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A Semi-Linear Problem for the Heisenberg Laplacian.

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ABSTRACT - We study the Dirichlet problem in a bounded domain Ω, for the equation $-\Delta_H u + (q-\lambda)u = au^p$ where Δ_H is the sub-elliptic operator usually called Heisenberg Laplacian and a changes sign in Ω. Precisely, we give some necessary and sufficient conditions on the function a and on λ for the existence of positive solutions.

1. Introduction and results.

Let Ω be a bounded smooth domain of $\mathbb{R}^{2n+1}$, $\xi := (x_1, .., x_n, y_1, .., y_n, t) := (x, y, t)$.

We denote by H^n the vector space $\mathbb{R}^{2n+1}$ endowed with the group action:

$$\xi_o \circ \xi = \left(x + x_o, y + y_o, t + t_o + 2\sum_{i=1}^{n}(x_i y_{o_i} - y_i x_{o_i})\right).$$

H^n is a Lie group and the corresponding Lie Algebra of left-invariant

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vector fields is generated by

$$\begin{cases} X_i^1 = \dfrac{\partial}{\partial x_i} + 2y_i \dfrac{\partial}{\partial t}\,, \\ X_i^2 = \dfrac{\partial}{\partial y_i} - 2x_i \dfrac{\partial}{\partial t}\,, \\ T = \dfrac{\partial}{\partial t}\,. \end{cases}$$

The second order self-adjoint operator:

$$\Delta_H := \sum_{i=1}^{n} (X_i^1)^2 + (X_i^2)^2$$

i.e.

$$\Delta_H = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2)\frac{\partial^2}{\partial t^2}$$

is usually called the Heisenberg Laplacian.

Observe that $\Delta_H u = \operatorname{div}(A\nabla u)$, where A is the following $(2n+1) \times \times (2n+1)$ matrix:

$$\begin{bmatrix} I & 0 & 2y^T \\ 0 & I & -2x^T \\ 2y & -2x & 4(x^2+y^2) \end{bmatrix},$$

I is the $(n \times n)$ identity matrix and $x^2 + y^2 = \sum_{i=1}^{n} x_i^2 + y_i^2$. Therefore the Gauss-Green formula holds:

$$\int_{\Omega} \Delta_H uv\, d\xi = -\int_{\Omega} \nabla_H u \nabla_H v\, d\xi + \int_{\partial\Omega} vA\nabla u \cdot \nu\, d\sigma, \tag{1.1}$$

where $\nabla_H u$ is the $2n$-vector $(X_1^1 u, X_1^2 u, \ldots, X_n^1 u, X_n^2 u)$.

Observe that A is a positive semi-definite matrix with $\det(A) \equiv 0$ for all $(x, y, t) \in H^n$ and $\operatorname{rank}(A) \equiv 2n$.

Furthermore, X_i^1, X_j^2 satisfy $[X_i^1, X_j^2] = -4T\delta_{ij}$, so the vector fields X_i^j and their first order commutators span the whole Lie Algebra. Therefore, the vector fields satisfy the Hormander condition of order one (see [2], [9]). This implies the hypoellipticity of Δ_H (i.e. if $-\Delta_H u \in C^\infty$ then $u \in C^\infty$ (see [9])).

Folland and Stein in [6] introduced $\overset{\circ}{S}{}_1^2(\Omega)$, the analogue of $W_o^{1,2}$, naturally related to the vector fields X_i^j. Namely, $\overset{\circ}{S}{}_1^2(\Omega)$ is the closure of $C_o^1(\Omega)$ with respect to the norm:

$$\|u\|_{S_2^1}^2 = \int_\Omega |\nabla_H u|^2 + |u|^2 d\xi .$$

Let, moreover, Γ^β be the analogue of the classical Holder functions space introduced in [6], see Definition 2.2.

Semi-linear equations for the Heisenberg Laplacian similar to (1.2) but with the function a of constant sign, have been studied by Garofalo and Lanconelli in [7].

The main results in the present paper are some necessary and sufficient conditions for the existence of a solution $u \in \Gamma^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ for $0 < \alpha < 1$ of the following problem:

$$\begin{cases} -\Delta_H u + (q - \lambda) u = a u^p, & u > 0 \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where λ is a real parameter, a and q belong to Γ^β for some $\beta > 0$ and $p > 1$. We assume that a changes sign in Ω.

Observe that, from the definition of Γ^β, $u \in \Gamma^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ for $0 < \alpha < 1$ implies that u is a classical solution of (1.2).

Let λ_1 be the principal eigenvalue of $-\Delta_H + q$, precisely there exists $\phi > 0$ such that:

$$-\Delta_H \phi + q\phi = \lambda_1 \phi , \qquad \phi = 0 \quad \text{on } \partial\Omega . \tag{1.3}$$

Problem (1.2) with the classical Laplacian instead of the Δ_H, was studied by Berestycki, Capuzzo Dolcetta, Nirenberg in [3]. We will give similar sufficient and necessary conditions on λ and a, to obtain the existence of solutions of (1.2).

Precisely, we obtain the following theorems:

THEOREM 1.1. *Assume* (1.3) *holds and* (1.2) *has a solution. Then the following conditions are satisfied:*

(i) $\int_\Omega a\phi^{p+1} d\xi < 0$, *if* $\lambda > \lambda_1$,

(ii) $\Omega^+ := \{x \in \Omega;\, a(x) > 0\} \neq \emptyset$, *if* $\lambda < \lambda_1$,

(iii) $\Omega^+ \neq \emptyset$, $\Omega^- := \{x \in \Omega;\, a(x) < 0\} \neq \emptyset$ *and* $\int_\Omega a\phi^{p+1} d\xi < 0$ *if* $\lambda = \lambda_1$.

THEOREM 1.2. *Assume* (1.3), $1 < p < \frac{Q+2}{Q-2}$, *where* $Q = 2n + 2$ and

(i) $\int_{\Omega} a\phi^{p+1} d\xi < 0$, if $\lambda \geq \lambda_1$,

(ii) $\Omega^+ \neq \emptyset$ *and* $\Omega^- \neq \emptyset$.

Then there exists λ^* *such that* (1.2) *has a solution for* $\lambda_1 \leq \lambda < \lambda^*$, *while no solution exist for* $\lambda > \lambda^*$.

The proofs of Theorem 1.1 and 1.2, given in section 3, are based on variational methods similar to those used in [3] and on the characteristic features and properties of the Heisenberg laplacian.

In the next section we detail these properties, in particular we prove a Hopf type Lemma and we state an embedding theorem due to Folland and Stein [6], see also Garofalo and Lanconelli [7].

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2. On the Heisenberg Laplacian.

From the definition of the Lie Group acting on $\mathbb{R}^{2n+1}$ and thus from the definition of the X_i^j, it is evident that the t direction plays a particular role. We are in an anisotropic space, therefore the concept of dilation is modified. Precisely, there is a «natural» group of dilations on H^n introduced by Folland and Stein (see [5], [6]) for which the X_j^i are homogeneous, given by:

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t) \qquad \forall \lambda > 0 . \tag{2.1}$$

Therefore Δ_H is homogeneous of degree 2 with respect to δ_λ, precisely:

$$\Delta_H \circ \delta_\lambda = \lambda^2 \delta_\lambda \circ \Delta_H .$$

In order to have a distance from the origin which is homogeneous of degree zero with respect to the dilation (2.1), we define, as in [5]:

$$d(\xi, 0) = \left(\left(\sum_{i=1}^{n} x_i^2 + y_i^2 \right)^2 + t^2 \right)^{1/4} .$$

Using the group action, we obtain a «natural» metric in H^n by defining:

$$d(\xi_1, \xi_2) = d(\xi_2^{-1} \circ \xi_1, 0).$$

We will denote by

$$B_H(\eta, r) = \{\xi \in \mathbb{R}^{2n+1} : d(\xi, \eta) < r\}$$

the Heisenberg ball, also called «Boule de Korànyi», which will play the role of the euclidean ball in H^n.

$Q = 2n + 2$ is called the «homogeneous» dimension of H^n and will play the same role as the euclidean in the uniformly elliptic theory. In particular, for example, the Lebesgue measure of $B_H(0, R)$ is proportional to the Q-th power of R.

The fondamental solution of the Heisenberg Laplacian is constructed similarly to the fondamental solution of the Laplacian but with the intrinsic distance defined as above. Precisely, it is easy to check that

$$\Delta_H(d(\xi, \xi_o)^{2-Q}) = \delta_{\xi_o}(\xi).$$

Observe that for any operator

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$$

with $(a_{ij}(x))$ positive semi-definite matrix, the weak Maximum Principle holds (see [7]). Moreover, if L is in divergence form and is generated by vector fields satisfying the Hormander condition, then the Strong Maximum Principle (see [2]) holds.

In the proof of Theorem 1.2 we need a version of the Hopf lemma for the Heisenberg laplacian. Let us first give the following definition which generalizes the interior sphere condition property.

DEFINITION 2.1. *Let $\Omega \subset \mathbb{R}^{2n+1}$. Ω satisfies the interior Heisenberg's sphere condition at $\xi_o \in \partial\Omega$ if there exist a constant $R > 0$ and $\eta \in \Omega$ such that the Heisenberg ball $B_H(\eta, R) \subseteq \Omega$ and ξ_o belongs to the boundary of the ball.*

LEMMA 2.1. *Let ξ_o be a point of $\partial\Omega$ where the interior Heisenberg's sphere condition is satisfied. If*

1) *$u \in \Gamma^{2+a}(\Omega)$ for some $a > 0$ and is continuous in ξ_o,*

2) *$-\Delta_H u + cu \geqslant 0$ in Ω where c is bounded in Ω,*

3) *$u(\xi) > u(\xi_o) := 0$ for $\xi \in B_H(\xi_o, R) \cap \Omega$ for some $R > 0$,*

then, for any n exterior direction to $\partial\Omega$ at ξ_o,

$$\limsup_{h\to 0^+} \frac{u(\xi_o)-u(\xi_o-hn)}{h}<0$$

and if it exists,

$$\frac{\partial u(\xi_o)}{\partial n}<0.$$

Moreover $A\nabla u(\xi_o)\cdot\nu(\xi_o)<0$ when ν, the exterior normal to $\partial\Omega$ at ξ_o, is not orthogonal to the vector fields X_i^j, $j=1,2$, $i=1,\ldots,n$.

Proof. Let $\xi_o\in\partial\Omega$ and let $\eta=(x',y',t')$ and $R>0$ be as in the Definition 2.1. Observe that $\partial B_H(\eta,R)$ is tangent to $\partial\Omega$ at ξ_o.

Similarly to Serrin in [12], see also [8], we will eliminate the zero order term.

Indeed, let n be an exterior direction to $\partial\Omega$ at ξ_o, and choose $w=e^{-K(x_1-x_1')^2}u$, with $K>0$. It is easy to check that w satisfies $-\Delta_H w-4K(x_1-x_1')X_1^1 w\geqslant 0$ as soon as K is sufficiently large.

If we prove that w satisfies

$$0>\frac{\partial w(\xi_o)}{\partial n}$$

then the statement follows as

$$\frac{\partial w(\xi_o)}{\partial n}=e^{-K((x_o)_1-x_1')^2}\frac{\partial u(\xi_o)}{\partial n}.$$

For $d(\xi,\eta)=r$ we define

$$v(\xi)=e^{-aR^2}-e^{-ar^2}$$

for $\varrho<r<R$. It is easy to check that, for functions depending only on the distance from the origin r, we have:

$$\Delta_H v(r)=\Psi(\xi)\left[v_{rr}+\frac{Q-1}{r}v_r\right]$$

where $\Psi=\dfrac{\sum(x_i^2+y_i^2)}{r^2}$ is a positive function, homogeneous of degree zero with respect to the dilation defined in (2.3) and where v_r is the derivative of v with respect to r.

Thus, as v depends only on the distance from η and Δ_H and X_1^1 are invariants with respect to the group action, (2.2) still holds with $r = d(\eta^{-1} o \xi, 0)$. Precisely, using hypothesis 2):

$$-\Delta_H v - 4K(x_1 - x_1')X_1^1 v =$$

$$= \{\Psi(\eta^{-1} o \xi)[4a^2 r^2 - 2Qa] - 8Kar(x_1 - x_1')X_1^1 r\} e^{-ar^2} \geq 0$$

for a sufficiently large.

Therefore in $B_H(\eta, R) \backslash B_H(\eta, \varrho)$, $-\Delta_H(w + \varepsilon v) - 4K(x_1 - x_1') \cdot X_1^1(w + \varepsilon v) \geq 0$ and on $\partial B_H(\eta, R)$, $w + \varepsilon v \geq 0$. Furthermore, for ε sufficiently small, $w + \varepsilon v \geq 0$ on $\partial B_H(\eta, \varrho)$. Thus, from the weak maximum principle, we obtain that

$$w + \varepsilon v \geq 0 \quad \text{in } B_H(\eta, R) \backslash B_H(\eta, \varrho).$$

Now observe that $w(\xi_o) = -\varepsilon v(\xi_o) = 0$. Furthermore, for any $n \cdot \nu > 0$ and for small $h > 0$,

$$w(\xi_o - hn) \geq -\varepsilon v(\xi_o - hn).$$

Therefore, as v_r is strictly positive, the first part of the lemma is proved.

To end the proof it is enough to check that $A\nu$ is an exterior direction at ξ_o when ν is not orthogonal to X_i^j, $j = 1, 2$, $i = 1, \ldots, n$.

But this is immediate from the fact that

$$A(\xi_o)\nu \cdot \nu = \left(\sum_{i=1}^n ((x_0)_i - x_i')^2 + ((y_0)_i - y_i')^2 \right) R.$$

REMARK 1. Observe that, differently from the uniformly elliptic case, $A \nabla u \cdot \nu$ may be zero. As an example, suppose that u is constant on the boundary so that $\forall \xi_o \in \partial\Omega$, $\nabla u(\xi_o)$ has the same direction as the normal $\nu(\xi_o)$. Thus, when ν is orthogonal to X_i^j, $j = 1, 2$, $i = 1, \ldots, n$, $A \nabla u \cdot \nu = 0$.

REMARK 2. Clearly Lemma 2.1 holds also for the operator

$$-\Delta_H + B \cdot \nabla_H + c$$

if $B := (k_1^1(x_1 - x_1'), k_1^2(y_1 - y_1'), \ldots, k_n^1(x_n - x_n'), k_n^2(y_n - y_n'))$ where k_j^i are bounded functions.

Let us recall the definitions of the functional spaces needed (see [4], [6]). For this purpose, let $I = (i_1, \ldots, i_k)$, for $1 \leq i_j \leq 2n$ and $j = 1, \ldots, k$, a multi-index and set $|I| = k$. Denote by X_I the operator $X_{i_1} \circ X_{i_2} \circ \ldots \circ X_{i_k}$ where X_{i_j} is one of the vector fields X_i^1, X_i^2 $(i = 1, \ldots, n)$.

Observe that $T = \partial/\partial t$ is a linear combination of X_I with $|I| = 2$.

DEFINITION 2.2. *For* $0 < \beta < 1$

$$\Gamma^{\beta} = \left\{ f \in L^{\infty} C^{o} : \sup_{\xi, \eta} \frac{|f(\eta \circ \xi) - f(\eta)|}{d(\xi, 0)^{\beta}} < \infty \right\};$$

for $\beta = 1$

$$\Gamma^{\beta} = \left\{ f \in L^{\infty} C^{o} : \sup_{\xi, \eta} \frac{|f(\eta \circ \xi) + f(\eta \circ \xi^{-1}) - 2f(\eta)|}{d(\xi, 0)^{\beta}} < \infty \right\};$$

for $\beta = k + a$ *with* $k = 1, 2, \ldots$ *and* $0 < a \leq 1$

$$\Gamma^{\beta} = \{ f \in \Gamma^{a} : X_I f \in \Gamma^{a} for |I| \leq k \}.$$

Let moreover $\mathring{S}_k^q$, for $1 \leq q \leq +\infty$ and k a positive integer, be the set of functions $f \in L^q$ such that $X_I f \in L^q$ for $|I| \leq k$.

As mentioned in the introduction, we need a theorem analogous to the standard Sobolev embedding theorems, for the spaces $\mathring{S}_1^2(\Omega)$, $\mathring{S}_2^q(\Omega)$.

THEOREM 2.1. $\mathring{S}_1^2(\Omega)$ *is compactly embedded in* $L^p(\Omega)$ *for* $1 \leq p < \frac{2Q}{Q-2}$ *where* $Q = 2n + 2$.

For $1 \leq q \leq +\infty$, $k = 1; 2$ *and* $\beta = k - Q/q$, *if* β *is not an integer* $\mathring{S}_k^q(\Omega) \subset \Gamma^{\beta}$, *if* β *is an integer* $\mathring{S}_k^q(\Omega) \subset \Gamma^{\beta - \varepsilon}$ *for any* $\varepsilon > 0$.

The proof is given in [6], see also [7] for the first statement.

3. Proofs.

Theorem 1.1 is a corollary of the following technical lemma.

LEMMA 3.1. *Assume that* (1.3) *holds, then for any solution* u *of* (1.2) *and* $\gamma \geq 0$ *the following is true:*

$$(3.1) \qquad \int_{\Omega} a u^{p-\gamma} \phi^{1+\gamma} d\xi = (\lambda_1 - \lambda) \int_{\Omega} u^{1-\gamma} \phi^{1+\gamma} d\xi - \gamma \int_{\Omega} Ag \cdot g \, d\xi$$

where

$$g = \phi^{\frac{\gamma - 1}{2}} u^{\frac{1-\gamma}{2}} \nabla\phi - \phi^{\frac{1+\gamma}{2}} u^{\frac{-1-\gamma}{2}} \nabla u.$$

PROOF. It is easy to check that u and ϕ satisfy the conditions of the Hopf's Lemma 2.1, thus there exists a constant $t > 0$ such that in a neighbourhood of $\partial\Omega$, $u \geqslant t\phi$. Therefore $u^{-\gamma}\phi^{1+\gamma}$ is bounded.

Let us multiply (1.2) by $u^{-\gamma}\phi^{1+\gamma}$, integrate and apply the divergence theorem to obtain:

$$\int_\Omega a u^{p-\gamma}\phi^{1+\gamma}\,d\xi = \int_\Omega (q-\lambda)u^{1-\gamma}\phi^{1+\gamma}\,d\xi + \int_\Omega A\nabla u\cdot\nabla(u^{-\gamma}\phi^{1+\gamma})\,d\xi .$$

But,

$$\int_\Omega A\nabla u\cdot\nabla(u^{-\gamma}\phi^{1+\gamma})\,d\xi = (1+\gamma)\int_\Omega u^{-\gamma}\phi^{\gamma}A\nabla u\cdot\nabla\phi\,d\xi - $$
$$-\gamma\int_\Omega u^{-\gamma-1}\phi^{1+\gamma}A\nabla u\cdot\nabla u\,d\xi;$$

so,

$$\int_\Omega a u^{p-\gamma}\phi^{1+\gamma}\,d\xi = \int_\Omega (q-\lambda)u^{1-\gamma}\phi^{1+\gamma}\,d\xi + \tag{3.2}$$
$$+(1+\gamma)\int_\Omega u^{-\gamma}\phi^{\gamma}A\nabla u\cdot\nabla\phi\,d\xi - \gamma\int_\Omega u^{-1-\gamma}\phi^{1+\gamma}A\nabla u\cdot\nabla u\,d\xi .$$

On the other hand (1.3), after multiplication by $u^{1-\gamma}\phi^{\gamma}$ and integration by parts yields:

$$\int_\Omega q u^{1-\gamma}\phi^{1+\gamma}\,d\xi = -\gamma\int_\Omega u^{1-\gamma}\phi^{\gamma-1}A\nabla\phi\cdot\nabla\phi\,d\xi + \tag{3.3}$$
$$+\lambda_1\int_\Omega u^{1-\gamma}\phi^{1+\gamma}\,d\xi - (1-\gamma)\int_\Omega u^{-\gamma}\phi^{\gamma}A\nabla\phi\cdot\nabla u\,d\xi .$$

Hence, combining (3.2) and (3.3), and using the symmetry of A, we get:

$$\int_\Omega a u^{p-\gamma}\phi^{1+\gamma}\,d\xi = (\lambda_1-\lambda)\int_\Omega u^{1-\gamma}\phi^{1+\gamma}\,d\xi +$$
$$+2\gamma\int_\Omega u^{-\gamma}\phi^{\gamma}A\nabla u\cdot\nabla\phi d\xi - \gamma\int_\Omega u^{-1-\gamma}\phi^{1+\gamma}A\nabla u\cdot\nabla u\,d\xi$$
$$-\gamma\int_\Omega u^{1-\gamma}\phi^{\gamma-1}A\nabla\phi\cdot\nabla\phi\,d\xi = (\lambda_1-\lambda)\int_\Omega u^{1-\gamma}\phi^{1+\gamma}\,d\xi - \gamma\int_\Omega Ag\cdot g\,d\xi$$

and the claim follows.

PROOF OF THEOREM 1.1. Let us observe that $\gamma = 0$ in (3.1) implies that $\Omega^+ \neq \emptyset$ when $\lambda < \lambda_1$ and both Ω^+ and Ω^- are not empty for $\lambda = \lambda_1$.

To complete the proof, we choose $\gamma = p$ and (3.1) becomes:

$$\int_\Omega a\phi^{1+p}\, d\xi = (\lambda_1 - \lambda)\int_\Omega u^{1-p}\phi^{1+p}\, d\xi - p\int_\Omega Ag\cdot g\, d\xi . \tag{3.4}$$

Therefore, as A is positive semi-definite, condition (i) holds.

For $\lambda = \lambda_1$ we still have to prove that the right hand side of (3.4) is strictly negative.

Suppose, by contradiction, that $Ag\cdot g = 0$. As it can easily be computed, this implies that $\nabla_H(u/\phi) = 0$. The Hormander condition guaranties that all functions satisfying $\nabla_H w = 0$, in a connected domain, are necessarily constants. Therefore, $u = C\phi$ for some $C > 0$. We have reached a contradiction since $C\phi$ is not a solution of (1.2). This concludes the proof.

The next lemmas will be used in the proof of Theorem 1.2.

LEMMA 3.2. *Let* $u \in \mathring{S}^2_1(\Omega)$ *be a solution of*

$$\varDelta_H u - u = Vu + g$$

with $V \in L^{Q/2}(\Omega)$ *and* $g \in L^2(\Omega) \cap L^q(\Omega)$ *for some* $q \in [2, \infty)$. *If*

$$B_Q \| V^- \|_{L^{Q/2}(\Omega)} < 1$$

with B_Q *the Sobolev constant of the embedding inequality of* $\mathring{S}^2_1$ *into* L^q, *then* $u \in L^{q(1+2/(Q-2))}$.

This lemma is due to Garofalo and Lanconelli, the proof may be found in ([7]).

LEMMA 3.3. *Let the function* $a \in \Gamma^\beta(\Omega)$ *satisfy* $\Omega^+ \neq \emptyset$ *with* Ω^+ *as in Theorem* 1.1, *and set*

$$M = \sup\left\{ \int_\Omega a|u|^{p+1} d\xi;\ u \in S_\lambda \right\} \tag{3.5}$$

where

$$S_\lambda = \left\{ u \in \mathring{S}^2_1(\Omega);\ I(u) := \int_\Omega |\nabla_H u|^2 d\xi + \int_\Omega (q - \lambda) u^2 d\xi = 1 \right\}.$$

Then, $S_\lambda \neq \emptyset$ *and* $M > 0$.

PROOF. Let $\xi_o \in \Omega^+$. Consider the set $\Omega^* = (\xi_o)^{-1} \circ \Omega$. Thus $0 \in \Omega^{*+}$. Take for a fixed R, any $\psi \in C_o^\infty(B_H(0; R))$ and set $u_\varepsilon(x, y, t) = (1/\varepsilon^{Q/2})\psi(x/\varepsilon, y/\varepsilon, t/\varepsilon^2)$. The following holds:

$$u_\varepsilon \in C_o^\infty(B_H(0; \varepsilon R));$$

$$\int_{B_H(0;\varepsilon R)} u_\varepsilon^2 \, d\xi = \int_{B_H(0;R)} \psi^2 \, d\xi;$$

$$\int_{B_H(0;\varepsilon R)} |\nabla_H u_\varepsilon|^2 \, d\xi = \frac{1}{\varepsilon^2} \int_{B_H(0;R)} |\nabla_H \psi|^2 \, d\xi.$$

Choose $R > 0$ such that $B_H(\xi_o; R) \subset \Omega^+$ and ψ the principal eigenfunction of $-\Delta_H$ in $B_H(\xi_o; R)$ with Dirichlet condition on ∂B_H. Denote by μ^* the corresponding eigenvalue and take ε sufficiently small that $\operatorname{supp} u_\varepsilon \subset \Omega^{*+}$ and

$$\frac{\mu_*}{\varepsilon^2} \geqslant \sup |q - \lambda| + \frac{1}{2\|\psi\|_{L^2}^2}.$$

Then the following holds:

$$\int_{B_H(0;\varepsilon R)} |\nabla_H u_\varepsilon|^2 \, d\xi = \frac{\mu^*}{\varepsilon^2} \int_{B_H(0;R)} \psi^2 \, d\xi \geqslant \sup |q - \lambda| \int_{B_H(0;\varepsilon R)} u_\varepsilon^2 \, d\xi + \frac{1}{2}.$$

Thus

$$\int_{B_H(0;\varepsilon R)} |\nabla_H u_\varepsilon|^2 \, d\xi + \int_{B_H(0;\varepsilon R)} (q - \lambda) u_\varepsilon^2 \, d\xi \geqslant \frac{1}{2}.$$

Observe that X_i^j are left invariant with respect to the group action and $(\xi_o)^{-1} \circ B_H(\xi_o; R) = B_H(0; R)$; so

$$\int_{B_H(\xi_o;\varepsilon R)} |\nabla_H u_\varepsilon|^2 \, d\xi + \int_{B_H(\xi_o;\varepsilon R)} (q - \lambda) u_\varepsilon^2 \, d\xi \geqslant \frac{1}{2}.$$

Extend now u_ε to 0 in $\Omega \backslash B_H(\xi_o; \varepsilon R)$ and call it $\tilde{u}_\varepsilon$. It is easy to check that

$$\int_\Omega |\nabla_H \tilde{u}_\varepsilon|^2 \, d\xi + \int_\Omega (q - \lambda) \tilde{u}_\varepsilon^2 \, d\xi \geqslant \frac{1}{2}$$

so either $\widetilde{u}_\varepsilon \in S_\lambda$ or there exists $k > 1$ such that $k\widetilde{u}_\varepsilon \in S_\lambda$. Moreover,

$$\int_\Omega a|\widetilde{u}_\varepsilon|^{p+1}d\xi = \int_{B_H(\xi_o;\, \varepsilon R)} a|u_\varepsilon|^{p+1}d\xi > 0\,.$$

We have thus proved that $M > 0$ and $S_\lambda \neq \emptyset$.

PROOF OF THEOREM 1.2. The proof is divided in four steps:

Step 1: for large $\lambda > \lambda_1$ (1.2) has no solution;

Step 2: if there exists a solution of (1.2) for a certain $\lambda' > \lambda_1$ then (1.2) has a solution for any $\lambda_1 < \lambda \leqslant \lambda'$;

Step 3: for $\lambda > \lambda_1$ sufficiently small, there exists $\widetilde{u}$ solution of (1.2);

Step 4: (1.2) has a solution also for $\lambda = \lambda_1$.

For the first step, let $\xi_o \in \Omega^+$ and $R > 0$ such that $B_H(\xi_o; R) \subset \Omega^+$ and, as in Lemma 3.3, denote by μ^* and $\psi > 0$ in B_H, respectively, the principal eigenvalue and the corresponding eigenfunction of $-\Delta_H$ with Dirichlet condition on ∂B_H:

$$\begin{cases} -\Delta_H \psi = \mu^* \psi & \text{in } B_H, \\ \psi = 0 & \text{on } \partial B_H\,. \end{cases} \tag{3.6}$$

Suppose there exists u solution of (1.2) for $\lambda \geqslant \|q\|_{L^\infty} + \mu^*$. Then it satisfies:

$$\begin{cases} -\Delta_H u \geqslant (\lambda - q)u & \text{in } B_H\,, \\ u > 0 & \text{on } \partial B_H\,, \end{cases} \tag{3.7}$$

and $A\nabla\psi\cdot\nu < 0$ on ∂B_H except at the two points where the outward normal is in the t-axis direction, as a consequence of Lemma 2.1. Hence,

$$0 < \int_{\partial B_H} -uA\nabla\psi\cdot\nu\, d\sigma = \int_{B_H}(-u\Delta_H\psi + \psi\Delta_H u)\,d\xi \leqslant \int_{B_H}(\mu^* + q - \lambda)u\psi\, d\xi$$

so, as $\lambda \geqslant \|q\|_{L^\infty} + \mu^*$, the integral in the right hand side is negative and we have reached a contradiction.

The second step is accomplished by a sub and supersolution argument. Namely, let $\lambda' > \lambda_1$ and denote by $w = u_{\lambda'}$ a solution of (1.2) for

$\lambda = \lambda'$, which exists by assumption. For $\lambda_1 < \lambda < \lambda'$, w satisfies:

$$\begin{cases} -\Delta_H w + (q - \lambda) w \geqslant a w^p & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

so, $\forall \lambda_1 < \lambda < \lambda'$, w is a supersolution of (1.2).

On the other hand, $\varepsilon\phi$ is a subsolution of (1.2), provided ε is small enough. Indeed,

$$-\Delta_H \varepsilon\phi + (q - \lambda)\varepsilon\phi = (\lambda_1 - \lambda)\varepsilon\phi \leqslant a\varepsilon^p \phi^p$$

for ε small enough, as a is bounded in Ω.

As the maximum principle holds for Δ_H and, from the Hopf lemma, there exists a small $\varepsilon > 0$ such that $\varepsilon\phi \leqslant w$, we can use the standard procedure (see e.g. [1]) to construct a solution u of (1.2) for any $\lambda_1 < \lambda < \lambda'$.

As for the third step, let us show that the variational problem

$$M = \max\left\{ \int_\Omega a|u|^{p+1} d\xi;\ u \in S_\lambda \right\} \tag{3.8}$$

has a solution. For this purpose, let $\{u_n\} \subset S_\lambda$ be a maximizing sequence for (3.8). We first prove that $\{u_n\}$ is bounded in $\mathring{S}_1^2$.

Let us decompose u_n in the following way:

$$u_n = t_n \phi + v_n \tag{3.9}$$

with $t_n \in \mathbb{R}$ and v_n orthogonal to ϕ in $L^2(\Omega)$.

Thus, from the definition of ϕ,

$$\int_\Omega \nabla_H \phi \nabla_H v + (q - \lambda)\phi v\, d\xi = (\lambda_1 - \lambda)\int_\Omega \phi v\, d\xi$$

for all v in $\mathring{S}_1^2$. Choosing v respectively equal to ϕ and to v_n, we obtain for $u_n \in S_\lambda$:

$$1\int_\Omega |\nabla_H u_n|^2 + (q - \lambda) u_n^2\, d\xi = \tag{3.10}$$

$$= (\lambda_1 - \lambda) t_n^2 + \int_\Omega |\nabla_H v_n|^2 + (q - \lambda) v_n^2\, d\xi\,.$$

We have normalized ϕ such that $\|\phi\|_{L^2} = 1$.

We will choose λ small enough that:

$$\lambda_2 - \lambda \geqslant \frac{\lambda_2}{2} > 0$$

and

$$\|q\|_{L^\infty} - \lambda > 0 ,$$

where λ_2 is the second eigenvalue of $-\Delta_H + q$.

From the Harnack inequality, which holds also for the Δ_H (see [11]), it is immediate to get that the first eigenvalue λ_1 of $-\Delta_H + q$ is simple. Therefore, as v_n is orthogonal to ϕ and the set of eigenfunctions $\{\phi_k\}$ is complete in L^2, the following equality holds:

$$-\Delta_H v_n + (q - \lambda) v_n = \sum_{k=2}^{\infty} a_n^k (\lambda_k - \lambda) \phi_k$$

where $a_n^k = \langle v_n , \phi_k \rangle_{L^2}$.

Therefore, for our choice of λ:

$$I(v_n) = \sum_{k=2}^{\infty} (a_n^k)^2 (\lambda_k - \lambda) \geqslant \frac{\lambda_2}{2} \|v_n\|_{L^2}^2 \tag{3.11}$$

and

$$I(v_n) \leqslant \int_\Omega |\nabla_H v_n|^2 \, d\xi + (\|q\|_{L^\infty} - \lambda) \int_\Omega v_n^2 \, d\xi \leqslant C \|v_n\|_{S_2^1}^2 . \tag{3.12}$$

Suppose by contradiction that $\{u_n\}$ is not bounded in $\overset{\circ}{S}{}_1^2$.

Assume first that $\|v_n\|_{S_2^1}$ is bounded. This implies that $I(v_n)$ and t_n are bounded too, as (3.12) holds and, from (3.10),

$$(\lambda - \lambda_1) t_n^2 = I(v_n) - 1 .$$

We have reached a contradiction as, from (3.9), $\|u_n\|_{S_2^1} \leqslant t_n \|\phi\|_{S_2^1} + \|v_n\|_{S_2^1}$.

Therefore v_n diverges in S_1^{2o}. This implies that $I(v_n)$ diverges. Indeed either $\|v_n\|_{L^2}$ is bounded and $I(v_n) \geqslant \int_\Omega |\nabla_H v_n|^2 - C$ diverges, or $\|v_n\|_{L^2}$ diverges and, from (3.11), $I(v_n)$ does also. Furthermore, for large n,

$$M - \varepsilon \leqslant \int_\Omega a |u_n|^{p+1} d\xi = |t_n|^{p+1} \int_\Omega a \left| \phi + \frac{v_n}{t_n} \right|^{p+1} d\xi .$$

But, from (3.10),

$$\int_\Omega a\left|\phi + \frac{v_n}{t_n}\right|^{p+1} d\xi \leqslant \int_\Omega a\phi^{p+1} d\xi + C\int_\Omega \left|\frac{v_n}{t_n}\right|^{p+1} d\xi \leqslant$$

$$\leqslant \int_\Omega a\phi^{p+1} d\xi + C\,\frac{I(v_n)^{(p+1)/2}}{t_n^{p+1}} \leqslant \int_\Omega a\phi^{p+1} d\xi + \tilde{C}(\lambda - \lambda_1)^{(p+1)/2} < 0$$

for $\lambda - \lambda_1 > 0$ sufficiently small, from condition (i). We get a contradiction since, by Lemma 3.3, $M > 0$.

As u_n is bounded in $\overset{\circ}{S}{}^2_1$, there exists a weakly convergent subsequence in $\overset{\circ}{S}{}^2_1$, converging strongly in L^2 and in L^{p+1} in view of Theorem 2.1, since $1 < p < (Q+2)/(Q-2)$.

So

$$\int_\Omega a|u_n|^{p+1} d\xi \to \int_\Omega a|u|^{p+1} d\xi = M > 0\,.$$

In particular $u \not\equiv 0$.

We still have to check that $u \in S_\lambda$. From weak lower continuity

$$I(u) \leqslant 1\,.$$

On the other hand, if we suppose $I(u) \leqslant 0$ we obtain a contradiction, applying the argument above to $u = t\phi + v$. Thus $I(u) > 0$. Now if $I(u) < < 1$ then $\delta u \in S_\lambda$ for some $\delta > 1$; hence:

$$M \geqslant \int_\Omega a|\delta u|^{p+1} d\xi = \delta^{p+1}\int_\Omega a|u|^{p+1} d\xi = \delta^{p+1} M$$

which is absurd.

Thus $u \in S_\lambda$.

Moreover, $|u|$ has the same properties as u, so we may assume $u \geqslant 0$.

Now, a standard argument shows that there exists a Lagrange multiplier τ such that u is a weak solution of

$$\begin{cases} -\Delta_H u + (q - \lambda)u = \tau a(x) u^p\,, & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3.13}$$

By Lemma 3.3, $M > 0$ so that $\tau > 0$. Hence $\tilde{u} = \tau^{1/(p-1)} u$ is a weak solution of (1.2).

In order to show that $\widetilde{u}$ is a classical solution of (1.2) we use Lemma 3.2. More precisely, for $\varepsilon > 0$ that will be chosen conveniently later, let $K = \varepsilon^{-p+(Q+2)/(Q-2)}$ and η be a $C^\infty(\Omega)$ such that

$$\eta(t) = \begin{cases} 0 & t > K, \\ 1 & t < \dfrac{K}{2}, \end{cases}$$

We choose $V = a(\xi)u^{p-1}(\eta(u)-1)$ and $g = -\eta(u)a(\xi)u^p + (q(\xi) - \lambda - 1)u$. Then it is easy to see that

$$\|V\|_{L^{Q/2}(\Omega)} \leqslant C\varepsilon\|u\|_{L^{2Q/(Q-2)}}$$

but, by Theorem 2.1, $u \in L^{2Q/(Q-2)}$ and, moreover, $\|u\|_{L^{2Q/(Q-2)}} \neq 0$ since $M > 0$. Thus we can choose $\varepsilon \leqslant (C\|u\|_{L^{2Q/(Q-2)}}B_Q)^{-1}$ and V satisfies the hypotheses of Lemma 3.2.

Furthermore, if $u \in L^q$ for some $q > 2$ then g also is in L^q. But, from Theorem 2.1, $u \in L^q$ for some $2 < q < 2Q/(Q-2)$ and we can apply Lemma 3.2 to obtain $u \in L^{q(1+2/(Q-2))}$.

Repeating this argument we get that

$$u \in \bigcap_{2 \leqslant p} L^p. \tag{3.14}$$

Therefore $u \in \overset{\circ}{S}{}^q_2$ in the interior of Ω for each $q \geqslant 2p$. From Theorem 2.1, we obtain that $u \in \Gamma^\beta$ for $\beta = 2 - Q/q > 0$. Using again regularity results, as u is a solution of (1.2), $u \in \Gamma^{2+\alpha}$ in the interior of Ω for some $0 < \alpha < 1$.

The continuity of u up to the boundary is a direct consequence of Theorem 3.1 of [10].

Moreover, as $u \geqslant 0$ satisfies the equation (1.2) in the classical sense and a and q are bounded, there exists a constant $K > 0$ such that $-\Delta_H u + Ku \geqslant 0$ in Ω and $u = 0$ on $\partial\Omega$. Thus the strong maximum principle implies the strict positivity of u in Ω.

The proof of the statement in the fourth step is as follows. Let us consider a sequence $\lambda_n \searrow \lambda_1$ and the corresponding solutions u_n of (1.2), which exist as proved in the previous two steps. The family $\{u_n\}$ is bounded in $\overset{\circ}{S}{}^2_1$, as we have proven; so we may extract a subsequence, that we still denote by u_n converging weakly in $\overset{\circ}{S}{}^2_1$ and strongly in L^{p+1} to u as a consequence of the embedding theorem. Such a u is a weak solution of the equation (1.2), with $\lambda = \lambda_1$. Repeating a bootstrap argument, it is easy to check the regularity of u and consequently its strict positivity.

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