Rendiconti
del
Seminario Matematico
della
Università di Padova

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*Rendiconti del Seminario Matematico della Università di Padova*, tome 94 (1995), p. 155-163

<http://www.numdam.org/item?id=RSMUP_1995__94__155_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 94 (1995)

# Minimal Immersions of Surfaces into $n$-Dimensional Space Forms.

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ABSTRACT - Using the motion of the ellipse of curvature we study minimal immersions of surfaces into $n$-dimensional space forms. In this paper we obtain an extension of Theorem 2 of [9]. Also, we obtain some inequalities relating the integral of the normal curvature with topological invariants.

## 1. Introduction.

Let $M$ be an oriented surface which is isometrically immersed into an orientable $n$-dimensional space form $Q^n(c)$, $n \geq 4$, where $Q^n(c)$ stands for the sphere $S^n(c)$ of radius $1/c$, the Euclidean space $\mathbb{R}^n$ or the hyperbolic space $H^n(c)$, according to $c$ is positive, zero or negative. If the normal curvature tensor $R^\perp$ of the immersion is nowhere zero, then exists an orthogonal bundle splitting $NM = (NM)^* \oplus (NM)^0$ of the normal bundle $NM$ of the immersion, where $(NM)^0$ consists of the normal directions that annihilate $R^\perp$ and $(NM)^*$ is a 2-plane subbundle of $NM$.

Let $K$ and $K_N$ be the Gaussian and the normal curvature of $M$. Let $K^*$ be the intrinsic curvature of $(NM)^*$.

We shall make use of the *curvature ellipse* of $x: M \to Q^n(c)$, which is, for each $p$ in $M$ the subset of $N_pM$ given by

$$\varepsilon_p = \{B(X, X) \in N_pM;\ X \in T_pM \text{ and } \|X\| = 1\}$$

where $B$ is the second fundamental form of the immersion. The first result of this paper is an extension of Theorem 2 of Rodriguez-Guadalupe [9] to the case when $M$ is not homeomorphic to the sphere $S^2$.

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THEOREM 1. Let $x\colon M \to S^n(1)$ be a minimal immersion of a complete oriented surface $M$ into the unit sphere $S^n(1)$ with $R^\perp \neq 0$ and $K \geqslant 0$. If $2K \geqslant K^*$ at every point, then $K^*$, the normal curvature $K_N$ and the Gaussian curvature $K$ of $M$ are constant.

REMARKS. (1) If $K > 0$, then we obtain a minimal $S^2$ of constant curvature in $S^n(1)$. These were classified by Do Carmo-Wallach [4]. Itoh [6] and Asperti-Ferus-Rodriguez [1] have a similar theorem.

(2) For $K = 0$ we obtain a «flat» minimal torus. These were studied by Kenmotsu [7], [8].

The second result of this paper is the following.

THEOREM 2. Let $x\colon M \to S^n(1)$ be a minimal immersion of a complete oriented surface $M$ into the unit sphere $S^n(1)$. If $K \geqslant 0$ at every point, then either $K \equiv 0$ or the ellipse is a circle.

The following theorem relates an inequality betwen the integral of the normal curvature with topological invariants.

THEOREM 3. Let $x\colon M \to Q^n(c)$ be a minimal immersion of a compact oriented surface $M$ into an oriented $n$-dimensional space form $Q^n(c)$ of constant curvature $c$ with $R^\perp \neq 0$. Then we have

$$\int_M K_N \, dM \geqslant 4\pi \mathcal{X}(M) \tag{1.1}$$

the equality holds if and only if $(M \sim S^2)\, n = 4$.

COROLLARY 1. Let $x\colon M \to S^n(1)$ be a minimal immersion of a compact oriented surface $M$ into the unit sphere $S^n(1)$ with $R^\perp \neq 0$. Then we have

$$\text{Area } (M) \geqslant 6\pi \mathcal{X}(M) \tag{1.2}$$

the equality holds if and only if $(M \sim S^2)\, n = 4$.

REMARK. Of course (1.2) has interest only when $M \sim S^2$, otherwise $\mathcal{X}(M) \leqslant 0$ and (1.2) becomes trivial.

The proofs of the above results are presented in section 4.

I want to thank Professor Asperti for bringing [2] and [3] for my attention.

## 2. Preliminaries.

Let $M$ be a surface immersed in a Riemannian manifold $Q^n$. For each $p$ in $M$, we use $T_pM$, $TM$, $N_pM$ and $NM$ to denote the tangent space of $M$ at $p$, the tangent bundle of $M$, the normal space of $M$ at $p$ and the normal bundle of $M$, respectively. We choose a local field of orthonormal frames $e_1, e_2, \ldots, e_n$ in $Q^n$ such that restricted to $M$, the vectores $e_1, e_2$ are in $T_pM$ and $e_3, \ldots, e_n$ are in $N_pM$. We shall make use the following convention on the ranges of indices:

$$1 \leqslant A, B, C, \ldots \leqslant n, \qquad 1 \leqslant i, j, k \leqslant 2,$$

$$3 \leqslant \alpha, \beta, \gamma, \ldots \leqslant n$$

and we shall agree that reapeated indices are summed over the respective ranges. With respect to the frame field of $Q^n$ chosen above, let $\omega^1, \omega^2, \ldots, \omega^n$ be the field of dual frames. Then the structure equations of $Q^n$ are given by.

$$(2.1) \quad d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0,$$

$$(2.2) \quad d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \phi_{AB}, \qquad \phi_{AB} = \frac{1}{2} \sum_{C,D} K_{ABCD}\, \omega_C \wedge \omega_D,$$

$$K_{ABCD} + K_{ABDC} = 0.$$

If we restrict these forms to $M$. Then

$$(2.3) \qquad \omega_\alpha = 0$$

since $0 = d\omega_\alpha = -\sum \omega_{\alpha i} \wedge \omega_i$, by Cartan' s lemma we may write

$$(2.4) \qquad \omega_{i\alpha} = \sum h^\alpha_{ij} \omega_j, \qquad h^\alpha_{ij} = h^\alpha_{ji},$$

From these formulas, we obtain

$$(2.5) \quad d\omega_i = -\sum \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$(2.6) \quad d\omega_{ij} = -\sum \omega_{ik} \wedge \omega_{k_j} + \Omega_{ij}, \qquad \Omega_{ij} = \frac{1}{2} \sum R_{ijkl}\, \omega_k \wedge \omega_l,$$

$$(2.7) \quad R_{ijkl} = K_{ijkl} + \sum_\alpha (h^\alpha_{ik} h^\alpha_{j\ell} - h^\alpha_{il} h^\alpha_{jk}),$$

$$(2.8) \quad d\omega_{\alpha\beta} = -\sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}, \quad \Omega_{\alpha\beta} = \frac{1}{2} \sum R_{\alpha\beta kl}\, \omega_k \wedge \omega_l,$$

$$(2.9) \quad R_{\alpha\beta kl} = K_{\alpha\beta kl} + \sum_i (h^\alpha_{ik} h^\beta_{il} - h^\alpha_{il} h^\beta_{ik}).$$

The Riemannian connection of $M$ is defined by $(\omega_{ij})$. The form $(\omega_{\alpha\beta})$ defines a connection $\nabla^{\perp}$ in the normal bundle of $M$. We call

$$B = \sum_{\alpha, i, j} h_{ij}^{\alpha} \omega_i \omega_j e_{\alpha} \tag{2.10}$$

the *second fundamental form* of $M$. The *mean curvature vector* is given by

$$H = \sum_{\alpha} \left( \sum_{i} h_{ii}^{\alpha} \right) e_{\alpha} \tag{2.11}$$

An immersion is said to be minimal if $H = 0$.

Let $R^{\perp}$ be the curvature tensor associated with $\nabla^{\perp}$. Let $\{e_1, e_2\}$ be a tangent frame, if we denote $B_{ij} = B(e_i, e_j)$; $i, j = 1, 2$ then it is easy to see that

$$R^{\perp}(e_1, e_2) = (B_{11} - B_{22}) \wedge B_{12} . \tag{2.12}$$

An interesting notion in the study of surfaces in higher codimension is that of the *ellipse of curvature* defined as $\{B(X, X) \in N_p M : \langle X, X \rangle = = 1\}$. To see that it is an ellipse, we just have to look at the following formula, for

$$\begin{cases} X = \cos\theta e_1 + \sin\theta e_2 , \\ B(X, X) = H + \cos 2\theta u + \sin 2\theta v , \end{cases} \tag{2.13}$$

where $u = (B_{11} - B_{22})/2$, $v = B_{12}$ and $\{e_1, e_2\}$ is a tangent frame. So we see that, as $X$ goes once around the unit tangent circle, $B(X, X)$ goes twice around the ellipse. Of course this ellipse could degenerate into a line segment or a point. Everywhere the ellipse is not a circle we can choose $\{e_1, e_2\}$ orthonormal such that $u$ and $v$ are perpendicular. When this happens they will coincide with the semi-axes of the ellipse.

From (2.12) it follows that if $R^{\perp} \neq 0$ then $u$ and $v$ are linearly independent and we can define a 2-plane subbundle $(NM)^*$ of the normal bundle $NM$. This plane inherits a Riemannian connection from that of $NM$. Let $R^*$ be its curvature tensor and define its curvature $K^*$ by

$$d\omega_{34} = -K^* \omega_1 \wedge \omega_2 \tag{2.14}$$

if $\{e_3, e_4\}$ locally generates $(NM)^*$.

Now, if $\xi$ is perpendicular to $(NM)^*$, then from (2.12), $R^{\perp}(e_1, e_2)\xi = 0$. Hence, it makes sense to define the normal curvature as

$$K_N = \langle R^{\perp}(e_1, e_2) e_4, e_3 \rangle \tag{2.15}$$

where $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are orthonormal oriented bases of $T_pM$ and $N_pM$, respectively. If $TM$ and $(NM)^*$ are oriented, then $K_N$ is globally defined. In codimension 2, $NM = (NM)^*$ and $K_N$ has a sign. In higher codimension, if $R^\perp \neq 0, (NM)^*$ is globally defined and oriented if $TM$ is. In this case, it is shown in [1] that $\mathcal{X}(NM)^* = 2\mathcal{X}(M)$, where $\mathcal{X}(NM)^*$ denote the Euler characteristic of the plane bundle $(NM)^*$ and $\mathcal{X}(M)$ denote the Euler characteristc of the tangle bundle $TM$.

## 3. Minimal immersions with $R^\perp \neq 0$.

In this section we assume that $M$ has non-zero normal curvature tensor $R^\perp$. Also if $M$ is orientable, then we will always choose orientations in $TM$ and in $(NM)^*$ such that $K_N$ is positive. We have

PROPOSITION 1.1. Let $x: M \to Q^n(c)$ be a minimal immersion of an oriented surface $M$ into an orientable $n$-dimensional space form $Q^n(c)$ of constant curvature $c$. Then we have

$$\Delta(\log |K_N - K + c|) = 2(2K - K^*) \tag{3.1}$$

if $(K - c)^2 - K_N^2 > 0$, and consequently

$$\Delta(\log |K_N + K - c|) = 2(2K + K^*). \tag{3.2}$$

PROOF. By Itoh [6] there exists isothermal coordinates $\{x_1, x_2\}$ such that putting $X_i = \partial/\partial x_i$, $i = 1, 2$ then $u = B(X_1, X_1) = -B(X_2, X_2)$ and $v = B(X_1, X_2)$ are the semi-axes of the ellipse at every point where $(K - c)^2 - K_N^2 \neq 0$. Moreover we observe that $|X_i|^2 = E = ((K - c)^2 - K_N^2)^{-1/4}$, $i = 1, 2$. If we denote $\lambda = < u, u >^{1/2}$ and $\mu = \langle v, v\rangle^{1/2}$ and following the same arguments that [10] we have

$$\lambda^2 - \mu^2 = 1, \tag{3.3}$$

$$\lambda^2 + \mu^2 = -(K - c)E^2, \tag{3.4}$$

$$2\lambda\mu = K_N E^2. \tag{3.5}$$

If $(K - c)^2 - K_N^2 > 0$ from (3.4) and (3.5) we obtain

$$\lambda + \mu = (K_N - K + c/K_N + K - c)^{1/4}. \tag{3.6}$$

Let $e_3 = \lambda^{-1}u$ and $e_4 = \mu^{-1}v$ an oriented frame in $(NM)^*$. Now, following the same computations that [10] we get

$$\omega_{34}(X_1) = -X_2(f), \tag{3.7}$$

$$\omega_{34}(X_2) = X_1(f), \tag{3.8}$$

where $f = \log|\lambda + u|$.

Hence, we have

$$\omega_{34} = -X_2(f)dX_1 + X_1(f)dX_2. \tag{3.9}$$

Deriving (3.9) and using (2.14) we get

$$\begin{aligned} -K^*\omega_1\wedge\omega_2 &= d\omega_{34}E^{-1} \\ &= (-X_2X_2(f)dX_2\wedge dX_1 + X_1X_1(f)dX_1\wedge dX_2)E^{-1} = \\ &= (X_1X_1(f) + X_2X_2(f))E^{-1}dX_1\wedge dX_2) = \tilde{\Delta}(f)E^{-1}\omega_1\wedge\omega_2 \end{aligned} \tag{3.10}$$

where $\tilde{\Delta}$ denotes de Laplacian of the «flat» metric. We know $\tilde{\Delta}(f) = E\Delta(f)$, where $\Delta$ is the Laplacian of the surface. Hence, from (2.18) and (2.22) we get

$$\Delta(\log|K_N - K + c/K_N + K - c|) = -4K^*. \tag{3.11}$$

Using $E = (K-c)^2 - K_N^2)^{-1/4}$ and the Gaussian curvature $K$ given by the equation

$$K = -\frac{1}{2}E^{-1}\tilde{\Delta}\log E. \tag{3.12}$$

we obtain

$$\Delta(\log|K_N - K + c|) + \Delta(\log|K_N + K - c|) = 8K \tag{3.13}$$

From (3.11) and (3.13) we get the equations (3.1) and (3.2).

COROLLARY 1. Let $x\colon M \to Q^n(c)$ be a minimal immersion with $K^* > 0$. Then the ellipse is a circle.

PROOF. Suppose that the ellipse is not a circle then from (3.11) $\Delta(\log|K_N - K + c/K_N + K - c|) < 0$. So $K_N > 0$ implies that $\log\dfrac{|K_N - K + c|}{|K_N + K - c|}$ is subharmonic and bounded from below. Therefore $\log\dfrac{|K_N - K + c|}{|K_N + K - c|}$ is constant and this implies that $K^* \equiv 0$. This is a contradiction.

COROLLARY 2. Let $x\colon M \to Q^n(c)$ be a minimal immersion of a compact surface $M$ with $2K > K^*$. Then the ellipse is a circle.

PROOF. Suppose that the ellipse is not a circle then from (3.1) $\Delta(\log|K_N - K + c|) > 0$. So we have that $\log|K_N - K + c|$ is subharmonic and bounded from above and therefore is constant. This implies that $2K = K^*$. This is a contradiction.

## 4. Proof of Theorems.

PROOF OF THEOREM 1. First we consider the case when the ellipse is not a circle, i.e., $(K-1)^2 - K_N^2 > 0$. Now if $2K \geqslant K^*$ then from (3.1) follows that $\Delta(\log|K_N - K + 1|) \geqslant 0$. So we have that $\log|K_N - K + 1|$ is subharmonic and bounded from above. Then

$$K_N - K + 1 = \text{constant} \tag{4.1}$$

and $2K = K^*$. On the other hand from (3.2) we get $\Delta(\log|K_N + K - 1|) = 2(2K + K^*) = 8K \geqslant 0$. Similarly from above we have

$$K_N + K - 1 = \text{constant}. \tag{4.2}$$

From (4.1) and (4.2) follows that $K^*$, $K_N$ and $K$ are constant.

In the case that ellipse is a circle the theorem follows by Rodriguez-Guadalupe [9]. This completes the proof of theorem.

PROOF OF THEOREM 2. Suppose that the ellipse is not a circle. From (3.13) we obtain

$$\Delta(\log|K_N - K + 1/K_N + K - 1|) = 8K \geqslant 0\,. \tag{4.3}$$

From (3.5), Rodriguez-Guadalupe ([9], p. 9) and $K \geqslant o$ implies $2\lambda\mu E^{-2} = K_N \leqslant 1$. So from (3.4) we have $(\lambda^2 + \mu^2)E^{-2} = 1$. Therefore we get

$$0 \leqslant (\lambda + \mu)^2 E^{-2} = K_N - K + 1\,,$$

$$= (\lambda^2 + \mu^2)E^{-2} + 2\lambda\mu E^{-2} \leqslant 2\,.$$

This implies that $|K_N - K + 1|$ is bounded from above. Similarly $|K_N + K - 1|$ is bounded from above, too. Then $\log(|K_N - K + 1/K_N + K - 1|)$ is subharmonic and bounded from above and therefore is constant. From (4.3) follows that $K \equiv 0$.

PROOF OF THEOREM 3. From Asperti ([2], Prop. 3.6) we have

$$K^* = K_N - \frac{\|B^2\|^2}{2K_N} \tag{4.4}$$

where $B^2$ is the 3th fundamental form of $M$. From (4.4) we obtain

$$K_N \geqslant K^* \tag{4.5}$$

Integrating (4.5) over $M$ and applying Ferus-Rodriguez-Asperti ([1], Th. 1) we get

$$\int_M K_N\, dM \geqslant \int_M K^*\, dM = 2\pi\mathcal{X}(NM)^* = 4\pi\mathcal{X}(M). \tag{4.6}$$

If $\int_M K_N\, dM = 4\pi\mathcal{X}(M)$ then $K_N = K^*$ and from (4.4) $B^2 \equiv 0$. By Erbacher [5] the codimension is two and $n = 4$.

PROOF OF COROLLARY 1. If $\text{Area}(M) = 6\pi\mathcal{X}(M)$ then $\mathcal{X}(M) > 0$ and, actually Area $(M) = 12\pi$. It follows from Asperti ([3], p. 60) that

$$12\pi \geqslant 2\pi(s+1)(s+2) \tag{4.7}$$

where $s$ is sucht that $n = 2 + 2s$. It is clear now from (4.7) that $s = 1$ and $n = 4$.

On the other hand, if $n = 4$ and $x\colon M \to S^4(1)$ is a minimal two sphere with $R^\perp \neq 0$, then $K_N = K^*$ and by theorem 3 above and Corollary 1 of Rodriguez-Guadalupe ([9]) we have

$$\text{Area}(M) = 2\pi\mathcal{X}(M) + 2\pi\mathcal{X}(NM) = 2\pi\mathcal{X}(M) + \int_M K_N\, dM = 6\pi\mathcal{X}(M).$$

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Manoscritto pervenuto in redazione il 6 giugno 1994.