see in particular [M2] and the results in [J2], [N1], [N2], [CDS], [CJS2], [S3], [OT], [C], [D]); hence, it is natural to ask whether the *Gevrey regularity* for eq. (3.1) holds.

On the other hand, O. Oleinik (see [O1]) proved the well-posedness in $\boldsymbol{C}^\infty$ of the Cauchy problem for any *weakly hyperbolic* linear equation of the form

$$L(u) \equiv u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(t, x) u_{x_i})_{x_j} + \\ + \sum_{j=1}^{n} b_j(t, x) u_{x_j} + b_0(t, x) u_t + c(t, x) u = g(t, x), \tag{7.1}$$

where the coefficients are C^∞ functions satisfying (4.1), such that

$$(2'.1) \qquad 0 \leqslant \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leqslant \lambda|\xi|^2 \qquad (\forall \xi \in \boldsymbol{R}^n)$$

and for positive constants A, $B > 0$, $\forall \xi \in \boldsymbol{R}^n$

$$(8.1) \quad B\cdot\left(\sum_{j=1}^{n} b_j(t, x)\xi_j\right)^2 \leqslant A\cdot\sum_{i,j} a_{i,j}(t, x)\xi_i\xi_j + \sum_{i,j} \partial_t a_{i,j}(t, x)\xi_i\xi_j\,.$$

Thus it is natural to pose the question whether a result of analytic regularity (or, more generally, of *Gevrey regularity*), similar to those proved in [S1] and [S2], may hold for eq. (3.1) under the Oleinik's condition (8.1), instead of the assumptions i) or ii) of [S2].

In this paper, we consider a real solution, $u(t, x)\colon [0, T) \times \boldsymbol{R}^n_x \to \boldsymbol{R}$, of the *semilinear* equation

$$(9.1) \qquad L(u) = f(t, x, u)\,,$$

where $L(u)$ is the linear *weakly hyperbolic* operator defined in (7.1), which satisfies (2'.1), (4.1) and (8.1). Fixed $s \geqslant 1$, we suppose that

$$(10.1) \qquad a_{ij},\, b_j,\, \partial_t b_0,\, c \in \boldsymbol{C}^0([0, T);\, \gamma^{(s)}(\boldsymbol{R}^n_x))\,,$$

while the nonlinear term, $f(t, x, u)\colon [0, T) \times \boldsymbol{R}^n_x \times \boldsymbol{R}_u \to \boldsymbol{R}$, satisfies

$$(11.1) \qquad f \in \boldsymbol{C}^0([0, T);\, \gamma^{(s)}(\boldsymbol{R}^n_x \times \boldsymbol{R}_u))\,.$$

Under these hypotheses, we are able to answer the previous questions and more precisely, we prove the following result.

THEOREM 1. *Consider eq.* (9.1), *under the assumptions* (2'.1), (4.1), (8.1), (10.1) *and* (11.1). *Then a solution* $u(t, x)\colon [0, T) \times \boldsymbol{R}^n_x \to \boldsymbol{R}$ *of class* $\boldsymbol{C}^2$, *belongs to* $\boldsymbol{C}^2([0, T);\, \gamma^{(s)}(\boldsymbol{R}^n_x))$ *as soon as*

$$(12.1) \qquad u(0, x)\,, \qquad u_t(0, x) \in \gamma^{(s)}(\boldsymbol{R}^n_x)\,.$$

REMARK. We give a direct proof of Thm. 1 only in the case $s > 1$. Indeed in the conclusive part of the proof (see § 5), in order to localize our result (see Prop. 1), we use functions with compact support. On the other hand, for the analytic case ($s = 1$), we can resort to Thm. 2 of [S2] (see condition ii) above). In fact, applying our result, we first demonstrate that the solution $u(t, x)$ belongs to $C^2([0, T); \gamma^{(s)}(\boldsymbol{R}_x^n))$ for any $s > 1$, thus $u(t, x)$ satisfies the hypotheses of Thm. 2 of [S2].

Moreover, we can also apply the methods of § 5 in the analytic case, using a suitable sequence of C^∞ compactly supported functions $\{\chi_N\}$ instead of a fixed $\chi \in \gamma^s(\boldsymbol{R}^n) \cap C_0^\infty(\boldsymbol{R}^n)$ (if $s > 1$). See Remark 5.

REMARK. We observe that the Oleinik condition (8.1) works only if the linear operator L takes the form (7.1), which is not preserved (in general) by coordinate transformations. Hence, we cannot apply in our proof the same *geometric* techniques used in [AM] (see Lemma 2.2 and 3.1) in order to prove the analytic regularity of the solution.

Some Remarks and Notations.

We denote by $\gamma^{(s)}(\boldsymbol{R}_x^n)$, with $s \geqslant 1$, the space of *Gevrey functions* of order s, that is the space of C^∞ functions $v(x)$ such that

$$|D_x^\alpha v(x)| \leqslant C_K A_K^{|\alpha|} |\alpha|!^s \qquad \alpha \in \boldsymbol{N}^n, \qquad x \in K,$$

for all compact sets $K \subset \boldsymbol{R}_x^n$. Besides, throughout this work we will consider the spaces $\gamma_{L^2}^{(s)}(\boldsymbol{R}_x^n)$ and $\gamma_{L^\infty}^{(s)}(\boldsymbol{R}_x^n)$, defined in the obvious way.

Now, we will consider some aspects of the Cauchy problem for eq. (9.1).

First of all, by defining,

$$\begin{cases} \tilde{u}(t, x) = u(t, x) - u_0(x) - tu_1(x), \\ \tilde{f}(t, x, v) = f(t, x, v + u_0 + tu_1) - L(u_0(x) + tu_1(x)), \end{cases} \tag{13.1}$$

where $u_0(x)$ and $u_1(x)$ are the initial data of $u(t, x)$, we can confine ourselves to a particular case of Thm. 1, namely:

$$L(\tilde{u}) = \tilde{f}(t, x, \tilde{u}), \qquad \tilde{u}(0, x) = \tilde{u}_t(0, x) = 0. \tag{14.1}$$

Indeed, the nonlinear term $\tilde{f}(t, x, v)$ satisfies the same hypotheses of $f(t, x, v)$.

Assuming $u(t, x) \in C^2([0, T); H^k(\boldsymbol{R}_x^n))$, (with k sufficiently large) and taking the initial data in $\gamma_{L^2}^{(s)}(\boldsymbol{R}_x^n)$, we can prove that $C^2([0, T); \gamma_{L^2}^{(s)}(\boldsymbol{R}_x^n))$. See Prop. 1.

Taking $s = 1$ in the statement of Thm. 1, we obtain the of *analytic regularity*. Moreover, if the coefficients are analytic in the variable t one can also derive the analyticity in t of the solution $u(t, x)$, by applying the classic Cauchy-Kovalewski theorem.

It is sufficient to assume that Oleinik's condition (8.1) holds only *locally* in $[0, T) \times \boldsymbol{R}_x^n$. That is for any compact set $K \subset [0, T) \times \boldsymbol{R}_x^n$ there exist constants $A = A_K$, $B = B_K$ such that (8.1) holds for all $\xi \in \boldsymbol{R}^n$.

An essential step in the proof of Thm. 1, is that the eq. (9.1) has the *uniqueness property*. For a detailed proof of the local $\boldsymbol{C}^\infty$ *well-posedness* of the Cauchy problem for eq. (9.1), we refer to [DM] (see also the proof of the *Sobolev estimates* in Appendix A).

Finally, applying Thm. 1 and $\boldsymbol{C}^\infty$*-well posedness* proved in [DM], it follows the that the Cauchy problem for eq. (7.1) and (9.1) is well-posed in the *Gevrey* classes $\gamma^{(s)}$ of order $s \geqslant 1$; see Prop. 1 of § 4 for more details.

This is the layout of the paper: in § 2 and § 3 we prove the basic *Gevrey estimates* for the linear and nonlinear equations respectively; in § 4 we introduce the *Gevrey-energies* and prove a result of *global regularity* in the space $\gamma_{L^2}^{(s)}(\boldsymbol{R}_x^n)$ (see Prop. 1); in § 5 we prove the statement of Thm. 1 (localizing the result of the previous section). Finally in the Appendix A, we give the *local estimates* of the $\boldsymbol{H}^k$*-norms* and prove the $\boldsymbol{C}^\infty$ *regularity* of the solution.

Acknowledgments. We would like to thank S. Spagnolo for many useful discussions about the subject of this paper.

2. Global Gevrey estimates for linear equations $L(u) = g(t, x)$.

In this section we shall derive the energy estimates for the linear Cauchy problem:

$$(1) \qquad L(u) \equiv u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(t, x)\, u_{x_i})_{x_j} + \\ + \sum_{j=1n} b_j(t, x)\, u_{x_j} + b_0(t, x)\, u_t + c(t, x)\, u = g(t, x),$$

$$(2) \qquad u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x),$$

provided eq. (1) satisfies the condition of weak hyperbolicity, to be precise $a_{i,j}(t, x) = a_{j,i}(t, x)$,

$$(3) \qquad \lambda |\xi|^2 \geqslant \sum_{i,j}^{n} a_{ij}(t, x)\, \xi_i \xi_j \geqslant 0 \qquad (\lambda > 0),$$

and the Oleinik's condition holds, ensuring the well posedness in $\boldsymbol{C}^\infty$ of the linear eq. (1) (see [O1]): for the constants $A, B > 0$

$$(4)\quad B\cdot\left(\sum_{j=1}^{n} b_j(t,x)\,\xi_j\right)^2 \leq A\cdot\sum_{i,j} a_{i,j}(t,x)\,\xi_i\xi_j + \sum_{i,j}\partial_t a_{i,j}(t,x)\,\xi_i\xi_j\,,$$

for all $(t, x) \in \boldsymbol{R}_+ \times \boldsymbol{R}^n$, $\xi \in \boldsymbol{R}^n$. Concerning the coefficients of the linear operator in (1) we assume that $a_{ij}(t, x)$, $b_j(t, x)$, $c(t, x)$, $\partial_t b_0(t, x)$, as function of the variable $x \in \boldsymbol{R}^n$, belong to some Gevrey class of order $s \geq 1$, more precisely we will take the coefficients in $\boldsymbol{C}^0([0, T]; \gamma^{(s)}_{L^\infty}(\boldsymbol{R}^n))$. Hence, the following upper bounds hold

$$(5)\qquad |\partial_x^\alpha w(t,x)| \leq C_0 \Lambda_0^{|\alpha|}\,|\alpha|!^s \qquad (\alpha \in \boldsymbol{N}^n)\,,$$

for $w = a_{ij}, b_j, \partial_t b_0, c$ and for some constants $C_0, \Lambda_0 \geq 0$. Finally we require that the initial data $u_0(x)$, $u_1(x)$ and $g(t, x)$ should belong to $\boldsymbol{H}^\infty(\boldsymbol{R}^n_x)$. Taking these assumptions into account and defining

$$(6)\qquad \tilde{g}(t,x) = g(t,x) - L(u_0(x) + tu_1(x))\,,$$

we shall restrict ourselves to the case:

$$(1')\quad L(u) \equiv u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(t,x)\,u_{x_i})_{x_j} + \sum_{j=1}^{n} b_j(t,x)\,u_{x_j} + b_0(t,x)\,u_t + c(t,x)\,u = \tilde{g}(t,x)\,,$$

$$(2')\quad u(0,x) = 0\,, \qquad u_t(0,x) = 0\,.$$

We introduce the following definitions:

$$G_\tau = [0,\tau]\times\boldsymbol{R}^n \qquad (\tau > 0)\,,$$

$$[u, v]_{G_\tau} = \int_{G_\tau} u(t,x)\,v(t,x)\,dx\,dt\,,$$

$$(u, v)_{t=\tau} = \int_{\boldsymbol{R}^n} u(\tau,x)\,v(\tau,x)\,dx$$

and we use the customary notations $D^\alpha = \partial_{x_1}^{\alpha_1}\cdot\ldots\cdot\partial_{x_n}^{\alpha_n}$. Moreover, in the foregoing proof we shall adopt the convention of implicit summation over repeated indices.

Let $u(t, x)$ be a regular solution of (1′), (2′) on $[0, T) \times \boldsymbol{R}^n$. Given τ ($0 \leqslant \tau < T$), we define the function

(7)
$$w(t, x) = \int_t^\tau u(\sigma, u)\, d\sigma ,$$

and for $j \geqslant 1$ we introduce the j-th energies of a solution $u(t, x)$ to Pb. (1′), (2′) by setting

(8)
$$E_j(\tau) = \sum_{|\alpha| = j-1} E_\alpha(\tau), \qquad (j \geqslant 1)$$

(9)
$$F_1(\tau) \equiv E_1(\tau); \qquad F_j(\tau) = \sum_{|\alpha| = j-1} E_\alpha(\tau) + \sum_{|\beta| = j-2} \tilde{E}_\beta(\tau), \qquad (j \geqslant 2)$$

where,

(10)
$$E_\alpha(\tau) = [D^\alpha u, D^\alpha u\, e^{j\theta t}]_{G_\tau} + j^2 [D^\alpha w, D^\alpha w\, e^{j\theta t}]_{G_\tau}, \qquad (|\alpha| = j-1),$$

(11)
$$\tilde{E}_\beta(\tau) = [D^\beta u_t, D^\beta u_t\, e^{j\theta t}]_{G_\tau}, \qquad (|\beta| = j-2),$$

with $\theta > 0$ a constant which will be chosen in the proof of Lemma 1. In order to estimate E_α we observe that $D^\alpha w(\tau, x) = 0$ $\forall \alpha \in \boldsymbol{N}^n$, thus

(12)
$$\frac{d}{d\tau} E_\alpha = (D_u^\alpha, D^\alpha u\, e^{j\theta t})_{t=\tau} + 2j^2 [D^\alpha w, D^\alpha u(\tau)\, e^{j\theta t}]_{G_\tau} \leqslant$$
$$\leqslant \left[1 + \frac{1}{\theta}(1 - e^{-j\theta\tau})\right] \cdot (D^\alpha u, D^\alpha u\, e^{j\theta t})_{t=\tau} + j^3 [D^\alpha w, D^\alpha w\, e^{j\theta t}]_{G_\tau},$$

hence we have to estimate

(13)
$$\frac{d}{d\tau} [D^\alpha u, D^\alpha u\, e^{j\theta t}]_{G_\tau} \equiv (D^\alpha u, D^\alpha u\, e^{j\theta t})_{t=\tau} .$$

To this end, using the fact that

$$w_t = -u, \qquad u(0, x) = u_t(0, x) = 0,$$

and integrating by parts, we have the following identities (see also [O1], [DM]):

(14)
$$[D^\alpha u_{tt}, D^\alpha w\, e^{j\theta t}]_{G_\tau} =$$
$$= \frac{1}{2} (D^\alpha u, D^\alpha u\, e^{j\theta t})_{t=\tau} + \left[D^\alpha u\, e^{j\theta t}, j^2\theta^2 D^\alpha w - \frac{3}{2} j\theta D^\alpha u\right]_{G_\tau},$$

$$(15)\qquad [(a_{ij}D^{\alpha}u_{x_i})_{x_j}, D^{\alpha}w\, e^{j\theta t}]_{G_\tau} =$$

$$= -\frac{1}{2}(a_{ij}D^{\alpha}w_{x_i}, D^{\alpha}w_{x_i})_{t=0} - \frac{1}{2}[(j\theta a_{ij} + \partial_t a_{ij})D^{\alpha}w_{x_i}, D^{\alpha}w_{x_j}\, e^{j\theta t}]_{G_\tau},$$

$$(16)\qquad [D^{\alpha}(b_0 u_t), D^{\alpha}w\, e^{j\theta t}]_{G_\tau} =$$

$$= -[D^{\alpha}(\partial_t b_0 u), D^{\alpha}w\, e^{j\theta t}]_{G_\tau} + [D^{\alpha}(b_0 u)e^{j\theta t}, D^{\alpha}u - j\theta D^{\alpha}w]_{G_\tau},$$

$$(17)\qquad [b_i D^{\alpha}u_{x_i}, D^{\alpha}w\, e^{j\theta t}]_{G_\tau} =$$

$$= -[\partial_{x_i} b_i D^{\alpha}u, D^{\alpha}w\, e^{j\theta t}]_{G_\tau} - [b_i D^{\alpha}w_{x_i}, D^{\alpha}u\, e^{j\theta t}]_{G_\tau}.$$

Defining,

$$(18)\qquad A_0 = -\sum_{i,j=1}^{n}\partial_{x_j}(a_{ij}\partial_{x_i}), \qquad B_0 = \sum_{j=1}^{n} b_j \partial_{x_j}$$

and applying the operator D^{α} to each term of $(1')$, we find

$$(19)\qquad (\partial_t^2 + A_0 + B_0)D^{\alpha}u + D^{\alpha}(b_0 u_t + cu) = K_{\alpha} + \Gamma_{\alpha} + D^{\alpha}\tilde{g}$$

where

$$(20)\qquad K_{\alpha} = A_0 D^{\alpha}u - D^{\alpha}A_0 u,$$

$$(21)\qquad \Gamma_{\alpha} = B_0 D^{\alpha}u - D^{\alpha}B_0 u.$$

Hence from the equality

$$(22)\qquad [(\partial_t^2 + A_0 + B_0)D^{\alpha}u + D^{\alpha}(b_0 u_t + cu), D^{\alpha}w\, e^{j\theta t}]_{G_\tau} =$$

$$= [K_{\alpha} + \Gamma_{\alpha} + D^{\alpha}\tilde{g}, D^{\alpha}w\, e^{j\theta t}]_{G_\tau},$$

using the identities (14), (15), (16), (17) we have

$$(23)\qquad \frac{1}{2}(D^{\alpha}u, D^{\alpha}u\, e^{j\theta t})_{t=\tau} = -\left[D^{\alpha}u\, e^{j\theta t}, j^2\theta^2 D^{\alpha}w - \frac{3}{2}j\theta D^{\alpha}u\right]_{G_\tau} -$$

$$-\frac{1}{2}[(j\theta a_{ij} + \partial_t a_{ij})D^{\alpha}w_{x_i}, D^{\alpha}w_{x_j}\, e^{j\theta t}]_{G_\tau} -$$

$$-\frac{1}{2}(a_{ij}D^{\alpha}w_{x_i}, D^{\alpha}w_{x_j})_{t=0} - [D^{\alpha}(b_0 u)e^{j\theta t}, D^{\alpha}u - j\theta D^{\alpha}w]_{G_\tau} +$$

$$+[\partial_{x_i} b_i D^{\alpha}u, D^{\alpha}w\, e^{j\theta t}]_{G_\tau} + [b_i D^{\alpha}w_{x_i}, D^{\alpha}u\, e^{j\theta t}]_{G_\tau} +$$

$$+[D^{\alpha}((\partial_t b_0 - c)u), D^{\alpha}w\, e^{j\theta t}]_{G_\tau} + [K_{\alpha} + \Gamma_{\alpha} + D^{\alpha}\tilde{g}, D^{\alpha}w\, e^{j\theta t}]_{G_\tau}.$$

Since $B > 0$ we can estimate the last term in (17) as follow:

(24) $$[b_i D^\alpha w_{x_i}, D^\alpha u e^{j\theta t}]_{G_\tau} \leqslant$$

$$\leqslant \frac{B}{2}[b_i D^\alpha w_{x_i}, b_i D^\alpha w_{x_i} e^{j\theta t}]_{G_\tau} + \frac{1}{2B}[D^\alpha u, D^\alpha u e^{j\theta t}]_{G_\tau}.$$

Moreover, integrating by parts we have

(25) $$[D^\alpha u, D^\alpha w e^{j\theta t}]_{G_\tau} = \frac{1}{2}(D^\alpha w, D^\alpha w)_{t=0} + \frac{j\theta}{2}[D^\alpha w, D^\alpha w e^{j\theta t}]_{G_\tau}$$

hence, from (3), (5), (23) and taking the Olienik's condition into account, we obtain

(26) $$\frac{1}{2}(D^\alpha u, D^\alpha u e^{j\theta t})_{t=\tau} \leqslant$$

$$\leqslant \frac{1}{2}(3j\theta + B^{-1} + C_0 \Lambda_0)[D^\alpha u, D^\alpha u e^{j\theta t}]_{G_\tau} -$$

$$-\frac{1}{2}(j^3\theta^3 - C_0\Lambda_0)[D^\alpha w, D^\alpha w e^{j\theta t}]_{G_\tau} -$$

$$-\frac{1}{2}(j\theta - A)[a_{ij}D^\alpha w_{x_i}, D^\alpha w_{x_j} e^{j\theta t}]_{G_\tau} - [D^\alpha(b_0 u)e^{j\theta t}, D^\alpha u - j\theta D^\alpha w]_{G_\tau} +$$

$$+[D^\alpha((\partial_t b_0 - c)u), D^\alpha w e^{j\theta t}]_{G_\tau} + [K_\alpha + \Gamma_\alpha + D^\alpha \tilde{g}, D^\alpha w e^{j\theta t}]_{G_\tau},$$

thus, assuming $\theta \geqslant 1 + \sqrt[3]{2C_0\Lambda_0}$, we have

$$1 + \frac{1}{\theta}(1 - e^{-j\theta\tau}) \leqslant 2$$

and from (10), (12) and (26) it follows that

(27) $$\frac{1}{2}\frac{d}{d\tau}E_\alpha \leqslant (3j\theta + B^{-1} + C_0\Lambda_0)[D^\alpha u, D^\alpha u e^{j\theta t}]_{G_\tau} -$$

$$-\frac{1}{2}j^3\theta^3[D^\alpha w, D^\alpha w e^{j\theta t}]_{G_\tau} - (j\theta - A)[a_{ij}D^\alpha w_{x_i}, D^\alpha w_{x_i} e^{j\theta t}]_{G_\tau} -$$

$$-2[D^\alpha(b_0 u)e^{j\theta t}, D^\alpha u - j\theta D^\alpha w]_{G_\tau} + 2[D^\alpha((\partial_t b_0 - c)u), D^\alpha w e^{j\theta t}]_{G_\tau} +$$

$$+2[K_\alpha + \Gamma_\alpha + D^\alpha \tilde{g}, D^\alpha w e^{j\theta t}]_{G_\tau}.$$

We can now prove the following estimate.

LEMMA 1. *Let $u(t, x)$ be a regular solution of (1′), (2′) (to be precise, we can assume $u(t, x) \in C^2([0, T); H^\infty(R^n))$); then, there exists*

$\theta_0 \geqslant A$, such that for $\theta \geqslant \theta_0$ and $\Lambda > \Lambda_0$, the following inequality holds

$$(28)\qquad \frac{1}{2}\frac{d}{d\tau}E_j \leqslant Cj!^s(j+1)^\sigma \sum_{h=0}^{j-1} \frac{(\Lambda e^{\theta\tau/2})^{j-h}\sqrt{E_{h+1}}}{h!^s(h+1)^\sigma(h+2)^\sigma}\sqrt{E_j} + \\ +2\sum_{|\alpha|=j-1}[D^\alpha \tilde{g}, D^\alpha w e^{j\theta t}]_{G_\tau} \qquad (j \geqslant 1)$$

where $C = C(n, A, B, C_0, \Lambda_0, \Lambda, \theta, \tau, s)$, $\sigma = s-1$.

PROOF. We will use here Lemma 2.2 and 2.3 of [D] (see also [AS] for the case $s = 1$). Taking assumption (5) into account and applying Lemma 2.3 of [D], we easily obtain that for any $\Lambda > \Lambda_0$, there exists a constant $C_1 = C_1(n, C_0, \Lambda_0, \Lambda)$ such that

$$(29)\qquad \Big(\sum_{|\alpha|=j-1}[D^\alpha(b_0 u), D^\alpha(b_0 u)e^{j\theta t}]_{G_\tau}\Big)^{1/2}, \\ \Big(\sum_{|\alpha|=j-1}[D^\alpha((\partial_t b_0 - c)u), D^\alpha((\partial_t b_0 - c)u)e^{j\theta t}]_{G_\tau}\Big)^{1/2} \leqslant \\ \leqslant C_1(j-1)!^s\sum_{h=0}^{j-1}\frac{\Lambda^{j-h-1}}{h!^s}\Big(\sum_{|\beta|=h}[D^\beta u, D^\beta u e^{j\theta t}]_{G_\tau}\Big)^{1/2}.$$

Hence, observing that

$$(30)\qquad \sum_{|\beta|=h}[D^\beta u, D^\beta u e^{j\theta t}]_{G_\tau} \leqslant e^{(j-h-1)\theta\tau}E_{h+1}, \qquad (0 \leqslant h \leqslant j-1)$$

and taking $C_2 = (2+\theta)C_1$, we deduce the following estimate

$$(31)\qquad \sum_{|\alpha|=j-1}[D^\alpha(b_0 u)e^{j\theta t}, D^\alpha u - j\theta D^\alpha w]_{G_\tau} + \\ +[D^\alpha((\partial_t b_0 - c)u), D^\alpha w e^{j\theta t}]_{G_\tau} \leqslant \\ \leqslant C_2(j-1)!^s\sum_{h=0}^{j-1}\frac{(\Lambda e^{\theta\tau/2})^{j-h-1}\sqrt{E_{h+1}}}{h!^s}\sqrt{E_j}.$$

In order to continue we must now estimate the term

$$(32)\qquad \sum_{|\alpha|=j-1}[K_\alpha + \Gamma_\alpha, D^\alpha w e^{j\theta t}]_{G\tau},$$

we define

$$\widetilde{K}_\alpha = \sum_{|\gamma| > 1, \gamma \leq \alpha} \binom{\alpha}{\gamma} (D^\gamma a_{ij} D^{\alpha - \gamma} u_{x_i})_{x_j}, \tag{33}$$

thus, writing

$$[K_\alpha + \Gamma_\alpha, D^\alpha w e^{j\theta t}]_{G_\tau} = \tag{34}$$

$$= \sum_{\eta=1}^{n} \alpha_\eta [\partial_{x_\eta} a_{ij} D^{\alpha - 1_\eta} u_{x_i}, D^\alpha w_{x_j} e^{j\theta t}]_{G_\tau} + [\widetilde{K}_\alpha + \Gamma_\alpha, D^\alpha w e^{j\theta t}]_{G_\tau},$$

where for $\alpha_\eta \geq 1$, $\alpha - 1_\eta = (\alpha_1, \ldots, \alpha_\eta - 1, \ldots, \alpha_n)$, we can consider separately the terms of (32) of order $\leq j - 1$ and those of order j. Using now Lemma 2.2 of [D] (in particular the estimates of the terms I_α, II_α) and taking (5) into account, for any arbitrary $\Lambda > \Lambda_0$, we can find a constant $C_3 = C_3(n, C_0, \Lambda_0, \Lambda)$ such that

$$\left(\sum_{|\alpha| = j-1} [(\widetilde{K}_\alpha + \Gamma_\alpha), (\widetilde{K}_\alpha + \Gamma_\alpha) e^{j\theta t}]_{G_\tau} \right)^{1/2} \leq \tag{35}$$

$$\leq C_3 (j+1)!^s \sum_{h=0}^{j-1} \frac{(\Lambda e^{\theta\tau/2})^{j-h-1}}{h!^s (h+1)^\sigma (h+2)^\sigma} \sqrt{E_{h+1}}.$$

Hence, from (10) and (35) we have

$$\sum_{|\alpha| = j-1} [\widetilde{K}_\alpha + \Gamma_\alpha, D^\alpha w e^{j\theta t}]_{G_\tau} \leq \tag{36}$$

$$\leq C_3 j!^s (j+1)^\sigma \sum_{h=0}^{j-1} \frac{(\Lambda e^{\theta\tau/2})^{j-h-1}}{h!^s (h+1)^\sigma (h+2)^\sigma} \sqrt{E_{h+1}} \sqrt{E_j}.$$

Finally, we have to estimate the first term in the right side of (34). Integrating by parts (for $\alpha_\eta \geq 1$) we find the identity

$$[\partial_{x_\eta} a_{ij} D^{\alpha - 1_\eta} u_{x_i}, D^\alpha w_{x_j} e^{j\theta t}]_{G_\tau} = -[\partial^2_{x_\eta x_i} a_{ij} D^{\alpha - 1_\eta} u, D^\alpha w_{x_j} e^{j\theta t}]_{G_\tau} + \tag{37}$$

$$+ [\partial^2_{x_\eta x_\eta} a_{ij} D^{\alpha - 1_\eta} u, D^{\alpha - 1_\eta} w_{x_i x_j} e^{j\theta t}]_{G_\tau} + [\partial_{x_\eta} a_{ij} D^\alpha u, D^{\alpha - 1_\eta} w_{x_i x_j} e^{j\theta t}]_{G_\tau},$$

now, integrating by parts again, we deduce the estimate

$$[\partial_{x_\eta} a_{ij} D^{\alpha - 1_\eta} u_{x_i}, D^\alpha w_{x_j} e^{j\theta t}]_{G_\tau} \leq \tag{38}$$

$$\leq [\partial^3_{x_\eta x_i x_j} a_{ij} D^{\alpha - 1_\eta} u, D^\alpha w e^{j\theta t}]_{G_\tau} + [\partial^2_{x_\eta x_i} a_{ij} D^{\alpha - 1_\eta} u_{x_j}, D^\alpha w e^{j\theta t}]_{G_\tau} -$$

$$-[\partial^3_{x_\eta x_\eta x_i} a_{ij} D^{\alpha - 1_\eta} u,\ D^{\alpha - 1_\eta} w_{x_j}\, e^{j\theta t}]_{G_\tau} - [\partial^2_{x_\eta x_\eta} a_{ij} D^{\alpha - 1_\eta} u_{x_i},\ D^{\alpha - 1_\eta} w_{x_j}\, e^{j\theta t}]_{G_\tau} +$$

$$+ \frac{1}{2}[D^\alpha u,\ D^\alpha u\, e^{j\theta t}]_{G_\tau} + \frac{1}{2}[\partial_{x_\eta} a_{ij} D^{\alpha - 1_\eta} w_{x_i x_j},\ \partial_{x_\eta} a_{ij} D^{\alpha - 1_\eta} w_{x_i x_j}\, e^{j\theta t}]_{G_\tau},$$

hence, multiplying by α_η in (38) and summing over $1 \leq \eta \leq n$, $|\alpha| = j - 1$, we have

$$(39) \qquad \sum_{|\alpha| = j - 1} \sum_{\eta = 1}^{n} \alpha_\eta [\partial_{x_\eta} a_{ij} D^{\alpha - 1_\eta} u_{x_i},\ D^\alpha w_{x_j}\, e^{j\theta t}]_{G_\tau} \leq$$

$$\leq C_4 (j \sqrt{E_j} + \sqrt{E_{j-1}}) \sqrt{E_j} +$$

$$+ \frac{1}{2} \sum_{|\alpha| = j - 1} \sum_{\eta = 1}^{n} \alpha_\eta [\partial_{x_\eta} a_{ij} D^{\alpha - 1_\eta} w_{x_i x_j},\ \partial_{x_\eta} a_{ij} D^{\alpha - 1_\eta} w_{x_i x_j}\, e^{j\theta t}]_{G_\tau}$$

where $C_4 = C_4(n, C_0, \Lambda_0, \theta, \tau, s)$. It remains to estimate the last term in (39). Using the condition of weak hyperbolicity we can apply the following inequality, due to Oleinik (see [O2] Lemma 4):

$$(40) \qquad \left(\sum_{i,j=1}^{n} \partial_{x_\eta} a_{ij} b_{ij} \right)^2 \leq C(n) \sup_{\substack{1 \leq i,j \leq n \\ |\alpha| = 2}} \| D^\alpha a_{ij} \|_{L^\infty} \cdot \sum_{i,j,p=1}^{n} a_{ij} b_{ip} b_{jp},$$

which holds for every $n \times n$ symmetric matrix $[b_{ij}]$. Thus, for $\alpha_\eta \geq 1$ we obtain

$$(41) \qquad \sum_{i,j=1}^{n} [\partial_{x_\eta} a_{ij} D^{\alpha - 1_\eta} w_{x_i x_j},\ \partial_{x_\eta} a_{ij} D^{\alpha - 1_\eta} w_{x_i x_j}\, e^{j\theta t}]_{G_\tau} \leq$$

$$\leq C_5 \sum_{i,j,p=1}^{n} [a_{ij} D^{\alpha - 1_\eta} w_{x_i x_p},\ D^{\alpha - 1_\eta} w_{x_j x_p}\, e^{j\theta t}]_{G_\tau},$$

where $C_5 = C_5(n, C_0, \Lambda_0, s) = C(n) C_0 2^s \Lambda_0^2$. Now, from (41) we deduce that

$$(42) \qquad \sum_{|\alpha| = j - 1} \sum_{\eta = 1}^{n} \alpha_\eta [\partial_{x_\eta} a_{ij} D^{\alpha - 1_\eta} w_{x_i x_j},\ \partial_{x_\eta} a_{ij} D^{\alpha - 1_\eta} w_{x_i x_j}\, e^{j\theta t}]_{G_\tau} \leq$$

$$\leq n^2 C_5\, j \sum_{|\alpha| = j - 1} [a_{ij} D^\alpha w_{x_i},\ D^\alpha w_{x_j}\, e^{j\theta t}]_{G_\tau}.$$

Taking (27), (39) and (42) into account, we define

$$\theta_0 = \max\{A + n^2 C_5, 1 + \sqrt[3]{2C_0 \Lambda_0}\}, \tag{43}$$

then, for $\theta \geqslant \theta_0$ we have

$$2 \sum_{|\alpha| = j-1} \sum_{\eta=1}^{n} a_\eta [\partial_{x_\eta} a_{ij} D^{\alpha - 1_\eta} u_{x_i}, D^\alpha w_{x_j} e^{j\theta t}]_{G_\tau} - \sum_{|\alpha| = j-1} (j\theta - A)[a_{ij} D^\alpha w_{x_i}, D^\alpha w_{x_j} e^{j\theta t}]_{G_\tau} \leqslant \leqslant 2C_4 (j\sqrt{E_j} + \sqrt{E_{j-1}})\sqrt{E_j} \tag{44}$$

hence, using (27), (31), (36), (44) we easily obtain that estimate (28) holds. Q.E.D.

In order to estimate $\widetilde{E}_\beta(\tau)$, we observe that

$$\frac{d}{d\tau} \widetilde{E}_\beta(\tau) \equiv \frac{d}{d\tau} [D^\beta u_t, D^\beta u_t e^{j\theta t}]_{G_\tau} = (D^\beta u_t, D^\beta u_t e^{j\theta t})_{t=\tau}, \tag{45}$$

moreover, using the fact that $u(0, x) = u_t(0, x) = 0$, integrating by parts we, have the following identities:

$$[D^\beta u_{tt}, D^\beta u_t e^{j\theta t}]_{G_\tau} = = \frac{1}{2} (D^\beta u_t, D^\beta u_t e^{j\theta t})_{t=\tau} - \frac{j\theta}{2} [D^\beta u_t, D^\beta u_t e^{j\theta t}]_{G_\tau}, \tag{46}$$

$$[(a_{ij} D^\beta u_{x_i})_{x_j}, D^\beta u_t e^{j\theta t}]_{G_\tau} = = \frac{1}{2} [(j\theta a_{ij} + \partial_t a_{ij}) D^\beta u_{x_i}, D^\beta u_{x_j} e^{j\theta t}]_{G_\tau} - \frac{1}{2} (a_{ij} D^\beta u_{x_i}, D^\beta u_{x_j} e^{j\theta t})_{t=\tau}. \tag{47}$$

Now, applying the operator D^β to each term of (1′), with the same notations as before, if follows that

$$(\partial_t^2 + A_0) D^\beta u + D^\beta (B_0 u + b_0 u_t + cu) = K_\beta + D^\beta \widetilde{g}, \tag{48}$$

hence, from the equality

$$[(\partial_t^2 + A_0) D^\beta u + D^\beta (B_0 u + b_0 u_t + cu), D^\beta u_t e^{j\theta t}]_{G_\tau} = = [K_\beta + D^\beta \widetilde{g}, D^\beta u_t e^{j\theta t}]_{G_\tau}, \tag{49}$$

using the identities (46), (47) we have

$$\frac{1}{2}(D^\beta u_t, D^\beta u_t\, e^{j\theta t})_{t=\tau} = \tag{50}$$

$$= \frac{j\theta}{2}[D^\beta u_t, D^\beta u_t\, e^{j\theta t}]_{G_\tau} + \frac{1}{2}[(j\theta a_{ij} + \partial_t a_{ij})D^\beta u_{x_i}, D^\beta u_{x_j}\, e^{j\theta t}]_{G_\tau} -$$

$$-\frac{1}{2}(a_{ij}D^\beta u_{x_i}, D^\beta u_{x_j}\, e^{j\theta t})_{t=\tau} - [D^\beta(B_0 u + b_0 u_t + cu), D^\beta u_t\, e^{j\theta t}]G_\tau +$$

$$+[K_\beta + D^\beta \tilde{g}, D^\beta u_t\, e^{j\theta t}]_{G_\tau}.$$

Now, defining

$$\tilde{E}_j(\tau) = \sum_{|\beta| = j-2} \tilde{E}_\beta(\tau), \qquad (j \geq 2) \tag{51}$$

we are in a position to prove the following Lemma.

LEMMA 2. *With the same hypotheses and notations of Lemma* 1, *for any* $\Lambda > \Lambda_0$ *we can find a constant* $C = C(n, C_0, \Lambda_0, \Lambda, \theta, \tau, s)$ *such that for* $j \geq 2$ *the following estimate holds:*

$$\frac{1}{2}\frac{d}{d\tau}\tilde{E}_j(\tau) \leq C\left(jF_j + (j-1)!^s \sum_{h=0}^{j-2} \frac{(\Lambda e^{\theta\tau/2})^{j-h-1}\sqrt{F_{h+1}}}{h!^s}\sqrt{\tilde{E}_j} + \right. \tag{52}$$

$$\left. + j!^s \sum_{h=0}^{j-2} \frac{(\Lambda e^{\theta\tau/2})^{j-h-1}}{h!^s(h+1)^\sigma(h+2)^\sigma}\sqrt{E_{h+1}}\sqrt{\tilde{E}_j}\right) + \sum_{|\beta|=j-2}[D^\beta\tilde{g}, D^\beta u_t\, e^{j\theta t}]_{G_\tau}$$

where $\sigma = s - 1$.

PROOF. It is easy to see that for $j \geq 2$ we have

$$\sum_{|\beta|=j-2}[(j\theta a_{ij} + \partial_t a_{ij})D^\beta u_{x_i}, D^\beta u_{x_j}\, e^{j\theta t}]_{G_\tau} \leq C_6(n, C_0, \Lambda_0, \theta) j E_j \tag{53}$$

moreover, using Lemma 2.3 of [D] we deduce that for any $\Lambda > \Lambda_0$ there exists a constant $C_7 = C_7(n, C_0, \Lambda_0, \Lambda)$ such that

$$\left(\sum_{|\beta|=j-2}[D^\beta(B_0 u + cu), D^\beta(B_0 u + cu)\, e^{j\theta t}]_{G_\tau}\right)^{1/2} \leq \tag{54}$$

$$\leq C_7 (j-1)!^s \sum_{h=0}^{j-1} \frac{(\Lambda e^{\theta\tau/2})^{j-h-1}}{h!^s}\sqrt{E_{h+1}},$$

$$(55)\quad \left(\sum_{|\beta|=j-2}[D^\beta(b_0 u_t), D^\beta(b_0 u_t)e^{j\theta t}]_{G_\tau}\right)^{1/2} \leqslant$$

$$\leqslant C_7(j-2)!^s \sum_{h=0}^{j-2} \frac{(\Lambda e^{\theta\tau/2})^{j-h-2}}{h!^s} \sqrt{\widetilde{E}_{h+2}}\,.$$

Finally, applying Lemma 2.2 of [D], for $\Lambda > \Lambda_0$ we can find a constant $C_8 = C_8(n, C_0, \Lambda_0, \Lambda)$ such that

$$(56)\quad \left(\sum_{|\beta|=j-2}[K_\beta, K_\beta e^{j\theta t}]_{G_\tau}\right)^{1/2} \leqslant$$

$$\leqslant C_8 j\left(\sum_{|\beta|=j-2}[-(a_{ij}D^\beta u_{x_i})_{x_j}, D^\beta u\, e^{j\theta t}]_{G_\tau}\right)^{1/2} +$$

$$+ C_8 j!^s \sum_{h=0}^{j-2} \frac{\Lambda^{j-h-1}}{h!^s(h+1)^\sigma(h+2)^\sigma}\left(\sum_{|\beta|=h}[D^\beta u, D^\beta u\, e^{j\theta t}]_{G_\tau}\right)^{1/2},$$

now, integrating by parts the first term in the right side of (56) and using (3), we have

$$(57)\quad \sum_{|\beta|=j-2}[-(a_{ij}D^\beta u_{x_i})_{x_j}, D^\beta u\, e^{j\theta t}]_{G_\tau} =$$

$$= \sum_{|\beta|=j-2}[a_{ij}D^\beta u_{x_i}, D^\beta u_{x_j}\, e^{j\theta t}]_{G_\tau} \leqslant$$

$$\leqslant \lambda \sum_{|\beta|=j-2}\sum_{i=1}^{n}[D^\beta u_{x_i}, D^\beta u_{x_i}\, e^{j\theta t}]_{G_\tau} \leqslant n\lambda E_j$$

hence, we deduce that

$$(58)\quad \left(\sum_{|\beta|=j-2}[K_\beta, K_\beta e^{j\theta t}]_{G_\tau}\right)^{1/2} \leqslant$$

$$\leqslant C_8\, j(n\lambda)^{1/2}\sqrt{E_j} + C_8 j!^s \sum_{h=0}^{j-2} \frac{(\Lambda e^{\theta\tau/2})^{j-h-1}}{h!^s(h+1)^\sigma(h+2)^\sigma}\sqrt{E_{h+1}}\,.$$

Taking inequalities (53), (54), (55), (58) into account and recalling the definitions (8), (9) we obtain the estimate (52). Q.E.D.

Putting together the results of Lemma 1 and Lemma 2, we have the following estimate:

LEMMA 3. *Let $u(t, x)$ be a regular solution of* (1′), (2′) (*we assume $u(t, x) \in \boldsymbol{C}^2([0, T); \boldsymbol{H}^\infty(\boldsymbol{R}^n))$ as before*); *then, taking θ_0 as in Lemma* 1 (*see* (43)), *for any $\theta \geqslant \theta_0$ and $\Lambda > \Lambda_0$, there exists a constant*

$C = C(n, A, B, C_0, \Lambda_0, \Lambda, \theta, \tau, s)$ *such that*

(59) $$\frac{1}{2}\frac{d}{d\tau}F_j \leq Cj!^s(j+1)^\sigma \sum_{h=0}^{j-1} \frac{(\Lambda e^{\theta\tau/2})^{j-h}\sqrt{F_{h+1}}}{h!^s(h+1)^\sigma(h+2)^\sigma}\sqrt{F_j} +$$

$$+2\sum_{|\alpha|=j-1}[D^\alpha\widetilde{g}, D^\alpha w e^{j\theta t}]_{G_\tau} + \sum_{|\beta|=j-2}[D^\beta\widetilde{g}, D^\beta u_t e^{j\theta t}]_{G_\tau} \qquad (j \geq 2).$$

PROOF. Observing that $F_1(\tau) \equiv E_1(\tau)$ and that $F_j(\tau) = E_j(\tau) + \widetilde{E}_j(\tau)$, for $j \geq 2$, the proof follows immediately from (28) and (52). Q.E.D.

3. Gevrey estimates for the semilinear equation $L(u) = g(t, x) + f(t, x, u)$.

As seen in the introduction, we can confine ourselves to the case of Pb. (14.1). Hence, we consider here a regular solution $u(t, x)$ of the following *semilinear* Cauchy problem

(60) $$L(u) = g(t, x) + f(t, x, u), \qquad u(0, x) = u_t(0, x) = 0,$$

where $L(u)$ is defined as in (1) and satisfies hypotheses (3), (4) and (5); the function $g(t, x): [0, T] \times \mathbf{R}^n \to \mathbf{R}$ belongs to $C^0([0, T]; \gamma_{L^2}^{(s)}(\mathbf{R}^n))$, with $\gamma_{L^2}^{(s)}(\mathbf{R}^n)$ the space defined by the condition

$$v(x) \in \gamma_{L^2}^{(s)}(\mathbf{R}^n) \Leftrightarrow \exists C_v, \Lambda_v \geq 0: \|D^\alpha v(x)\|_{L^2(\mathbf{R}^n)} \leq C_v \Lambda_v^{|\alpha|}|\alpha|!^s \quad (\forall \alpha \in \mathbf{N}^n).$$

As to the nonlinear term, we shall assume that $f(t, x, u): [0, T] \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ belongs to the space $C^0([0, T]; \gamma_{L^\infty}^{(s)}(\mathbf{R}^{n+1}))$ and vanishes for $u = 0$; more precisely, $f(t, x, u)$ will satisfy the following assumptions:

(61) $$\begin{cases} f(t, x, 0) = 0, & \forall (t, x) \in [0, T] \times \mathbf{R}^n, \\ |D_x^\alpha \partial_u^\nu f(t, x, u)| \leq C_f M^{|\alpha|} P^\nu |\alpha|!^s \nu!^s & \text{in } [0, T] \times \mathbf{R}^n \times \mathbf{R}, \end{cases}$$

for some constants $C_f, M, P \geq 0$. Introducing the notations

(62) $$\mathcal{E}_j(f) = \sum_{|\alpha|=j-1}[D^\alpha f - D_x^\alpha f, D^\alpha w e^{j\theta t}]_{G_\tau}, \qquad (j \geq 1),$$

(63) $$\widetilde{\mathcal{E}}_j(f) = \sum_{|\beta|=j-2}[D^\beta f - D_x^\beta f, D^\beta u_t e^{j\theta t}]_{G_\tau}, \qquad (j \geq 2),$$

and taking $\theta \geqslant \theta_0$, $\Lambda > \Lambda_0$, we can rewrite the estimate (59) (for eq. (60)) in the following form

$$(64) \quad \frac{1}{2}\frac{d}{d\tau}F_j \leqslant$$
$$\leqslant Cj!^s(j+1)^\sigma \sum_{h=0}^{j-1} \frac{(\Lambda e^{\theta\tau/2})^{j-h}\sqrt{F_{h+1}}}{h!^s(h+1)^\sigma(h+2)^\sigma}\sqrt{F_j} + 2\mathcal{E}_j(f) + \widetilde{\mathcal{E}}_j(f) +$$
$$+2\sum_{|\alpha|=j-1}[D^\alpha g + D^\alpha_x f, D^\alpha w\, e^{j\theta t}]_{G_\tau} +$$
$$+\sum_{|\beta|=j-2}[D^\beta g + D^\beta_x f, D^\beta u_t\, e^{j\theta t}]_{G_\tau} \qquad (j \geqslant 2).$$

To begin with, thanks to the assumptions on $g(t, x)$ we have

$$(65) \qquad \|D^\alpha g(t,\cdot)\|_{L^2(\mathbf{R}^n)} \leqslant C_g M_g^{|\alpha|}\,|\alpha|!^s,$$

for the constants C_g, $M_g \geqslant 0$, hence, we easily deduce that for $\mathfrak{M} > M_g\, e^{\theta\tau/2}$ there exists a constant $\widetilde{C}_g \geqslant 0$, such that

$$(66) \quad 2\sum_{|\alpha|=j-1}[D^\alpha g, D^\alpha w\, e^{j\theta t}]_{G_\tau} +$$
$$+\sum_{|\beta|=j-2}[D^\beta g, D^\beta u_t\, e^{j\theta t}]_{G_\tau} \leqslant \widetilde{C}_g \mathfrak{M}_g^j j!^s \sqrt{F_j}.$$

Moreover, using (61), we have

$$(67) \qquad |D^\alpha_x f(t, x, u)| \leqslant C_f M^{|\alpha|} L\,|\alpha|!^s\,|u|$$

thus, proceeding in the same way as in the estimate (66), we find that

$$(68) \quad 2\sum_{|\alpha|=j-1}[D^\alpha_x f(t, x, u), D^\alpha w\, e^{j\theta t}]_{G_\tau} +$$
$$+\sum_{|\beta|=j-2}[D^\beta_x f(t, x, u), D^\beta u_t\, e^{j\theta t}]_{G_\tau} \leqslant \widetilde{C}_f \mathfrak{M}_f^j j!^s \sqrt{F_1}\sqrt{F_j},$$

for the constants $\widetilde{C}_f \geqslant 0$ and $\mathfrak{M}_f > M\, e^{\theta\tau/2}$.

Estimates for $\mathcal{E}_j(f)$ and $\widetilde{\mathcal{E}}_j(f)$.

We recall here that, by Leibniz' formula we have, for $|\alpha| > 0$,

$$(69) \quad D^\alpha f(t, x, u) = D^\alpha_x(t, x, u) +$$
$$+\sum_{\substack{1\leqslant \nu \leqslant |\mu| \\ \mu \leqslant \alpha}} \binom{\alpha}{\mu} \frac{D_x^{\alpha-\mu}\partial_u^\nu f(t, x, u)}{\nu!} \sum_{\substack{\beta_1+\ldots+\beta_\nu=\mu \\ 0<|\beta_i|}} \frac{\mu!}{\beta_1!\ldots\beta_\nu!} D^{\beta_1}u\ldots D^{\beta_\nu}u.$$

Since,

$$\binom{\alpha}{\mu} \leqslant \frac{|\alpha|!}{|\alpha-\mu|!|\mu|!} \qquad \text{and} \qquad \frac{\mu!}{\beta_1!\ldots\beta_\nu!} \leqslant \frac{|\mu!|}{|\beta_1|!\ldots|\beta_\nu|!},$$

if $\mu = \beta_1 + \ldots + \beta_\nu$, we deduce the following estimate for $\mathcal{E}_j(f)$

$$(70) \qquad \mathcal{E}_j(f) \leqslant (j-1)! \sum_{|\alpha|=j-1} \Bigg[\sum_{\substack{1 \leqslant \nu \leqslant |\mu| \\ \mu \leqslant \alpha}} \frac{|D_x^{\alpha-\mu} \partial_u^\nu f(t,x,u)|}{(j-1-|\mu|)!\nu!} \cdot$$

$$\cdot \sum_{\substack{\beta_1+\ldots+\beta_\nu=\mu \\ 0<|\beta_i|}} \frac{|D^{\beta_1}u|}{|\beta_1|!} \cdots \frac{|D^{\beta_\nu}u|}{|\beta_\nu|!}, |D^\alpha w| e^{j\theta t} \Bigg]_{G_\tau} .$$

Now, putting $\eta = \alpha - \mu$ and changing the order of summation over the indices ν, μ and η we can rewrite the sum in the right hand side of (70) in the following way

$$(71) \qquad \sum_{1 \leqslant \nu \leqslant |\mu| \leqslant j-1} \sum_{|\eta|=j-1-|\mu|} \Bigg[\frac{|D_x^\eta \partial_u^\nu f(t,x,u)|}{(j-1-|\mu|)!\nu!} \cdot$$

$$\cdot \sum_{\substack{\beta_1+\ldots+\beta_\nu=\mu \\ 0<|\beta_i|}} \frac{|D^{\beta_1}u| \ldots |D^{\beta_\nu}u|}{|\beta_1|! \ldots |\beta_\nu|!}, |D^{\mu+\eta} w| e^{j\theta t} \Bigg]_{G_\tau} .$$

Defining the integers, $h = |\mu|$ and $h_i = |\beta_i|$, for $1 \leqslant i \leqslant \nu$, we observe that

$$\sum_{|\mu|=h} \sum_{\substack{\beta_1+\ldots+\beta_\nu=\mu \\ 0<|\beta_i|}} = \sum_{\substack{h_1+\ldots+h_\nu=h \\ 0<h_i}} \sum_{|\beta_1|=h_1} \cdots \sum_{|\beta_\nu|=h_\nu},$$

moreover for every nonnegative symmetric function ξ defined on a symmetric set $\mathcal{B} \subseteq (N^n)^\nu$,

$$\sum_{(\beta_1,\ldots,\beta_\nu)\in\mathcal{B}} \xi(\beta_1,\ldots,\beta_\nu) \leqslant \nu \sum_{\substack{(\beta_1,\ldots,\beta_\nu)\in\mathcal{B} \\ |\beta_i| \leqslant |\beta_\nu|}} \xi(\beta_1,\ldots,\beta_\nu)$$

thus, we have

$$(72) \quad \mathcal{E}_j(f) \leqslant (j-1)! \sum_{1 \leqslant \nu \leqslant h \leqslant j-1} \sum_{|\eta| = j-1-h} \cdot$$

$$\cdot \left[\frac{|D_x^\eta \partial_u^\nu f(t,x,u)|}{(j-h-1)!\nu!} \nu \sum_{|\mu|=h} \sum_{\substack{\beta_1+\dots+\beta_\nu=\mu \\ 0<|\beta_i|\leqslant|\beta_\nu}} \frac{|D^{\beta_1}u| \dots |D^{\beta_\nu}u|}{|\beta_1|! \dots |\beta_\nu|!}, |D^{\mu+\eta}w| e^{j\theta t} \right]_{G_\tau} \leqslant$$

$$\leqslant (j-1)! \sum_{1 \leqslant \nu \leqslant h \leqslant j-1} \sum_{|\eta| = j-1-h} \cdot$$

$$\cdot \left[\frac{|D_x^\eta \partial_u^\nu f(t,x,u)|}{(j-h-1)!\nu!} \nu \sum_{\substack{h_1+\dots+h_\nu=h \\ 0<h_i\leqslant h_\nu}} \sum_{|\beta_1|=h_1} \dots \sum_{|\beta_\nu|=h_\nu} \frac{|D^{\beta_1}u| \dots |D^{\beta_\nu}u|}{h_1! \dots h_\nu!}, \right.$$

$$\left. |D^{\beta_1+\dots+\beta_\nu+\eta}w| e^{j\theta t} \right]_{G_\tau} .$$

Taking (61) into account, since $|\eta| = j-h-1$, it follows that

$$\frac{|D_x^\eta \partial_u^\nu f(t,x,u)|}{(j-h-1)!\nu!} \leqslant C_f M^{j-h-1} P^\nu (j-h-1)!^\sigma \nu!^\sigma$$

hence, substituting in (72) and changing the order of summation again,

$$(73) \quad \mathcal{E}_j(f) \leqslant (j-1)! C_f \sum_{1 \leqslant \nu \leqslant h \leqslant j-1} M^{j-h-1} P^\nu (j-h-1)!^\sigma \nu!^\sigma \nu \cdot$$

$$\cdot \left[\sum_{\substack{h_1+\dots+h_\nu=h \\ 0<h_i\leqslant h_\nu}} \sum_{|\eta|=j-h-1} \sum_{|\beta_1|=h_1} \dots \sum_{|\beta_\nu|=h_\nu} \frac{|D^{\beta_1}u| \dots |D^{\beta_\nu}u|}{h_1! \dots h_\nu!}, \right.$$

$$\left. |D^{\beta_1+\dots+\beta_\nu+\eta}w| e^{j\theta t} \right]_{G_\tau} .$$

Now, applying Schwartz' inequality and the definition of $E_j(\tau)$ we have

$$(74) \quad \sum_{|\eta|=j-h-1} \sum_{|\beta_\nu|=h_\nu} [|D^{\beta_1}u| \dots |D^{\beta_\nu}u|, |D^{\beta_1+\dots+\beta_\nu+\eta}w| e^{j\theta t}]_{G_\tau} \leqslant$$

$$\leqslant \sum_{|\eta|=j-h-1} \left(\sum_{|\beta_\nu|=h_\nu} [|D^{\beta_1}u| \dots |D^{\beta_\nu}u|, |D^{\beta_1}u| \dots |D^{\beta_\nu}u| e^{j\theta t}]_{G_\tau} \right)^{1/2} .$$

$$\cdot\left(\sum_{|\beta_\nu|=h_\nu}[D^{\beta_1+\ldots+\beta_\nu+\eta}w,\ D^{\beta_1+\ldots+\beta_\nu+\eta}w\,e^{j\theta t}]_{G_\tau}\right)^{1/2}\leqslant$$

$$\leqslant\frac{\sqrt{E_j}}{j}\left(\sum_{|\beta_\nu|=h_\nu}[\,|D^{\beta_1}u|\ldots|D^{\beta_\nu}u|,\ |D^{\beta_1}u|\ldots|D^{\beta_\nu}u|^{j\theta t}]_{G_\tau}\right)^{1/2}\cdot\sum_{|\eta|=j-h-1}1,$$

hence, using the elementary estimate

$$\sum_{|\eta|=k}1=\begin{bmatrix}n+k-1\\ n-1\end{bmatrix}\leqslant(k+1)^{n-1}\qquad(k\geqslant 0),$$

from (73) and (74) we deduce that

$$(75)\qquad \mathcal{E}_j(f)\leqslant(j-1)!\,C_f\,\frac{\sqrt{E_j}}{j}\sum_{1\leqslant\nu\leqslant h\leqslant j-1}\cdot$$

$$\cdot M^{j-h-1}P^\nu(j-h-1)!^\sigma\nu!^\sigma\nu(j-h)^{n-1}\sum_{\substack{h_1+\ldots+h_\nu=h\\ 0<h_i\leqslant h_\nu}}\ \sum_{|\beta_1|=h_1\ldots}\ \sum_{|\beta_{\nu-1}|=h_{\nu-1}}\cdot$$

$$\cdot\left(\sum_{|\beta_\nu|=h_\nu}\left[\frac{|D^{\beta_1}u|\ldots|D^{\beta_\nu}u|}{h_1!\ldots h_\nu!},\ \frac{|D^{\beta_1}u|\ldots|D^{\beta_\nu}u|}{h_1!\ldots h_\nu!}\,e^{j\theta t}\right]_{G_\tau}\right)^{1/2}.$$

Since in (75)

$$j=(j-h-1)+h_1+\ldots+h_{\nu-1}+(h_\nu+1)$$

we have

$$(76)\qquad \sum_{|\beta_\nu|=h_\nu}[\,|D^{\beta_1}u|\ldots|D^{\beta_\nu}u|,\ |D^{\beta_1}u|\ldots|D^{\beta_\nu}u|\,e^{j\theta t}]_{G_\tau}\leqslant$$

$$\leqslant e^{(j-h-1)\theta_\tau}\cdot\sup_{0\leqslant t\leqslant\tau}\|D^{\beta_1}u(t,\cdot)\|^2_{L^\infty}\,e^{h_1\theta t}\ldots\cdot$$

$$\cdot\sup_{0\leqslant t\leqslant\tau}\|D^{\beta_{\nu-1}}u(t,\cdot)\|^2_{L^\infty}\,e^{h_{\nu-1}\theta t}\sum_{|\beta_\nu|=h_\nu}[D^{\beta_\nu}u,\ D^{\beta_\nu}u\,e^{(h_\nu+1)\theta t}]_{G_\tau}.$$

To estimate the L^∞*-norm* of $D^{\beta_i}u(t,\cdot)$, using the Sobolev embedding theorem, we prove the following lemma:

LEMMA 4. Let $u(t,x)$ *be a regular function* (*to be precise in the following we will assume* $u(t,x)\in \boldsymbol{C}^1([0,T);\boldsymbol{H}^\infty(\boldsymbol{R}^n))$) *such that* $u(0,x)=0$, *then defining for* $j\geqslant 1$, $F_j(\tau)$ *as in* (8), (9) *we can find a* *constant* $C = C(n)$ *such that for any* $h \geqslant 0$,

$$(77)\quad \sum_{|\beta| = h} \sup_{0 \leqslant t \leqslant \tau} \|D^\beta u(t, \cdot)\|_{L^\infty}^2 e^{h\theta t} \leqslant$$

$$\leqslant C(n)[(1 + h\theta)(F_{h+1}(\tau) + F_{h+p+1}(\tau)) + F_{h+2}(\tau) + F_{h+p+2}(\tau)],$$

where $p = [n/2] + 1$.

Proof. Using the Sobolev embedding theorem, we have

$$(78)\quad \sup_{0 \leqslant t \leqslant \tau} \|D^\beta u(t, \cdot)\|_{L^\infty}^2 e^{h\theta t} \leqslant$$

$$\leqslant \tilde{C}(n)\left[\sup_{0 \leqslant t \leqslant \tau} \|D^\beta u(t, \cdot)\|_{L^2}^2 e^{h\theta t} + \sup_{0 \leqslant t \leqslant \tau} \sum_{|\eta| = p} \|D^\eta D^\beta u(t, \cdot)\|_{L^2}^2 e^{h\theta t}\right].$$

Deriving with respect to t, the function $t \to \|D^\beta u(t, \cdot)\|_{L^2}^2 e^{j\theta t}$, we find

$$(79)\quad \frac{d}{dt} \|D^\beta u(t, \cdot)\|_{L^2}^2 e^{j\theta t} =$$

$$= 2 e^{j\theta t} \int_{R^n} D^\beta u_t(t, x) D^\beta u(t, x)\, dx + j\theta \|D^\beta u(t, \cdot)\|_{L^2}^2 e^{j\theta t},$$

thus, integrating over $[0, t]$ the terms in the right hand side of (79) and using the assumption $u(0, x) = 0$, we deduce the inequality

$$(80)\quad \|D^\beta u(t, \cdot)\|_{L^2}^2 e^{j\theta t} \leqslant [D^\beta u_t D^\beta u_t\, e^{j\theta s}]_{G_t} + (1 + j\theta)[D^\beta u, D^\beta u\, e^{j\theta s}]_{G_t}.$$

Clearly, in the right hand side of (80) we have an increasing function of t; hence, summing for $|\beta| = j$ it follows that

$$(81)\quad \sum_{|\beta| = j} \sup_{0 \leqslant t \leqslant \tau} \|D^\beta u(t, \cdot)\|_{L^2}^2 e^{j\theta t} \leqslant \sum_{|\beta| = j} [D^\beta u_t, D^\beta u_t\, e^{j\theta t}]_{G_\tau} +$$

$$+ (1 + j\theta) \sum_{|\beta| = j} [D^\beta u, D^\beta u\, e^{j\theta t}]_{G_\tau} \leqslant F_{j+2}(\tau) + (1 + j\theta) F_{j+1}(\tau).$$

Now, (77) follows immediately from (78), applying (81) for $j = h$ and $j = h + p$, since

$$(82)\quad \sum_{|\beta| = h} \sup_{0 \leqslant t \leqslant \tau} \sum_{|\eta| = p} \|D^\eta D^\beta u(t, \cdot)\|_{L^2}^2 e^{h\theta t} \leqslant$$

$$\leqslant C'(n) \sum_{|\beta| = h + p} \sup_{0 \leqslant t \leqslant \tau} \|D^\beta u(t, \cdot)\|_{L^2}^2 e^{h\theta t}. \qquad \text{Q.E.D.}$$

To apply Lemma 4 to the estimate (76), we introduce the notations:

$$(83)\qquad \Gamma_h(\tau) \equiv \sqrt{1+h\theta}\left(\sqrt{F_{h+1}(\tau)} + \sqrt{F_{h+p+1}(\tau)}\right) + \\ + \sqrt{F_{h+2}(\tau)} + \sqrt{F_{h+p+2}(\tau)},$$

thus, for $1 \leqslant i \leqslant \nu - 1$, we have

$$(84)\qquad \sum_{|\beta_i| = h_i} \sup_{0 \leqslant t \leqslant \tau} \|D^{\beta_i} u(t, \cdot)\|_{L^\infty} e^{h_i(\theta t/2)} \leqslant \\ \leqslant (1+h_i)^{(n-1)/2} \left(\sum_{|\beta_i| = h_i} \sup_{0 \leqslant t \leqslant \tau} \|D^{\beta_i} u(t, \cdot)\|^2_{L^\infty} e^{h_i \theta t} \right)^{1/2} \leqslant \\ \leqslant C_\Gamma (1+h_i)^{(n-1)/2} \Gamma_{h_i}(\tau)$$

where $C_\Gamma = C_\Gamma(n)$ depends on the constant $C(n)$ of (77). Substituting in (76), we obtain

$$(85)\qquad \sum_{|\beta_1| = h_1} \cdots \sum_{|\beta_{\nu-1}| = h_{\nu-1}} \Bigg(\sum_{|\beta_\nu| = h_\nu} [\,|D^{\beta_1}u| \ldots |D^{\beta_\nu}u|, \\ |D^{\beta_1}u| \ldots |D^{\beta_\nu}u|\, e^{j\theta t}]_{G_\tau} \Bigg)^{1/2} \leqslant e^{(j-h-1)\theta\tau/2} \sqrt{E_{h_\nu+1}} \sum_{|\beta_1| = h_1} \cdot \\ \cdot \sup_{0 \leqslant t \leqslant \tau} \|D^{\beta_1} u(t, \cdot)\|_{L^\infty} e^{h_1(\theta t/2)} \ldots \sum_{|\beta_{\nu-1}| = h_{\nu-1}} \sup_{0 \leqslant t \leqslant \tau} \|D^{\beta_{\nu-1}} u(t, \cdot)\|^2_{L^\infty} \cdot \\ \cdot e^{h_{\nu-1}(\theta t/2)} \leqslant e^{(j-h-1)\theta\tau/2} C_\Gamma^{\nu-1} \cdot \\ \cdot \sqrt{E_{h_\nu+1}(\tau)} (1+h_1)^{(n-1)/2} \Gamma_{h_1}(\tau) \ldots (1+h_{\nu-1})^{(n-1)/2} \Gamma_{h_{\nu-1}}(\tau)$$

thus, we have the following estimate of $\mathcal{E}_j(\tau)$:

$$(86)\qquad \mathcal{E}_j(f) \leqslant (j-1)!\, C_f \frac{\sqrt{E_j}}{j} \cdot \\ \cdot \sum_{1 \leqslant \nu \leqslant h \leqslant j-1} (M e^{\theta\tau/2})^{j-h-1} P^\nu (j-h-1)!^\sigma \nu!^\sigma \nu (j-h)^{n-1} C_\Gamma^{\nu-1} \cdot \\ \cdot \sum_{\substack{h_1 + \ldots + h_\nu = h \\ 0 < h_i \leqslant h_\nu}} (h_1+1)^{(n-1)/2} \frac{\Gamma_{h_1}}{h_1!} \ldots (h_{\nu-1}+1)^{(n-1)/2} \frac{\Gamma_{h_{\nu-1}}}{h_{\nu-1}!} \frac{\sqrt{E_{h_\nu+1}}}{h_\nu!}.$$

By the same methods we can estimate $\tilde{\mathcal{E}}_j(\tau)$, we finally obtain:

$$(87)\quad \tilde{\mathcal{E}}_j(f) \leq (j-2)!\, C_f \sqrt{\tilde{E}_j}\, e^{\theta\tau/2} \cdot$$

$$\cdot \sum_{1 \leq \nu \leq h \leq j-2} (M e^{\theta\tau/2})^{j-h-2} P^{\nu} (j-h-2)!^{\sigma} \nu!^{\sigma} \nu (j-h-1)^{n-1} C_{\Gamma}^{\nu-1} \cdot$$

$$\cdot \sum_{\substack{h_1 + \ldots + h_\nu = h \\ 0 < h_i \leq h_\nu}} (h_1+1)^{(n-1)/2} \frac{\Gamma_{h_1}}{h_1!} \ldots (h_{\nu-1}+1)^{(n-1)/2} \frac{\Gamma_{h_{\nu-1}}}{h_{\nu-1}!} \frac{\sqrt{E_{h_\nu+1}}}{h_\nu!}\, .$$

REMARK 1. To conclude, we observe that, taking $\mathfrak{M}$, $\mathfrak{L}$,

$$(88)\qquad \mathfrak{M} > M \cdot e^{\theta\tau/2}\,, \qquad \mathfrak{L} > P C_{\Gamma}\,,$$

we can find a constant $C = C(n, M, P, C_\Gamma, \theta, \tau)$, such that, for $j - h \geq 0$, $v \geq 0$, the following inequalities hold

$$(89)\qquad (M e^{\theta\tau/2})^{j-h} (j-h)^{n-1} P^{\nu} \nu C_{\Gamma}^{\nu-1} \leq C \mathfrak{M}^{j-h} \mathfrak{L}^{\nu}\, .$$

Hence, we can prove the following lemma.

LEMMA 5. *Let $u(t, x)$ be a regular solution of Pb. (60); then, taking $\theta \geq \theta_0$ (as in Lemma 3), and*

$$(90)\qquad \mathfrak{M} > \max\{\Lambda_0, M, M_g\} \cdot e^{\theta\tau/2}\,, \qquad \mathfrak{L} > P C_{\Gamma} \sqrt{\theta}$$

we can find a constant $\mathcal{C} = \mathcal{C}(n, A, B, M, M_g, P, C_f, C_g, C_0, \Lambda_0, \mathfrak{M}, \mathfrak{L}, \theta, \tau, s)$, such that for $j \geq 1$ the following estimate holds

$$(91)\quad \frac{d}{d\tau}\sqrt{F_j} \leq \mathcal{C} j!^s (j+1)^{\sigma} \sum_{h=0}^{j-1} \frac{\mathfrak{M}^{j-h}\sqrt{F_{h+1}}}{h!^s (h+1)^{\sigma} (h+2)^{\sigma}} + \mathcal{C}\mathfrak{M}^j j!^s +$$

$$+ \mathcal{C}(j-2)! \sum_{1 \leq \nu \leq h \leq j-1} \mathfrak{M}^{j-h} \mathfrak{L}^{\nu} [(j-h-1)!\nu!]^{\sigma} \cdot$$

$$\cdot \sum_{\substack{h_1 + \ldots + h_\nu = h \\ 0 < h_i \leq h_\nu}} \frac{\tilde{\Gamma}_{h_1}}{h_1!} \cdots \frac{\tilde{\Gamma}_{h_{\nu-1}}}{h_{\nu-1}!} \frac{\sqrt{E_{h_\nu+1}}}{h_\nu!}\,,$$

where $\sigma = s - 1$, *and* $\widetilde{\Gamma}_h$ *is defined for* $h \geq 0$, *by*

$$(92)\qquad \widetilde{\Gamma}_h = (1+h)^{n/2}\left[\sqrt{F_{h+1}} + \sqrt{F_{h+p+1}} + \frac{\sqrt{F_{h+2}} + \sqrt{F_{h+p+2}}}{(1+h)^{1/2}}\right],$$

with $p = [n/2] + 1$.

PROOF. To begin with, we put together (64), (66), (68) and the estimates (86), (87) for $\mathcal{E}_j(\tau)$ and $\widetilde{\mathcal{E}}_j(\tau)$; thus, we obtain an expression which can be divided by $\sqrt{F_j}$. Then, taking (90) into account, and proceeding in the same way as in Remark 1, the estimate (91) follows immediately. Q.E.D.

4. Estimates for the energy of Gevrey type.

Following [S2] (see also [J1]), we define the energy functions

$$(93)\qquad \mathcal{E}^N(\tau) = \varrho(\tau) + \sum_{j=k+1}^{N} \frac{\varrho(\tau)^{j-k}}{j!^s} j^{ks} \sqrt{F_j(\tau)}, \qquad (N \geq k+1)$$

where $\varrho(\tau)$ is a certain strictly positive, decreasing function which will be defined in the following; $k \geq 0$ is an integer greater than or equal to

$$(94)\qquad k_0 \equiv \left[\frac{n}{2}\right] + 3 + \frac{n-1}{2s}.$$

Differentiating (93) termwise, we have

$$(95)\qquad \frac{d}{d\tau}\mathcal{E}^N = \varrho' + \sum_{j=k+1}^{N} \frac{\varrho^{j-k-1}}{(j-1)!^s} j^{ks-\sigma} \frac{j-k}{j} \varrho' \sqrt{F_j} + \sum_{j=k+1}^{N} \frac{\varrho(\tau)^{j-k}}{j!^s} j^{ks} (\sqrt{F_j})', \qquad (N \geq k+1).$$

In order to estimate $(\mathcal{E}^N)'$, we shall introduce (91) into (95). We obtain the following terms

$$(96)\qquad I^N = \mathcal{C} \sum_{j=k+1}^{N} \varrho(\tau)^{j-k} \mathcal{M}^j j^{ks},$$

$$(97)\qquad II^N = \mathcal{C} \sum_{j=k+1}^{N} \varrho(\tau)^{j-k} j^{ks} (j+1)^{\sigma} \sum_{h=0}^{j-1} \frac{\mathcal{M}^{j-h} \sqrt{F_{h+1}}}{h!^s (h+1)^{\sigma} (h+2)^{\sigma}},$$

$$(98) \quad III^N = \mathcal{C} \sum_{j=k+1}^{N} \frac{\varrho(\tau)^{j-k}}{(j-1)!^\sigma} j^{ks-\sigma-2} \sum_{1 \leq \nu \leq h \leq j-1} \frac{\mathcal{M}^{j-h} \mathcal{L}^\nu}{[(j-h-1)!\nu!]^{-\sigma}} \cdot$$

$$\cdot \sum_{\substack{h_1 + \ldots + h_\nu = h \\ 0 < h_i \leq h_\nu}} \frac{\tilde{\Gamma}_{h_1}}{h_1!} \cdots \frac{\tilde{\Gamma}_{h_{\nu-1}}}{h_{\nu-1}!} \frac{\sqrt{E_{h_\nu+1}}}{h_\nu!} .$$

Estimate of I^N.

We remark that, for $\iota \in \mathbf{N}$, $\iota \geq 1$

$$(99) \qquad \sum_{j=\iota}^{\infty} \delta^j j^k \leq C_k \delta^\iota \iota^k , \qquad \text{if} \quad 0 \leq \delta \leq \frac{1}{2}$$

hence, taking

$$(100) \qquad \varrho \leq \min \left\{ \frac{1}{2}, \frac{1}{2\mathcal{M}} \right\},$$

we obtain the estimate:

$$(101) \qquad I^n = C\varrho^{-k} \sum_{j=k+1}^{N} (\varrho \mathcal{M})^j j^{ks} \leq \mathcal{C} c(k, s, \mathcal{M}) \varrho .$$

Estimate of II^N.

To estimate II^N we introduce the notation

$$(102) \qquad \sqrt{\mathcal{F}_j(\tau)} = \max_{1 \leq i \leq j} \sqrt{F_i(\tau)}, \qquad (j \geq 1),$$

moreover, in the following we indicate with $c(k)$ various constants (which may depend also on $\mathcal{M}$, $\mathcal{L}$, s) obtained applying inequalities like (99). Now, if (100) holds, we find the estimate

$$(103) \quad II^N = \mathcal{C} c(k) \sqrt{\mathcal{F}_k} +$$

$$+ \mathcal{C} \sum_{h=k}^{N-1} \frac{\mathcal{M}^{-h} \sqrt{F_{h+1}}}{h!^s (h+1)^\sigma (h+2)^\sigma} \varrho^{-k} \cdot \sum_{j=h+1}^{N} \varrho^j \mathcal{M}^j j^{ks} (j+1)^\sigma \leq$$

$$\leq \mathcal{C} c(k) \varrho \sqrt{\mathcal{F}_k} + \mathcal{C} c(k) \sum_{j=k}^{N-1} \frac{\varrho^{h+1-k}}{h!^s} (h+1)^{ks-\sigma} \sqrt{F_{h+1}} .$$

Estimate of III^N.

To estimate III^N we put

$$III^N = III_1^N + III_2^N \tag{104}$$

where III_1^N groups all the terms in which $h_\nu < k$, and III_2^N the terms with $h_\nu \geq k$.

For the terms of III_1^N we have

$$\tilde{\Gamma}_{h_i} \leq 4(h_\nu + 1)^{n/2} \sqrt{\mathcal{F}_{h_\nu + p + 2}} \leq 4\, k^{n/2} \sqrt{\mathcal{F}_{k+p+1}}, \quad (1 \leq i \leq \nu - 1) \tag{105}$$

hence, we easily deduce that

$$\sum_{\substack{h_1 + \ldots + h_\nu = h \\ 0 < h_i \leq h_\nu < k}} \frac{\tilde{\Gamma}_{h_1}}{h_1!} \cdots \frac{\tilde{\Gamma}_{h_\nu - 1}}{h_{\nu-1}!} \frac{\sqrt{E_{h_\nu + 1}}}{h_\nu!} \leq \mathcal{L}_1^\nu (\sqrt{\mathcal{F}_{k+p+1}})^\nu, \tag{106}$$

for the constant $\mathcal{L}_1 = \mathcal{L}_1(k)$. Thus, using (106) and observing that for $\sigma \geq 0$, $1 \leq \nu \leq h \leq j - 1$,

$$\frac{(j-h-1)!^\sigma \nu!^\sigma}{(j-1)!^\sigma} \leq 1$$

we can estimate III_1^N with

$$\mathcal{C} \sum_{j=k+1}^{N} \varrho^{j-k} j^{ks-\sigma-2} \sum_{1 \leq \nu \leq h \leq j-1} \mathcal{M}^{j-h} (\mathcal{L}\mathcal{L}_1)^\nu (\sqrt{\mathcal{F}_{k+p+1}})^\nu. \tag{107}$$

Now, we consider separately the terms in (107) in which $h < k$ and those with $h \geq k$; if (100) holds, it follows that

$$\begin{aligned} III_1^N &\leq \mathcal{C} c(k) \varrho \sqrt{\mathcal{F}_{k+p+1}} (1 + \sqrt{\mathcal{F}_{k+p+1}})^{k-2} + \\ &\quad + \mathcal{C} \sum_{\substack{1 \leq \nu \leq h \leq N-1 \\ h \geq k}} \frac{(\mathcal{L}\mathcal{L}_1)^\nu}{\mathcal{M}^h \varrho^k} (\sqrt{\mathcal{F}_{k+p+1}})^\nu \sum_{j=h+1}^{N} (\varrho \mathcal{M})^j j^{ks} \leq \\ &\leq \mathcal{C} c(k) \varrho \sqrt{\mathcal{F}_{k+p+1}} (1 + \sqrt{\mathcal{F}_{k+p+1}})^{k-2} + \\ &\quad + \mathcal{C} c(k) \sum_{h=k+1}^{N} \varrho^{h-k} h^{ks} \sum_{\nu=1}^{h-1} (\mathcal{L}\mathcal{L}_1)^\nu (\sqrt{\mathcal{F}_{k+p+1}})^\nu. \end{aligned} \tag{108}$$

As before, we consider separately the terms with $\nu < k$ and the terms

$\nu \geq k$. We have

(109) $$III_1^N \leq \mathcal{C} c(k) \varrho \sqrt{\mathcal{F}_{k+p+1}} (1 + \sqrt{\mathcal{F}_{k+p+1}})^{k-2} + + \mathcal{C} c(k) \sum_{\nu=k}^{N-1} (\mathcal{L}\mathcal{L}_1)^\nu (\sqrt{\mathcal{F}_{k+p+1}})^\nu \varrho^{-k} \sum_{h=\nu+1}^{N} \varrho^h h^{ks},$$

hence, if (100) holds, we deduce that

(110) $$\sum_{\nu=k}^{N-1} (\mathcal{L}\mathcal{L}_1)^\nu (\sqrt{\mathcal{F}_{k+p+1}})^\nu \varrho^{-k} \sum_{h=\nu+1}^{N} \varrho^h h^{ks} \leq \leq c(k) \varrho \sum_{\nu=k}^{N-1} (\mathcal{L}\mathcal{L}_1)^\nu (\sqrt{\mathcal{F}_{k+p+1}})^\nu \varrho^{\nu-k} (\nu+1)^{ks}.$$

Finally, assuming

(111) $$\varrho \leq \frac{1}{2 \mathcal{L}\mathcal{L}_1 \sqrt{\mathcal{F}_{k+p+1}}},$$

we have

(112) $$III_1^N \leq \mathcal{C} C(k) \varrho \sqrt{\mathcal{F}_{k+p+1}}$$

where $C(k) = C(k, s, \mathcal{M}, \mathcal{L}\mathcal{L}_1, \sqrt{\mathcal{F}_{k+p+1}})$.

To estimate III_2^N we introduce the following notations

(113) $$\begin{cases} \eta(j) = \dfrac{\varrho(\tau)^{j-k}}{j!^s} j^{ks} \sqrt{F_j(\tau)}, & \text{for } j \geq k+1; \\ \eta(j) = \dfrac{\varrho}{k}, & \text{for } 1 \leq j \leq k, \end{cases}$$

moreover, we remark that, for $h_1, \ldots, h_\nu \geq 1$,

(114) $$\frac{(j-h-1)!^\sigma h_1!^\sigma \ldots h_\nu!^\sigma}{(j-1)!^\sigma} \nu!^\sigma \leq 1, \qquad (h = h_1 + \ldots + h_\nu \leq j-1)$$

hence

(115) $$III_2^N \leq \mathcal{C} \sum_{j=k+1}^{N} \varrho^{j-k} j^{ks-\sigma-2} \sum_{1 \leq \nu \leq h \leq j-1} \mathcal{M}^{j-h} \mathcal{L}^\nu \cdot \cdot \sum_{\substack{h_1 + \ldots + h_\nu = h \\ 0 < h_i \leq h_\nu,\, k \leq h_\nu}} \frac{\widetilde{\Gamma}_{h_1}}{h_1!^s} \cdots \frac{\widetilde{\Gamma}_{h_{\nu-1}}}{h_{\nu-1}!^s} \frac{\sqrt{F_{h_\nu+1}}}{h_\nu!^s} \qquad (h_\nu \geq k).$$

Recalling the definition of $\widetilde{\Gamma}_h$ (see (92)), and observing that, for $r \geqslant 1$

$$\frac{\sqrt{F_{h_i + r}}}{h_i!^s} \varrho^{h_i} \leqslant \eta(h_i + r)\, \varrho^{k - r} \frac{(h_i + r)^s \ldots (h_i + 1)^s}{(h_i + r)^{ks}} \text{ if } h_i + r > k, \tag{116}$$

$$\frac{\sqrt{F_{h_i + r}}}{h_i!^s} \varrho^{h_i} \leqslant \eta(h_i + r)\, \varrho^{h_i - 1} k \frac{\sqrt{\mathcal{F}_k}}{h_i!^s}, \qquad \text{if } 1 \leqslant h_i + r \leqslant k, \tag{117}$$

we easily find that taking $\varrho \leqslant 1$ and

$$k \geqslant k_0 \equiv p + 2 + \frac{n - 1}{2s}, \tag{94}$$

then, there exists a constant $\mathcal{L}_\eta = \mathcal{L}_\eta(k, \sqrt{\mathcal{F}_k})$ (which does not depend on h_i), such that

$$\frac{\widetilde{\Gamma}_{h_i}}{h_i!^s} \varrho^{h_i} \leqslant \frac{\mathcal{L}_\eta}{4} \sum_{j=1}^{2} \eta(h_i + j) + \eta(h_i + p + j) \leqslant \tag{118}$$

$$\leqslant \frac{\mathcal{L}_\eta}{4} [\eta(h_i + 1) + \eta(h_i + 2) + \eta(h_i + p + 1) + \eta(h_i + p + 2)]$$

for $1 \leqslant i \leqslant \nu - 1$. Since in (115), $h_\nu \geqslant k \geqslant p + 2$, we have

$$h_i + p + 2 \leqslant h_i + h_\nu \leqslant h \leqslant N - 1, \tag{119}$$

hence, having $\mathcal{E}^N = \eta(1) + \ldots + \eta(N)$, it follows that

$$\varrho^{h - k} \sum_{\substack{h_1 + \ldots + h_\nu = h \\ 0 < h_i \leqslant h_\nu,\, k \leqslant h_\nu}} \frac{\widetilde{\Gamma}_{h_1}}{h_1!^s} \cdots \frac{\widetilde{\Gamma}_{h_{\nu - 1}}}{h_{\nu - 1}!^s} \frac{\sqrt{F_{h_\nu + 1}}}{h_\nu!^s} = \tag{120}$$

$$= \sum_{h_\nu = k}^{h - \nu + 1} \frac{\sqrt{F_{h_\nu + 1}}}{h_\nu!^s} \varrho^{h_\nu - k} \sum_{\substack{h_1 + \ldots + h_{\nu - 1} = h - h_\nu \\ 0 < h_i \leqslant h_\nu}} \frac{\widetilde{\Gamma}_{h_1}}{h_1!^s} \varrho^{h_1} \cdots \frac{\widetilde{\Gamma}_{h_{\nu - 1}}}{h_{\nu - 1}!^s} \varrho^{h_\nu - 1} \leqslant$$

$$\leqslant \sum_{h_\nu = k}^{h - \nu + 1} \frac{\sqrt{F_{h_\nu + 1}}}{h_\nu!^s} \varrho^{h_\nu - k} \cdot \mathcal{L}_\eta^{\nu - 1} (\mathcal{E}^N)^{\nu - 1}.$$

Thus, we have

$$(121)\qquad III_2^N \leqslant \mathcal{C} \sum_{j=k+1}^{N} j^{ks-\sigma-2} \sum_{\substack{1\leqslant \nu \leqslant h-k+1\\ k\leqslant h\leqslant j-1}} (\mathcal{M}\varrho)^{j-h} \cdot$$

$$\cdot \mathcal{L}^{\nu}(\mathcal{L}_{\eta}\mathcal{E}^N)^{\nu-1} \sum_{h_\nu=k}^{h-\nu+1} \frac{\varrho^{h_\nu-k}\sqrt{F_{h_\nu+1}}}{h_\nu!^s}.$$

Moreover, taking into account that

$$(122)\qquad \frac{1}{(h_\nu+1)^{ks-\sigma}} \leqslant \frac{\nu^{ks-\sigma}}{h^{ks-\sigma}},$$

since $h_i \leqslant h_\nu$ for $1 \leqslant i \leqslant \nu$, changing the order of summation it follows that,

$$(123)\qquad III_2^N \leqslant \mathcal{C} \sum_{h_\nu=k}^{N-1} \frac{\varrho^{h_\nu-k}}{h_\nu!^s}(h_\nu+1)^{ks-\sigma}\sqrt{F_{h_\nu+1}}\,\cdot$$

$$\cdot\left\{\sum_{j=k+1}^{N} j^{ks-\sigma-2} \sum_{\substack{1\leqslant \nu\leqslant h-k+1\\ k\leqslant h\leqslant j-1}} \frac{(\mathcal{M}\varrho)^{j-h}\,\tilde{\mathcal{L}}^{\nu-1}}{h^{ks-\sigma}}(\mathcal{E}^N)^{\nu-1}\right\},$$

where $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}(\mathcal{L}, \mathcal{L}_\eta, k, s)$ satisfies

$$\tilde{\mathcal{L}}^{\nu-1} \geqslant \mathcal{L}^{\nu}\mathcal{L}_\eta^{\nu-1}\nu^{ks-\sigma}.$$

Changing again the order of summation, we have

$$(124)\qquad \sum_{j=k+1}^{N} j^{ks-\sigma-2} \sum_{\substack{1\leqslant \nu\leqslant h-k+1\\ k\leqslant h\leqslant j-1}} \frac{(\mathcal{M}\varrho)^{j-h}\tilde{\mathcal{L}}^{\nu}}{h^{ks-\sigma}}(\mathcal{E}^N)^{\nu-1} \leqslant$$

$$\leqslant \sum_{\substack{1\leqslant \nu\leqslant h-k+1\\ k\leqslant h\leqslant N-1}} \frac{\tilde{\mathcal{L}}^{\nu-1}}{h^{ks-\sigma}}(\mathcal{E}^N)^{\nu-1} \sum_{j=h+1}^{N} (\mathcal{M}\varrho)^{j-h} j^{ks-\sigma-2}$$

hence, using an inequality like (99), if (100) holds, the last sum in (124) can be estimated by

$$(125)\qquad c(k)\,\varrho\, h^{ks-\sigma-2}$$

thus, summing over h we easily see that

$$(126)\quad III_2^N \leq \mathcal{C}c(k) \sum_{j=k+1}^{N} \frac{\varrho^{j-k-1}}{(j-1)!^s} j^{ks-\sigma} \sqrt{F_j} \cdot \varrho \left\{ \sum_{\nu=0}^{N-k-1} \tilde{\mathcal{L}}^{\nu} (\mathcal{E}^N)^{\nu} \right\}.$$

Taking the estimates of I^N, II^N, III^N, into account we can prove the following lemma.

LEMMA 6. *With the same hypotheses of Lemma* 5, *let* $u(t, x)$ *be a regular solution to Pb.* (60) (*more precisely, we assume* $u(t, x) \in C^2([0, T); H^\infty(R^n))$) *and assume estimate* (91) *holds*; *then, taking* $k = [n/2] + 3 + (n-1)/2s$ *and defining for* $N > k$, *the energies functions*:

$$\mathcal{E}^N(\tau) = \varrho(\tau) + \sum_{j=k+1}^{N} \frac{\varrho(\tau)^{j-k}}{j!^s} j^{ks} \sqrt{F_j(\tau)}, \qquad (0 \leq \tau < T)$$

where $\varrho(\tau)$ *is a strictly positive function satisfying,*

$$(127)\quad \varrho(\tau) \leq \varrho_0(\tau) \equiv \min \left\{ \frac{1}{2}, \frac{1}{2\mathcal{M}}, \frac{1}{2\mathcal{L}\mathcal{L}_1 \sqrt{\mathcal{F}_{k+p+1}(\tau)}} \right\},$$

($p = [n/2] + 1$) (*with* $\mathcal{L}_1 = \mathcal{L}_1(k)$ *defined as in* (106)), *we can find two constants* $\tilde{\mathcal{C}} = \tilde{\mathcal{C}}(\mathcal{C}, \mathcal{M}, \mathcal{L}, s, k, \mathcal{F}_{k+p+1})$, $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}(\mathcal{L}, k, \mathcal{F}_k)$ (*which do not depend on* N), *such that the following inequality holds*

$$(128)\quad \frac{d}{d\tau} \mathcal{E}^N \leq \varrho' + \tilde{\mathcal{C}}\varrho +$$

$$+ \sum_{j=k+1}^{N} \frac{\varrho^{j-k-1}}{(j-1)!^s} j^{ks-\sigma} \sqrt{F_j} \left\{ \frac{j-k}{j} \varrho' + \tilde{\mathcal{C}}\varrho + \tilde{\mathcal{C}}\varrho \sum_{\nu=0}^{N-k-1} \tilde{\mathcal{L}}^{\nu} (\mathcal{E}^N)^{\nu} \right\},$$

where $\mathcal{F}_j$, *for* $j \geq 1$ *is defined as in* (102).

PROOF. Assuming $k = [n/2] + 3 + (n-1)/2s$, if condition (127) holds, then we can apply the estimates of the terms I^N, II^N and the estimate of the term $III^N = III_1^N + III_2^N$; hence (128) follows immediately, introducing (101), (103), (112), (126) into (95). Q.E.D.

REMARK 2. Since $F_j(0) = 0\ \forall j \geqslant 1$, for all $N > k$, we have

$$(129)\qquad \mathcal{E}^N(0) = \varrho(0) \quad\text{and}\quad \frac{d}{d\tau}\mathcal{E}^N\bigg|_{\tau=0} \leqslant \varrho'(0) + \widetilde{\mathcal{C}}\varrho(0).$$

Defining, for $0 \leqslant \tau < T$, the nondecreasing function

$$(130)\qquad y(\tau) = 2\cdot\left[1 + \mathcal{M} + \widetilde{\mathcal{L}} + \mathcal{L}\mathcal{L}_1\left(1 + \sum_{j=1}^{k+p+1} F_j(\tau)\right)\right],$$

clearly (see (102), (127)), we have

$$(131)\qquad \frac{1}{y(\tau)} < \min\left\{\varrho_0(\tau), \frac{1}{2\widetilde{\mathcal{L}}}\right\}.$$

Then, we define $\varrho(\tau)$, the solution of the linear differential equation,

$$(132)\qquad \begin{cases} \dfrac{\varrho'}{k+1} + 3\widetilde{\mathcal{C}}\varrho + y'(\tau)\varrho = 0, \\ \varrho(0) = e^{\{-(k+1)y(0)\}}, \end{cases}$$

hence, we find

$$(133)\qquad \varrho(\tau) \equiv exp\{-(k+1)[3\widetilde{\mathcal{C}}\tau + y(\tau)]\} \leqslant \frac{1}{(k+1)y(\tau)}\,.$$

thus, $\varrho(\tau)$ satisfies (127) and estimate (128) holds. Now, thanks to (129) and (132), it follows that

$$(134)\quad \mathcal{E}^N(0) = \exp\{-(k+1)y(0)\}, \quad\text{and}\quad \frac{d}{d\tau}\mathcal{E}^n\bigg|_{\tau=0} < 0 \quad \text{for } N > k;$$

moreover, assuming $\mathcal{E}^N(\tau) \leqslant 1/2\,\widetilde{\mathcal{L}}$ and applying (128), we have the following estimate

$$(135)\qquad \frac{d}{d\tau}\mathcal{E}^N(\tau) \leqslant$$

$$\leqslant \varrho' + \widetilde{\mathcal{C}}\varrho + \sum_{j=k+1}^{N} \frac{\varrho^{j-k-1}}{(j-1)!^s} j^{ks-\sigma}\sqrt{F_j}\left\{\frac{j-k}{j}\varrho' + 3\widetilde{\mathcal{C}}\varrho\right\} < 0\,.$$

Thus, taking $\varrho(\tau)$ as in (132) and using (131), (134) and (135) it is easy to deduce that

$$\mathcal{E}^N(\tau) \leqslant \frac{1}{2\widetilde{\mathcal{L}}} \qquad \text{for } 0 \leqslant \tau < T, \tag{136}$$

for any $N > k$.

REMARK 3. Fixed $s \geqslant 1$, assume that $f(x, u): \boldsymbol{R}_x^n \times \boldsymbol{R}_u \to \boldsymbol{R}$ belongs to $\gamma_{L^\infty}^{(s)}(\boldsymbol{R}_x^n \times \boldsymbol{R}_u)$ and vanishes for $u = 0$, that is

$$\begin{cases} f(x, 0) = 0\,, \\ |D_x^\alpha \partial_u^\nu f(x, u)| \leqslant C_f M^{|\alpha|} L^\nu |\alpha|!^s \nu!^s\,. \end{cases} \tag{137}$$

Given $u(x) \in \gamma_{L^2}^{(s)}(\boldsymbol{R}_x^n)$, which satisfies the estimates

$$\|D_x^\alpha u(x)\|_{L^\infty}\,, \qquad \|D_x^\alpha u(x)\|_{L^2} \leqslant C_u K^{|\alpha|} |\alpha|!^s\,, \tag{138}$$

we consider the composite function $f(x, u(x)): \boldsymbol{R}_x^n \to \boldsymbol{R}$. We will prove that $f(x, u(x)) \in \gamma_{L^2}^{(s)}(\boldsymbol{R}_x^n)$.

Taking into account of (138), for $\beta_1 + \ldots + \beta_\nu = \mu$, we have

$$\|D^{\beta_1} u \ldots D^{\beta_\nu} u\| \leqslant C_\mu^\nu K^{|\mu|} |\beta_1|!^s \ldots |\beta_\nu|!^s\,, \tag{139}$$

then, applying Leibniz' formula (69) and estimates (114),

$$\frac{|\alpha - \mu|!^\sigma |\beta_1|!^\sigma \ldots |\beta_\nu|!^\sigma}{|\alpha|!^\sigma} \nu!^\sigma \leqslant 1\,, \qquad (\text{for } \sigma = s - 1,\ |\beta_i| \geqslant 1) \tag{140}$$

we easily find that

$$\|D^\alpha f(x, u(x))\|_{L^2} \leqslant C_f C_u L M^{|\alpha|} |\alpha|!^s + \tag{141}$$
$$+ C_f |\alpha|!^s \cdot \sum_{\substack{1 \leqslant \nu \leqslant |\mu| \\ \mu \leqslant \alpha}} M^{|\alpha - \mu|} K^{|\mu|} (LC_u)^\nu \cdot \sum_{\substack{\beta_1 + \ldots + \beta_\nu = \mu \\ 0 < |\beta_i|}} 1\,,$$

Now, observe that

$$\sum_{\substack{\beta_1 + \ldots + \beta_\nu = \mu \\ 0 < |\beta_i|}} 1 \leqslant \sum_{\substack{h_1 + \ldots + h_\nu = |\mu| \\ 0 < h_i}} \left(\sum_{|\beta_1| = h_1} 1 \right) \cdots \left(\sum_{|\beta_\nu| = h_\nu} 1 \right), \tag{142}$$

and

$$\sum_{|\beta_i| = h_i} 1 = \begin{bmatrix} n + h_i - 1 \\ n - 1 \end{bmatrix} \leqslant (2h_i)^{n-1}\,, \qquad (h_i > 0) \tag{143}$$

hence,

$$(144)\qquad \sum_{\substack{\beta_1+\ldots+\beta_\nu=\mu\\ 0<|\beta_i|}} 1 \leqslant$$

$$\leqslant 2^{\nu(n-1)} \sum_{\substack{h_1+\ldots+h_\nu=|\mu|\\ 0<h_i}} (h_1\ldots h_\nu)^{n-1} \leqslant 2^{\nu(n-1)} \sum_{\substack{h_1+\ldots+h_\nu=|\mu|\\ 0<h_i}} \left(\frac{|\mu|}{\nu}\right)^{\nu(n-1)} \leqslant$$

$$\leqslant 2^{\nu(n-1)}\, e^{(n-1)|\mu|/e}\cdot \sum_{\substack{h_1+\ldots+h_\nu=|\mu|\\ 0<h_i}} 1 \leqslant 2^{\nu(n-1)}\, e^{(n-1)|\mu|/e}\cdot \begin{bmatrix} |\mu|-1\\ \nu-1\end{bmatrix}$$

since,

$$(145)\qquad \sum_{\substack{h_1+\ldots+h_\nu=|\mu|\\ 0<|h_i|}} = \begin{bmatrix} |\mu|-1\\ \nu-1\end{bmatrix} \quad\text{and}\quad h_1\ldots h_\nu \leqslant \left(\frac{|\mu|}{\nu}\right)^{\nu} \leqslant e^{|\mu|/e}.$$

Now, introducing (144) into (141) and summing for $1 \leqslant \nu \leqslant |\mu|$, we have

$$(146)\qquad \sum_{1\leqslant\nu\leqslant|\mu|} (2^{n-1}LC_u)^{\nu} \begin{bmatrix} |\mu|-1\\ \nu-1\end{bmatrix} = 2^{n-1}LC_u\cdot(1+2^{n-1}LC_u)^{|\mu|-1},$$

thus,

$$(147)\qquad \|D^\alpha f(x, u(x))\|_{L^2} \leqslant C_f C_u L M^{|\alpha|}\, |\alpha|!^s\cdot$$

$$\cdot\left[1+2^{n-1}\cdot\sum_{\mu\leqslant\alpha} \frac{K^{|\mu|}\, e^{(n-1)|\mu|/e}(1+2^{n-1}LC_u)^{|\mu|-1}}{M^{|\mu|}}\right].$$

Now, to complete the estimate of the $\gamma^{(s)}_{L^2}$-*norm* of the function $f(t, x, u(x))$, note that for $h \neq 1$,

$$(148)\qquad \sum_{\mu\leqslant\alpha} h^{|\mu|} = \frac{h^{\alpha_1+1}-1}{h-1}\cdots\frac{h^{\alpha_n+1}-1}{h-1},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \boldsymbol{N}^n$. Q.E.D.

Finally, we observe that in the same way, we can prove that if $f(t, x, u) \in \boldsymbol{C}^0([0, T); \gamma^{(s)}_{L^\infty}(\boldsymbol{R}^n_x \times \boldsymbol{R}_u))$ and $u(t, x) \in \boldsymbol{C}^0([0, T); \gamma^{(s)}_{L^2}(\boldsymbol{R}^n_x))$, then the composite function $f(t, x, u(x))$ belongs to $\boldsymbol{C}^0([0, T); \gamma^{(s)}_{L^2}(\boldsymbol{R}^n_x))$.

We are in a position to prove the following.

PROPOSITION 1. *Assume the hypotheses of Lemma* 5 *and Lemma* 6 *hold, then the solution* $u(t, x)$ *of Pb.* (60), *satisfies*

$$(149)\qquad u(t,\cdot) \quad \text{and} \quad u_t(t,\cdot) \in \gamma_{L^2}^{(s)}(\boldsymbol{R}^n), \qquad \forall t \in [0, T).$$

PROOF. Defining $\varrho(\tau)$ as in (132), thanks to Remark 2, the energies $\mathcal{E}^N(\tau)$ are uniformly bounded on $[0, T)$. More precisely, from the definition of $\mathcal{E}^N(\tau)$ and the estimate (136), if follows that

$$(150)\qquad \sqrt{E_j(\tau)} \leq \sqrt{F_j(\tau)} \leq \frac{1}{2\widetilde{\mathcal{L}}}\,\frac{j!^s \varrho(\tau)^{k-j}}{j^{ks}}, \qquad \text{for } j \geq k+1,$$

hence, from estimate (81) of the proof of Lemma 4, we deduce that $u(t,\cdot) \in \gamma_{L^2}^{(s)}$ because

$$(151)\qquad \sum_{|\alpha|=j}\ \sup_{0\leq t\leq \tau} \|D^\alpha u(t,\cdot)\|_{L^2}^2\, e^{j\theta t} \leq F_{j+2}(\tau) + (1+j\theta)F_{j+1}(\tau).$$

To prove that $u_t(t,\cdot) \in \gamma_{L^2}^{(s)}$, we observe that

$$(152)\qquad \frac{d}{d\tau}F_j(\tau) - \frac{d}{d\tau}E_j(\tau) = e^{j\theta\tau}\cdot \sum_{\beta=j-2} \|u_t(\tau,\cdot)\|_{L^2}^2, \qquad \text{for } j \geq 2,$$

hence, the thesis follows introducing the upper bounds (150) into estimates (28) and (59) with

$$(153)\qquad \widetilde{g}(t,x) = g(t,x) + f(t,x,u(x)),$$

and applying the result of Remark 3.

Continuity in Gevrey classes for $s \geq 1$.

To prove that the solution of Pb. (60), $u(t, x)$, belongs to $C^1([0,T);\ \gamma_{L^2}^{(s)}(\boldsymbol{R}_x^n))$, we can restrict ourselves to proving the continuity of $u(t, x)$ for $t = 0$. Hence, recalling that $u(0,x) = u_t(0,x) = 0$, we have to show that $u(t,\cdot)$, $u(t,\cdot) \to 0$ in $\gamma_{L^2}^{(s)}(\boldsymbol{R}^n)$ when $t \to 0$.

Taking $\varrho(\tau)$ as in (132), thanks to the results of Remark 2 (see in particular estimate (135)), we find that $\forall N \geq k+1$,

$$(154)\qquad \frac{d}{d\tau}\sum_{j=k+1}^{N} \frac{\varrho(\tau)^{j-k}}{j!^s}\, j^{ks}\sqrt{F_j(\tau)} = \frac{d}{d\tau}\mathcal{E}^N(\tau) - \varrho'(\tau) \leq -\varrho'(\tau),$$

thus we have,

$$(155)\qquad 0 \leqslant \sum_{j=k+1}^{N} \frac{\varrho(\tau)^{j-k}}{j!^s} j^{ks} \sqrt{F_j(\tau)} \leqslant \varrho(0) - \varrho(\tau), \qquad \forall N \geqslant k+1$$

hence, applying the same estimates of the proof of Prop. 1 and Remark 3, we can easily prove that

$$u(t,\cdot), \qquad u_t(t,\cdot) \to 0, \qquad \text{in } \gamma_{L^2}^{(s)} \qquad \text{as } t \to 0^+ .$$

Using estimates like (155), by standard arguments, we can prove the continuity from the right and from the left in $[0, T)$ of the functions $u(t, x)$ and $u_t(t, x)$. Finally, we recall that the transformation

$$t' = -t, \qquad x' = x$$

does not preserve the Oleinik's condition, hence, in some sense, the continuity from the right, does not imply the continuity from the left (see also the Remarks after Thm. 1).

5. – Regularity in Gevrey classes of order $s \geqslant 1$.

Let $u(t, x) \in \boldsymbol{C}^2([0, T); \boldsymbol{H}^\infty_{\text{loc}}(\boldsymbol{R}^n_x))$ be a solution of the semilinear equation

$$(156)\qquad L(u) = g(t, x) + f(t, x, u),$$

where the linear operator $L(u)$, defined as in (1), satisfies the condition of weak hyperbolicity (3),

$$(3)\qquad \lambda |\xi|^2 \geqslant \sum_{i,j}^{n} a_{ij}(t, x)\xi_i \xi_j \geqslant 0 \qquad (\lambda > 0),$$

and the Oleinik's condition (4). Now, we take the coefficients of the linear differential operator L, and the function $g(t, x)$ in the space $\boldsymbol{C}^0([0, T); \gamma^{(s)}(\boldsymbol{R}^n))$; more precisely we assume that the upper bounds (5) and (65) hold only locally, that is

$$(157)\qquad |\partial_x^\alpha w(t, x)| \leqslant C_K A_K^{|\alpha|} |\alpha|!^s \qquad \forall (t, x) \in K,$$

for $w = a_{ij}, b_j, \partial_t b_0, C, g$ and for all compact sets $K \subset [0, T) \times \boldsymbol{R}^n$ and $\alpha \in \boldsymbol{N}^n$.

Finally, we take $f(t, x, u)$: $[0, T) \times \boldsymbol{R}^n_x \times \boldsymbol{R}_u \to \boldsymbol{R}$ a function of $\boldsymbol{C}^0([0, T); \gamma^{(s)}(\boldsymbol{R}^n_u \times \boldsymbol{R}_u))$ which vanishes for $u = 0$; hence, $f(t, x, u)$

satisfies the following assumptions:

$$(61') \quad \begin{cases} f(t, x, 0) = 0, & \forall (t, x) \in [0, T) \times \boldsymbol{R}_x^n, \\ |D_x^\alpha \partial_u^\nu f(t, x, u)| \leq C_{K'} M_{K'}^{|\alpha|} L_{K'}^\nu |\alpha|!^s \nu!^s, & \forall (t, x, u) \in K', \end{cases}$$

for all compact sets $K' \subset [0, T) \times \boldsymbol{R}_x^n \times \boldsymbol{R}_u$ and $\alpha \in \boldsymbol{N}^n$.

Fixed $x_0 \in \boldsymbol{R}^n$ and $r_0, \lambda' \in \boldsymbol{R}$ such that

$$\lambda' > \lambda, \qquad 0 < r_0 < T\sqrt{\lambda'},$$

we consider the close backward cone $Q = Q(x_0, r_0)$ of $[0, T) \times \boldsymbol{R}_x^n$ defined by

$$(158) \qquad r(t) = r_0 - t\sqrt{\lambda'} \qquad \left(0 \leq t \leq \frac{r_0}{\sqrt{\lambda'}}\right),$$

$$(159) \qquad B_t = \{x \in \boldsymbol{R}^n : |x - x_0| \leq r(t)\},$$

$$(160) \qquad Q(x_0, r_0) = \left\{(t, x) \in \boldsymbol{R}_t \times \boldsymbol{R}^n : x \in B_t,\ 0 \leq t \leq \frac{r_0}{\sqrt{\lambda'}}\right\}.$$

Moreover, for any $\varepsilon \in \boldsymbol{R}$, $0 < \varepsilon < r_0$, we define the domain

$$(161) \quad Q^\varepsilon = \left\{(t, x) : \sqrt{\varepsilon^2 + |x - x_0|^2} - r(t) \leq 0,\ 0 \leq t \leq \frac{r_0 - \varepsilon}{\sqrt{\lambda'}}\right\},$$

observe that, taking in $[0, T) \times \boldsymbol{R}_x^n$ the induced topology, for $0 < \varepsilon_1 < < \varepsilon < r_0$, we have $Q^\varepsilon \subset\subset Q^{\varepsilon_1}$, and

$$Q^\varepsilon \subset\subset \operatorname{int}\{Q\}, \qquad \bigcup_{0 < \varepsilon < r_0} Q^\varepsilon = \operatorname{int}\{Q\}.$$

In the following two lemmas, we will assume $s > 1$ and we shall see that it is possible to find a weakly hyperbolic operator

$$(162) \qquad L^\varepsilon \equiv \partial_{tt}^2 - \sum_{i,j=1}^n \partial_{x_j}(a_{ij}^\varepsilon \partial_{x_i}) + \sum_{j=1}^n b_j^\varepsilon \partial_{x_j} + b_0^\varepsilon \partial_t + c^\varepsilon$$

satisfying the Oleinik's condition (4) *and the upper bounds* (5), *and two continuous functions, $f^\varepsilon(t, x, u)$ and $g^\varepsilon(t, x)$ which satisfy the estimate* (61) *and* (65), *such that the semilinear equation*

$$(163) \qquad L^\varepsilon(u) = g^\varepsilon(t, x) + f^\varepsilon(t, x, u)$$

has a global solution $u^\varepsilon(t, x) \in C^2([0, \infty); H^\infty(R^n))$, *with*

$$u^\varepsilon(t, x)\Big|_{Q^\varepsilon} = u(t, x)\Big|_{Q^\varepsilon}, \tag{164}$$

hence, the regularity of $u(t, x)$, *in the domain* Q^ε, *follows from the regularity of* $u^\varepsilon(t, x)$ (*thanks to Prop.* 1).

Thus, the proof of Thm. 1 is complete.

REMARK 4. It is easy to see that the analytic hypersurface

$$\partial Q^\varepsilon \equiv \left\{(t, x)\colon \sqrt{\varepsilon^2 + |x_0 - x|^2} - r(t) = 0,\ 0 \leqslant t \leqslant \frac{r_0 - \varepsilon}{\sqrt{\lambda'}}\right\}, \tag{165}$$

is a *non-characteristic* hypersurface, for any linear differential operator L_η of the form

$$L_\eta \equiv L - \sum_{i,j=1}^{n} \partial_{x_j}(\eta_{ij}\partial_{x_i}) \qquad (\eta_{ij} = \eta_{ji}), \tag{166}$$

if

$$(\lambda' - \lambda)|\xi|^2 > \sum_{i,j}^{n} \eta_{ij}(t, x)\xi_i\xi_j \geqslant 0 \qquad (\forall \xi \in R^n) \tag{167}$$

(see [M1], chapter 4, for a more detailed proof).

REMARK 5. We shall restrict ourselves to the case $s > 1$ because we are forced to use functions with compact support. To consider the analytic case it is sufficient to recall the Remark after Thm. 1, where we obtain the analytic regularity applying Thm. 2 of [S2]. On the other hand when $s = 1$, we can resort to a family $\{\chi_N\}_{N\geqslant 1}$ of suitable C^∞ compactly supported functions satisfying the conditions

$$|D^\alpha \chi_N(t, x)| \leqslant C^{|\alpha|}N^{|\alpha|}, \qquad \text{if } |\alpha| \leqslant N,$$

where C does not depend on N (see Lemma 2.2 of [H], see also [AM]). More precisely, to estimate the *Gevrey energy* $\mathcal{E}^N$, we can take a function like χ_{N+m}, with a fixed integer m, and then, we apply the same techniques of the case $s > 1$. See the proof below.

Construction of the linear operator L^ε *for* $s > 1$.

We recall that, contrary to the analytic functions, $\gamma^{(s)}(R^n)$, for $s > 1$, contains fairly large class of C^∞ functions with compact support. Thus, we can find $\eta \in \gamma^{(s)}(R)$ a non-decreasing function $(\eta' \geqslant 0)$, such that

$$\eta(h) = 0 \quad \text{if } h \leqslant 0, \qquad 0 < \eta(h) < 1 \quad \text{if } 0 < h < 1, \tag{168}$$
$$\eta(h) = 1 \quad \text{if } h \geqslant 1.$$

Then, for ε, $\mu \in \boldsymbol{R}$, with $0 < \varepsilon < r_0$, $0 < \mu$, we define

$$\eta_\varepsilon(t, x) = \eta\left(\frac{\sqrt{\varepsilon^2 + |x - x_0|^2} - r(t)}{\mu}\right) \quad \forall (t, x) \in [0, \infty) \times \boldsymbol{R}^n, \tag{169}$$

where $\mu > 0$ will be chosen sufficiently small, such that the function

$$1 - \eta\left(\frac{|x - x_0| - r(t)}{\mu}\right),$$

is compactly supported in $[0, T) \times \boldsymbol{R}^n$. Clearly, we have

$$\eta_\varepsilon(t, x)\Big|_{Q^\varepsilon} \equiv 0, \quad 0 < \eta_\varepsilon(t, x) \leqslant 1 \quad \forall (t, x) \in [0, \infty) \times \boldsymbol{R}^n \backslash Q^\varepsilon. \tag{170}$$

Taking into account of (168), for $0 < \varepsilon < r_0$, we define

$$L_{\eta_\varepsilon} = L - \frac{\lambda' - \lambda}{2} \sum_{i=1}^{n} \partial_{x_i}(\eta_\varepsilon \partial_{x_i}), \tag{171}$$

The differential operator L_{η_ε} satisfies the Oleinik's condition (4), since

$$\partial_t \eta_\varepsilon(t, x) = \eta'\left(\frac{\sqrt{\varepsilon^2 + |x - x_0|^2} - r(t)}{\mu}\right) \frac{\sqrt{\lambda'}}{\mu} \geqslant 0, \tag{172}$$

moreover, thanks to the assumptions (168) on $\eta(h)$, for any $\varepsilon_1 \in \boldsymbol{R}$, $0 < \varepsilon_1 < \varepsilon$, L_{η_ε} is a strictly hyperbolic operator in $[0, T) \times \boldsymbol{R}^n \backslash Q^{\varepsilon_1}$, since we have

$$Q^\varepsilon \subset\subset \operatorname{int}\{Q^{\varepsilon_1}\} \subset Q^{\varepsilon_1} \subset\subset \operatorname{int}\{Q(x_0, r_0)\}. \tag{173}$$

Fixed, for example, $\varepsilon_1 = \varepsilon/2$, we set

$$a^\varepsilon_{ij}(t, x) = a_{ij}(t, x)(1 - \eta_{\varepsilon_1}(t, x)) + \delta^j_i \frac{\lambda' - \lambda}{2} \eta_\varepsilon(t, x), \tag{174}$$

$$b^\varepsilon_j(t, x) = b_j(t, x)(1 - \eta_{\varepsilon_1}(t, x)), \tag{175}$$

$$c^\varepsilon(t, x) = c(t, x)(1 - \eta_{\varepsilon_1}(t, x)), \tag{176}$$

(where $\delta^j_i = 0$ if $i \neq j$, and $\delta^i_i = 1$), and then we define L^ε as in (162).

Now, we easily see that

$$\text{(177)} \qquad \frac{\lambda' + \lambda}{2} |\xi|^2 \geqslant \sum_{i,j}^{n} a_{ij}^{\varepsilon}(t, x)\xi_i\xi_j \geqslant 0 \qquad (\forall \xi \in \boldsymbol{R}^n),$$

thus ∂Q^{ε} is a non-characteristic hypersurface for L^{ε} (see [M1]); besides, the Oleinik's condition holds, since

$$\text{(178)} \qquad L^{\varepsilon}\Big|_{Q^{\varepsilon 1}} = L_{\eta_{\varepsilon}}\Big|_{Q^{\varepsilon 1}},$$

and

$$\text{(179)} \qquad 0 < \inf_{(t,x) \notin Q^{\varepsilon 1}} \eta_{\varepsilon}(t, x).$$

Finally, we observe that taking $\mu > 0$ sufficiently small, the upper bounds (5) hold for suitable chosen C_0, Λ_0, since the coefficients a_{ij}^{ε}, b_j^{ε}, $\partial_t b_0^{\varepsilon}$, c^{ε}, and $\nabla_x \eta_{\varepsilon}$ are *Gevrey functions* compactly supported in $[0, T) \times \boldsymbol{R}^n$. Q.E.D.

In the same way, we define

$$\text{(180)} \qquad \begin{cases} g^{\varepsilon}(t, x) = g(t, x)\cdot(1 - \eta_{\varepsilon_1}(t, x)), \\ F(t, x, u) = f(t, x, u)\cdot(1 - \eta_{\varepsilon_1}(t, x)), \end{cases}$$

clearly, $g^{\varepsilon}(t, x)$ satisfies the condition (65).

To go on, we must show that the equation

$$\text{(181)} \qquad L^{\varepsilon}(v) = g^{\varepsilon}(t, x) + F(t, x, v)$$

admits a solution $v(t, x)$, defined in a neighborhood of Q^{ε}, such that $v(t, x) \in \boldsymbol{C}^2([0, T_{\varepsilon}); \boldsymbol{H}_{\text{loc}}^{\infty})$, for some $T_{\varepsilon} \in \boldsymbol{R}$, $(r_0 - \varepsilon)/\sqrt{\Lambda'} < T_{\varepsilon} < T$, and

$$\text{(182)} \qquad v(t, x)\Big|_{Q^{\varepsilon}} = u(t, x)\Big|_{Q^{\varepsilon}}.$$

Let $\phi(t, x) \in \gamma_0^{(s)}([0, T) \times \boldsymbol{R}^n)$, such that $0 \leqslant \phi(t, x) \leqslant 1$, $\phi \equiv 1$ in a neighborhood of Q^{ε}, for example we can take $\phi(t, x) = 1 - \eta_{\varepsilon_1}(t, x)$. Then, for any $\tau \in \boldsymbol{R}$, $0 \leqslant \tau < (r_0 - \varepsilon)/\sqrt{\lambda'}$, we consider the Cauchy problem P_τ, defined by,

$$\text{(183)} \quad \begin{cases} L^\varepsilon(v) = g^\varepsilon(t, x) + F(t, x, v), \\ v(\tau, x) = u(\tau, x)\phi(\tau, x), \qquad v_t(\tau, x) = u_t(\tau, x)\phi(\tau, x), \end{cases}$$

for $t \geqslant \tau$.

REMARK 6. For any $0 \leqslant \tau < (r_0 - \varepsilon)/\sqrt{\lambda'}$, the Cauchy problem P_τ, has a unique local solution $v^\tau(t, x)$, such that

$$\text{(184)} \qquad v^\tau(t, x) \in \boldsymbol{C}^2([\tau, T(P_\tau)); \boldsymbol{H}_0^\infty(\boldsymbol{R}^n)),$$

where, $T(P_\tau) - \tau > 0$ is the life-span of the regular solution (see [DM]). Moreover, using the fact that $u(t, x)\phi(t, x)$, $u_t(t, x)\phi(t, x) \in \boldsymbol{C}^2([0, T); \boldsymbol{H}^\infty(\boldsymbol{R}^n))$, we can easily find a positive lower bound, say T', for the life-span of the problems $\{P_\tau\}_{0 \leqslant \tau < (r_0 - \varepsilon)/\sqrt{\lambda'}}$, that is

$$\text{(185)} \qquad 0 < T' < \inf_{0 \leqslant \tau < (r_0 - \varepsilon)/\sqrt{\lambda'}} (T(P_\tau) - \tau).$$

Using (185) and the finite speed of propagation property (see Appendix A), we can prove the following Lemma.

LEMMA 7. *There exists an open neighborhood U^ε of Q^ε in $[0, T) \times \boldsymbol{R}^n$, such that the eq.* (181) *has a solution $v(t, x) \in \boldsymbol{C}^2([0, T_\varepsilon); \boldsymbol{H}^\infty_{\mathrm{loc}})$ (with $(r_0 - \varepsilon)/\sqrt{\lambda'} < T_\varepsilon < T$), defined in $U^\varepsilon \cup [0, T') \times \boldsymbol{R}^n$, which satisfies* (182).

PROOF. Thanks to Remark 6 and (185), the solution $v^0(t, x)$ of the Cauchy problem P_0 is defined in the stripe $[0, T(P_0)) \times \boldsymbol{R}^n$, with $T(P_0) > T' > 0$. Now, if $T' > (r_0 - \varepsilon)/\sqrt{\lambda'}$, there is nothing else to prove. In the other case, fixed τ_1,

$$\text{(186)} \qquad \frac{T'}{2} < \tau_1 < T',$$

we consider the solution $v^{\tau_1}(t, x)$ of problem P_{τ_1}; taking (177) into account, and the finite speed of propagation property (see (3*a*)), it is easy to see that $v^0(t, x) = v^{\tau_1}(t, x)$ in the open conic section

$$\text{(187)} \quad D(\tau_1, T') = \left\{(t, x) : \tau_1 < t < T', \ |x - x_0| < \varrho(\tau_1) - (t - \tau_1)\sqrt{\frac{\lambda' + \lambda}{2}}\right\},$$

where $\varrho(\tau)$, for $0 \leqslant \tau < (r_0 - \varepsilon)/\sqrt{\lambda'}$, is defined by

$$\varrho(\tau) = \sqrt{r(\tau)^2 - \varepsilon^2}\,. \tag{188}$$

Hence, we can define $v(t, x)$ for $0 \leqslant t < \tau_1 + T'$, setting

$$\begin{cases} v(t, x) = v^0(t, x), & \text{for } (t, x) \in [0, T') \times \boldsymbol{R}^n, \\ v(t, x) = v^{\tau_1}(t, x), & \text{for } (t, x) \in D(\tau_1, \tau_1 + T'), \end{cases} \tag{189}$$

where $D(\tau_1, \tau_1 + T')$ is defined as in (187).

Clearly, proceeding in this way, we can define the solution $v(t, x)$ in a neighborhood U^ε of Q^ε. Moreover, since

$$\begin{gathered} L^\varepsilon \Big|_{Q^\varepsilon} = L \Big|_{Q^\varepsilon}, \qquad g^\varepsilon(t, x)\Big|_{Q^\varepsilon} = g(t, x)\Big|_{Q^\varepsilon}, \\ f(t, x, u)\Big|_{Q^\varepsilon} = F(t, x, u)\Big|_{Q^\varepsilon}, \end{gathered} \tag{190}$$

equality (182) holds. Q.E.D.

Construction of $f^\varepsilon(t, x, u)$ and $u^\varepsilon(t, x)$.

To begin with, we observe that the function $v(t, x)$, obtained in (189), is a local solution of the Cauchy problem P_0,

$$\begin{cases} L^\varepsilon(v) = g^\varepsilon(t, x) + F(t, x, v), \\ v(0, x) = u(0, x)\phi(0, x), \qquad v_t(0, x) = u_t(0, x)\phi(0, x), \end{cases} \tag{191}$$

defined in the open set $U^\varepsilon \cup [0, T') \times \boldsymbol{R}^n$. Now, it is easy to find a *non-characteristic* hypersurface, for the operator L^ε, defined by

$$t = \beta(x), \qquad \forall x \in \boldsymbol{R}^n \tag{192}$$

such that $\beta(x) \in \gamma^{(s)}(\boldsymbol{R}^n)$, $\beta(x) \geqslant 0$, $\beta(x) = 0$ for $|x - x_0| \geqslant r_0 + 1$,

$$\beta(x) > \frac{r_0 - \sqrt{\varepsilon^2 + |x - x_0|^2}}{\sqrt{\lambda'}}, \tag{193}$$

$$\text{and} \qquad (\beta(x), x) \in U^\varepsilon \cup [0, T') \times \boldsymbol{R}^n, \qquad \forall x \in \boldsymbol{R}^n$$

In this way, $Q^\varepsilon \subset\subset \{(t, x)\colon 0 \leqslant t < \beta(x)\}$ and L^ε is a strictly hyperbolic operator in the set $\{t \geqslant \beta(x)\}$, since (179) holds for any $\varepsilon_1 < \varepsilon$.

Taking $\psi(h) \in \gamma^{(s)}(\boldsymbol{R})$ such that,

$$\psi(h) = 1 \text{ for } h \leqslant 1, \qquad \psi(h) = 0 \text{ for } h \geqslant 2, \tag{194}$$

we observe that for any $\widetilde{\mu} > 0$ sufficiently small, the function $v(t, x)$ is also a local solution of the problem

$$(195)\qquad \begin{cases} L^{\varepsilon}(v) = g^{\varepsilon}(t, x) + F(t, x, v)\cdot\psi\left(\dfrac{t-\beta(x)}{\widetilde{\mu}}\right), \\ v(0, x) = u(0, x)\,\phi(0, x), \qquad v_t(0, x) = u_t(0, x)\,\phi(0, x), \end{cases}$$

since, $\psi((t-\beta(x))/\widetilde{\mu}) = 1$ for $t \leqslant \widetilde{\mu} + \beta(x)$. Now, we take $0 < \widetilde{\mu} < \widetilde{\mu}_0$, choosing $\widetilde{\mu}_0$ such that

$$(196)\qquad \{(t, x): 0 \leqslant t \leqslant 2\widetilde{\mu}_0 + \beta(x)\} \subset U^{\varepsilon} \cup [0, T') \times \boldsymbol{R}^n ,$$

and we will define,

$$(197)\qquad f^{\varepsilon}(t, x, u) = F(t, x, u)\cdot\psi\left(\frac{t-\beta(x)}{\widetilde{\mu}}\right).$$

Then we perform the transformation of variables

$$(198)\qquad t' = t - \beta(x), \qquad x' = x,$$

this transformation and its inverse are smooth in all $\boldsymbol{R}_t \times \boldsymbol{R}^n_x$, moreover we have,

$$(199)\qquad \frac{\partial}{\partial_{x_i}} = \frac{\partial}{\partial_{x_i'}} - \beta_{x_i}\frac{\partial}{\partial_{t'}} \quad (1 \leqslant i \leqslant n), \qquad \frac{\partial}{\partial_t} = \frac{\partial}{\partial_{t'}}$$

hence, substituting these expressions into (195), we find that the function

$$w(t', x') \equiv v(t' + \beta(x'), x')$$

is a local solution (defined at least in the stripe $[0, \widetilde{\mu}) \times \boldsymbol{R}^n$) of the Cauchy problem

$$(200a)\qquad \widetilde{L}^{\varepsilon}(w) = g(t' + \beta(x'), x') + F(t' + \beta(x'), x', w)\cdot\psi\left(\frac{t'}{\widetilde{\mu}}\right),$$

$$(200b)\qquad w(0, x') = v(\beta(x'), x'), \qquad w_{t'}(0, x') = v_t(\beta(x'), x'),$$

where $\widetilde{L}^{\varepsilon}$ is the linear differential operator, obtained from L^{ε}, by the transformation (198)-(199); that is, taking $L^{\varepsilon} \equiv L^{\varepsilon}(t, x, \partial_t, \partial_{x_i})$,

$$\widetilde{L}^{\varepsilon} \equiv \widetilde{L}^{\varepsilon}(t', x', \partial_{t'}, \partial_{x_i'}) = L^{\varepsilon}(t' + \beta(x'), x', \partial_{t'}, \partial_{x_i'} - \beta_{x_i}\partial_{t'}).$$

Besides, $\widetilde{L}^{\varepsilon}$ is strictly hyperbolic for $t' \geqslant 0$ (see [M1]), and the initial data (200*b*), $v(\beta(x'), x')$ and $v_t(\beta(x'), x')$, belong to $C_0^\infty(\boldsymbol{R}^n)$.

To conclude, we need the following lemma.

LEMMA 8. *Let $a(t, x, \partial_t, \partial_{x_i})$ be a linear differential operator, strictly hyperbolic with respect to the variable t; given $a_0(t, x)$, $a_1(t, x, u)$ regular functions such that,*

$$a_0(t, x) \in C^0([0, \infty); H^\infty(\boldsymbol{R}^n)); \qquad a_1(t, x, 0) = 0 \qquad (\forall t, x), \tag{201}$$

consider, for $\mu > 0$ *the Cauchy problem,*

$$\begin{cases} a(t, x, \partial_t, \partial_{x_i})u = a_0(t, x) + a_1(t, x, u)\cdot \psi\left(\dfrac{t}{\mu}\right), \\ u(0, x) = \varphi_0(x), \qquad u_t(0, x) = \varphi_1(x), \end{cases} \tag{202}$$

where the initial data, $\varphi_0(x), \varphi_1(x) \in C_0^\infty(\boldsymbol{R}^n)$ and $\psi(h)$ is defined as in (194). *Then, there exists $\mu_0 > 0$, such that* (202) *has a unique regular global solution for all $0 < \mu < \mu_0$.*

PROOF. Since the perturbing term $\psi(t/\mu)$, does not depend on the variable x, we can obtain (using merely the classical energy estimates and the regularity of $a_1(t, x, u)$) a positive lower bound for the life-span of the regular solution, say T^*, which does not depend on μ. Hence, taking

$$0 < \mu \leqslant \frac{T^*}{3}, \tag{203}$$

the solution, which we know to exist in $[0, T^*) \times \boldsymbol{R}^n$, can be extended to all $[0, \infty) \times \boldsymbol{R}^n$, since the eq. (202) becomes linear for $t \geqslant 2/3T^*$. Q.E.D.

Applying Lemma 8 to eq. (200*a*) we deduce that for $\widetilde{\mu} > 0$ sufficiently small, Pb. (200*a*), (200*b*) has global solution $w(t', x') \in C^2([0, \infty); H^\infty(\boldsymbol{R}^n))$ such that

$$w(t', x') = v(t' + \beta(x'), x') \qquad \text{for } 0 \leqslant t' \leqslant \widetilde{\mu}, \tag{204}$$

hence, going back (using the inverse transformation) to (195), we immediately see that, Pb. (195) has global smooth solution $u^\varepsilon(t, x)$; moreover, thanks to hypotheses (194) on $\psi(h)$ we have

$$u^\varepsilon(t, x)\Big|_{\{t \leqslant \beta(x) + \widetilde{\mu}\}} = v(t, x)\Big|_{\{t \leqslant \beta(x) + \widetilde{\mu}\}}, \tag{205}$$

hence, $u^\varepsilon(t, x)$ satisfies (164). Finally, observe that $u^\varepsilon(t, x) \in \boldsymbol{C}^2([0, \infty); \boldsymbol{H}^\infty(\boldsymbol{R}^n))$ is a global solution of the problem,

$$(195') \qquad \begin{cases} L^\varepsilon(v) = g^\varepsilon(t, x) + f^\varepsilon(t, x, v), \\ v(0, x) = v(0, x)\phi(0, x), \qquad v_t(0, x) = u_t(0, x)\phi(0, x), \end{cases}$$

and, without loss of generality, we can suppose that $f^\varepsilon(t, x, u)$ defined as in (197), satisfies the upper bounds (61).

This completes the proof of Thm. 1. Q.E.D.

Appendix A.

Assuming $u(t, x)$ a solution of class $\boldsymbol{C}^2$ of eq. (9.1), we will prove that $u(t, x) \in \boldsymbol{C}^\infty$. To this end, we need some *local* estimates of the $\boldsymbol{H}^k$*-norms* of the solution, at least in the *linear case*; then, we can obtain *a priori* upper bounds for the solution of the *nonlinear* equation which lead (by standard arguments) to the $\boldsymbol{C}^\infty$ regularity.

Local Sobolev estimates.

As is known, the linear Cauchy problem for eq. (7.1), namely

$$(1a) \qquad L(u) = g(t, x), \qquad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x)$$

where $u_0(x), u_1(x) \in \boldsymbol{H}^k(\boldsymbol{R}^n)$ and $g(t, x) \in \boldsymbol{C}^0([0, T); \boldsymbol{H}^k(\boldsymbol{R}^n))$, for some $k \geqslant 2$, has a unique solution (see [O1]),

$$(2a) \qquad u(t, x) \in \boldsymbol{C}^i([0, T); \boldsymbol{H}^{k-i}(\boldsymbol{R}^n)), \qquad i = 0, 1, 2$$

enjoying the finite speed of propagation property, with speed for $0 \leqslant t < T$ not greater than $\sqrt{\lambda}$ (see (2.1)), that is to say, fixed $(t_0, x_0) \in (0, T) \times \boldsymbol{R}^n$, the bahavior of the solution in the backward cone

$$(3a) \qquad Q(t_0, x_0) = \{(t, x) : 0 \leqslant t < t_0, \quad |x - x_0| < (t_0 - t)\sqrt{\lambda}\}$$

depends only on the behavior of $g(t, x)$ in $Q(t_0, x_0)$ and of the behavior of the initial data in the open disk $\{|x - x_0| < t_0\sqrt{\lambda}\}$. Furthermore, we have the estimate

$$(4a) \qquad \|u(t)\|^2_{\boldsymbol{H}^k(\boldsymbol{R}^n)} + \|u_t(t)\|^2_{\boldsymbol{H}^{k-1}(\boldsymbol{R}^n)} \leqslant$$
$$\leqslant C_k(t) \cdot \left(\|u_0\|^2_{\boldsymbol{H}^k(\boldsymbol{R}^n)} + \|u_1\|^2_{\boldsymbol{H}^k(\boldsymbol{R}^n)} + \int_0^t \|g(s)\|^2_{\boldsymbol{H}^k(\boldsymbol{R}^n)} ds \right)$$

where $C_k(t)$ is a continuous nondecreasing function which depends only on the coefficients of the linear operator $L(u)$ (see [DM] for a detailed proof).

Now, we prove that a local form of the estimate (4a) holds.

Fixed $x_0 \in \boldsymbol{R}^n$, $r_0 > 0$ (with $r_0/\sqrt{\lambda} < T$), we consider the backward cone $Q = Q(x_0, r_0)$ of $[0, T) \times \boldsymbol{R}^n$ defined by

$$\text{(5a)} \quad \begin{cases} r(t) = r_0 - t\sqrt{\lambda} \qquad \left(0 \leqslant t < \dfrac{r_0}{\sqrt{\lambda}}\right), \\ B_t = \{x \in \boldsymbol{R}^n : |x - x_0| < r(t)\}, \\ Q(x_0, r_0) = \left\{(t, x) \in \boldsymbol{R}_t \times \boldsymbol{R}^n : x \in B_t, \, 0 \leqslant t < \dfrac{r_0}{\sqrt{\lambda}}\right\}. \end{cases}$$

Taking $\eta(h) \in C_0^\infty(\boldsymbol{R})$ such that

$$\text{(6a)} \quad \eta(h) = 1 \text{ for } h \leqslant 1, \qquad 0 \leqslant \eta(h) \leqslant 1, \qquad \eta(h) = 0 \text{ for } h \geqslant 2,$$

we define for $0 \leqslant t < r_0/\sqrt{\lambda}$, $\varrho \geqslant 0$,

$$\text{(7a)} \quad \eta_k(t, \varrho) = \eta\left(4(k+1)\frac{\varrho - r(t)}{r(t)}\right).$$

Now, for functions $g(t, x)$ defined on $Q(x_0, r_0)$, making use of the spherical polar coordinate representation in $\boldsymbol{R}_x^n$, with the origin in the point x_0,

$$\text{(8a)} \quad x = (\phi, \varrho) = (\phi_1, \ldots, \phi_{n-1}, \varrho) \qquad (\varrho = |x - x_0|),$$

where $\varrho \geqslant 0$, $-\pi \leqslant \phi_2 \leqslant \pi$, $0 \leqslant \phi_1, \ldots, \phi_{n-1} \leqslant \pi$, we define (see [A], Thm. 4.26) the extension $Eg(t, x)$ defined on $[0, r_0/\sqrt{\lambda}) \times \boldsymbol{R}^n$ of the function $g(t, x)$, setting for $0 \leqslant t < r_0/\sqrt{\lambda}$,

$$\text{(9a)} \quad \begin{cases} Eg(t, \phi, \varrho) = g(t, \phi, \varrho), \qquad \text{if } 0 \leqslant \varrho < r(t), \\ Eg(t, \phi, \varrho) = \eta_k(t, \varrho) \cdot \sum\limits_{j=1}^{k+1} \zeta_j g(t, \phi, (j+1)r(t) - j\varrho), \quad \text{if } \varrho \geqslant r(t), \end{cases}$$

where, $\zeta_1, \ldots, \zeta_{k+1} \in \boldsymbol{R}$ satisfy the linear equation

$$\sum_{j=1}^{k+1} (-j)^m \zeta_j = 1 \qquad \text{for } m = 0, \ldots, k$$

If $g(t, x) \in C^0([0, r_0/\sqrt{\lambda}); C^k(\overline{B_t}))$, then it is easily verified that $Eg(t, x) \in C^0([0, r_0/\sqrt{\lambda}); C^k(\boldsymbol{R}^n))$. Moreover, from the definition of the extension operator E, we can easily deduce that

$$(10a) \qquad \|Eg(t, \cdot)\|_{H^k(\boldsymbol{R}^n)} \leq \varphi_k(t) \cdot \|g(t, \cdot)\|_{H^k(B_t)} \qquad \left(0 \leq t < \frac{r_0}{\sqrt{\lambda}}\right),$$

where

$$(11a) \qquad \varphi_k(t) \equiv \varphi(k, r_0, \lambda, \eta, t) \colon \left[0, \frac{r_0}{\sqrt{\lambda}}\right) \to [0, \infty)$$

is a continuous nondecreasing function (which does not depend on x_0).

Hence, the above inequality, (10a), extends to function $g(t, x) \in C^0([0, r_0/\sqrt{\lambda}); H^k(B_t))$. Finally, we observe that if $g(t, x) \in C^0([0, r_0/\sqrt{\lambda}); H^k_{\text{loc}}(\boldsymbol{R}^n))$, then the function $Eg(t, x)$ belongs to $C^0([0, r_0/\sqrt{\lambda}); H^k(\boldsymbol{R}^n))$.

Now, let $u(t, x) \in C^0([0, T); H^k_{\text{loc}}(\boldsymbol{R}^n))$ be a solution of Pb. (1a), with initial data $u_0(x)$, $u_1(x) \in H^k_{\text{loc}}(\boldsymbol{R}^n)$ and $g(t, x) \in C^0([0, T); H^k_{\text{loc}}(\boldsymbol{R}^n))$. Then, we consider the Cauchy problem

$$(12a) \qquad L(v) = Eg(t, x), \qquad v(0, x) = Eu_0(x), \qquad v_t(0, x) = Eu_1(x)$$

for $0 \leq t < r_0/\sqrt{\lambda}$ where the extension Eu_0, Eu_1 of the initial data are defined in the obvious way. Hence, the unique solution $v(t, x)$ of (12a) belongs to $C^0([0, r_0/\sqrt{\lambda}); H^k(\boldsymbol{R}^n))$ and satisfies (4a) on $[0, r_0/\sqrt{\lambda})$. Besides, thanks to the finite speed of propagation property,

$$(13a) \qquad v(t, x)\Big|_{Q(x_0, r_0)} = u(t, x)\Big|_{Q(x_0, r_0)},$$

thus, using (4a), (10a) and (13a), we deduce that

$$(14a) \qquad \|u(t)\|^2_{H^k(B_t)} + \|u_t(t)\|^2_{H^{k-1}(B_t)} \leq C_k(t)\,\varphi_k(t)^2 \cdot \left(\|u_0\|^2_{H^k(B_0)} + \|u_1\|^2_{H^k(B_0)} + \int_0^t \|g(s)\|^2_{H^k(B_s)} ds\right). \qquad \text{Q.E.D.}$$

The C^∞ regularity for the nonlinear equation.

Assuming $u(t, x)$ be a solution of class C^2 of the semi-linear eq. (9.1), on the backward cone $Q(x_0, r_0)$, such that

$$\sup_{(t,x) \in Q(x_0, r_0)} |u(t, x)| = C_{x_0, r_0} < \infty , \tag{15a}$$

we shall see that

$$u(t, \cdot), \qquad u_t(t, \cdot) \text{ are } C^\infty \text{ functions on } B_t \quad \left(0 \leq t < \frac{r_0}{\sqrt{\lambda}}\right). \tag{16a}$$

We sketch here the proof (referring to [S2], step 3 and step 4, for more details). Writing,

$$f(t, x, u) = g(t, x) + \tilde{f}(t, x, u),$$

where $\tilde{f}(t, x, 0) = 0$, using Leibniz' formula (see (69)) and the interpolations's inequality of Gagliardo and Nirenberg, we have

$$\|\tilde{f}(\cdot, u(t, \cdot))\|_{H^k(B_t)} \leq \tilde{\varphi}_k(t) \cdot \|u(t, \cdot)\|_{H^k(B_t)} \cdot \sum_{\nu=1}^{k} \|u(t, \cdot)\|_{L^\infty(B_t)}^{\nu-1} \tag{17a}$$

$(0 \leq t < r_0/\sqrt{\lambda})$ where $\tilde{\varphi}_k(t): [0, r_0/\sqrt{\lambda}) \to [0, \infty)$, is a continuous nondecreasing function (which depends on the constants of the Gagliardo and Nirenberg's inequalities for the domain B_t). Hence, from (14a) and (17a), we deduce

$$\|u(t)\|_{H^k(B_t)}^2 \leq C_k(t)\, \varphi_k(t)^2 \cdot \Bigg(\|u_0\|_{H^k(B_0)}^2 + \|u_1\|_{H^k(B_0)}^2 + \int_0^t \|g(s)\|_{H^k(B_s)}^2\, ds + \tag{18a}$$

$$+ \tilde{\varphi}_k(t)^2 \cdot \big(1 + \sup_{Q(x_0, r_0)} |u(t, x)|\big)^{2(k-1)} \int_0^t \|u(s)\|_{H^k(B_s)}^2\, ds \Bigg)$$

thus, for $k \geq 2$, applying Gronwall' inequality,

$$\|u(t)\|_{H^k(B_t)} \leq \psi_k(t) \qquad \left(0 \leq t < \frac{r_0}{\sqrt{\lambda}}\right), \tag{19a}$$

where $\psi_k(t) \equiv \psi(k, r_0, \lambda, \eta, u_0, u_1, g, \tilde{f}, C_{x_0, r_0}, t)$ is a continuous nondecreasing function on $[0, r_0/\sqrt{\lambda})$.

In this way, we find a priori upper bounds for the H^k-*norms* of the solution on the domain B_t which easily lead to the C^∞ regularity of $u(t, \cdot)$ and $u_t(t, \cdot)$. Q.E.D.

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Manoscritto pervenuto in redazione il 7 giugno 1994.