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Π-Normally Embedded Subgroups of Finite Soluble Groups.

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1. Introduction and statement of results.

All groups considered here are finite and soluble.

Let G be a group and let π be a set of primes. A subgroup H of G is said to be *π-normally embedded* in G if a Hall π-subgroup of H is a Hall π-subgroup of some normal subgroup of G. A Hall π-subgroup of a normal subgroup of G is a typical example of a π-normally embedded subgroup of G. It is clear that if H is π-normally embedded in G then H is p-normally embedded in G, in the sense of [3, Definition (7.1a)], for every prime $p \in \pi$ but the converse does not hold in general (see example 2 of [4]). A subgroup H of G is said to be *normally embedded* in G if H is p-normally embedded in G for all primes p.

Fischer, Lockett and Ti Yen (see [3, I; (7.9)]) proved that the set of all normally embedded subgroups of a group G into which a given Hall system of G reduces forms a sublattice of the subgroup lattice of G. This result is an easy consequence of the following Theorem:

THEOREM (Lockett [3]). *Let U and V be normally embedded subgroups of a group G into which a given Hall system Σ of G reduces. Then $UV = VU$, and UV and $U \cap V$ are normally embedded subgroups of G into which Σ reduces.*

The hypothesis «normally embedded» cannot be relaxed to simply «p-normally embedded» in the above Theorem in order to obtain the same result. It is enough to consider the group $G = \Sigma_4$, the symmetric

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group of degree 4, the Hall system

$$\Sigma = \{1,\ G, \langle(1\,2)(3\,4), (1\,3)(2\,4)\rangle\langle(1\,2)\rangle, \langle(1\,2\,3)\rangle\}$$

and the subgroups $U = \langle(1\,3)(2\,4)\rangle$ and $V = \langle(1\,2)\rangle$. It is clear that U does not permute with V. However U and V are 3-normally embedded subgroups of G into which Σ reduces.

Bearing in mind the result of Lockett, it turns out to be natural to wonder whether the set of all π-normally embedded subgroups of a group G is a sublattice of the subgroup lattice of G, that is, if U and V are π-normally embedded subgroups of G, is it true that $U \cap V$ and $\langle U, V\rangle$ are π-normally embedded subgroups of G ? The answer is negative in general as the next examples show:

EXAMPLE 1. Let $G = C_5 \,\mathrm{wr}\, \Sigma_4$, the wreath product of the cyclic group of order 5 with the symmetric group of degree 4. The wreath product is taken with respect to the natural representation of Σ_4 (of degree 4). The group G is expresible as a semidirect product $G = NG^*$, $N \cap G^* = 1$ where N is an elementary abelian group of order 5^4 and G^* is the symmetric group Σ_4. N is generated by elements a_1, a_2, a_3, a_4 of order 5 and $x^{-1} a_i x = a_{ix}$ ($x \in G^*$, where ix is the image of i under the permutation x of Σ_4). Let $U = G^*$ and $V = U^n$, where $n = a_1 + a_2 + a_3 \in N$. If $\pi = \{2, 3\}$, it is clear that U and V are π-normally embedded subgroups of G. However $U \cap V = C_U(n)$ (see for instance [3, A; (16.3)]) is not π-normally embedded in G because $C_U(n)$ is isomorphic to the symmetric group of degree 3.

EXAMPLE 2. Let X be the symmetric group of degree 4. It is known (see [3, B; (16.3)]) that X has an irreducible and faithful X-module W over $GF(3)$, the finite field of 3 elements. Let $G = [W]X$ be the corresponding semidirect product and take $\pi = \{3\}$. Consider $U \in \mathrm{Syl}_2(X)$ and $V = U^x$ for some $x \in X - N_X(U)$. It is clear that U and V are π-normally embedded subgroups of G but $X = \langle U, V\rangle$ is not π-normally embedded in G.

The aim of this paper is to give some sufficient conditions for $U \cap V$ and $\langle U, V\rangle$ to be π-normally embedded in G provided that U and V are π-normally embedded in G.

We prove the following results:

THEOREM 1. *Let π be a set of primes and let G be a group. Assume that U and V are π-normally embedded subgroups of G. Then $U \cap V$ is π-normally embedded in G provided that one of the following conditions is satisfied:*

i) *U permutes with V.*

ii) *There exists a Hall π-subgroup of G reducing into U and V.*

iii) *U is a subnormal subgroup of G.*

iv) *$U \cap V$ is a nilpotent subnormal subgroup of G.*

THEOREM 2. *Let π be a set of primes. Assume that U and V are π-normally embedded subgroups of a group G. Then $\langle U, V\rangle$ is π-normally embedded in G provided that one of the following conditions is satisfied:*

i) *U permutes with V.*

ii) *There exists a Hall system Σ of G which reduces into U and V.*

iii) *Either U or V is a subnormal subgroup of $\langle U, V\rangle$.*

Combining Theorem 1.0 (ii) and Theorem 2.0 (ii), we obtain the following generalization of [3, I;(7.9)].

COROLLARY 1. *Let Σ be a Hall system of a group G and let π be a set of primes. If U and V are π-normally embedded subgroups of G into which Σ reduces, then $U \cap V$ and $\langle U, V\rangle$ are π-normally embedded subgroups of G.*

We shall adhere to the notation used in [3]. This book will be the main reference for basic notation, terminology and results.

For the sake of completeness, we state two results used in proving our Theorems. Their proofs are analogous to those [3, I; (7.3)] and [3, I; (7.6)].

LEMMA 1. *Let U be a π-normally embedded subgroup of a group G. Let $K \trianglelefteq G$ and $H \leqslant G$. Then:*

i) *If $U \leqslant H$, then U is π-normally embedded in H.*

ii) *UK/K is a π-normally embedded subgroup of G/K.*

iii) *If $K \leqslant H$ and H/K is π-normally embedded in G/K, then H is π-normally embedded in G.*

iv) *$U \cap K$ is a π-normally embedded subgroup of G.*

LEMMA 2. *Let P_1 and P_2 be subgroups of a Hall π-subgroup of G and assume that P_1 and P_2 are π-normally embedded in G.*

Then $P_1P_2 = P_2P_1$ and P_1P_2 and $P_1 \cap P_2$ are π-normally embedded subgroups of G.

Recall that a Hall π-subgroup H of G reduces into a subgroup U of G if $H \cap U$ is a Hall π-subgroup of U.

LEMMA 3 [Theorem D [2]). *Assume that U and V are π-normally embedded subgroups of a group G. If a Hall π-subgroup G_π of G reduces into U and V, then G_π reduces into $U \cap V$.*

2. The proofs.

PROOF OF THEOREM 1. (i) If $UV = VU$, then it follows from [1 Lemma 1.3.2] that there exist a Hall π-subgroup U_π of U and a Hall π-subgroup V_π of V such that $U_\pi V_\pi = V_\pi U_\pi$ is a Hall π-subgroup of UV. So we have that

$$\frac{|U_\pi|\,|V_\pi|}{|U_\pi \cap V_\pi|} = |U_\pi V_\pi| = |UV|_\pi = \frac{|U|_\pi\,|V|_\pi}{|U \cap V|_\pi} = \frac{|U_\pi|\,|V_\pi|}{|U \cap V|_\pi},$$

where for any group X we denote $|X|_\pi$ the highest π-number dividing $|X|$. Consequently $|U \cap V|_\pi = |U_\pi \cap V_\pi|$, that is, $U_\pi \cap V_\pi$ is a Hall π-subgroup of $U \cap V$. From Lemma 2.0 we deduce that $U_\pi \cap V_\pi$ is a π-normally embedded subgroup of G. Then it is clear that $U \cap V$ is a π-normally embedded subgroup of G.

(ii) Assume now that there exists a Hall π-subgroup G_π of G such that G_π reduces into both U and V, that is, $U_\pi = G_\pi \cap U$ and $V_\pi = G_\pi \cap V$ are Hall π-subgroups of U and V respectively. By Lemma 2.0 it is clear that $U_\pi \cap V_\pi$ is a π-normally embedded subgroup of G.

On the other hand, it follows from Lemma 3.0 that G_π reduces into $U \cap V$, that is, $U_\pi \cap V_\pi = G_\pi \cap U \cap V$ is a Hall π-subgroup of $U \cap V$. Now it is clear that $U \cap V$ is a π-normally embedded subgroup of G.

(iii) By [3, I; (4.16)], we know that there exists a Hall π-subgroup G_π of G reducing into V. Since U is subnormal in G, we also have that G_π reduces into U by [3, I; (4.21)]. We are now in the hypothesis of (ii) to deduce that $U \cap V$ is a π-normally embedded subgroup of G.

(iv) Since $U \cap V$ is a nilpotent subnormal subgroup of G, it must be $U \cap V \leq F(G)$. Moreover U is a π-normally embedded subgroup of G and so is $U \cap F(G)$ by Lemma 1.0 (iv). It is clear that $U \cap F(G)$ is subnormal in G. Consequently it follows from (iii) that $U \cap V = U \cap V \cap F(G)$ is a π-normally embedded subgroup of G. ■

REMARKS AND EXAMPLES. None of the stated hypothesis in the Theorem 1.0 can be dispensed with in order to obtain the same result, as the following examples show:

1) Recall Example 1. There we consider a group G with two π-normally embedded subgroups U and V such that $U \cap V$ is not π-normally embedded in G. Notice that in this example U does not permute with V, U is not a subnormal subgroup and there is no Hall π-subgroup of G reducing into both U and V. So Theorem 1.0 fails whether we do not assume any of the hypothesis (i), (ii) or (iii).

2) We see next that in Theorem 1.0 (iv) the hypothesis that $U \cap V$ is a nilpotent subgroup is essential.

Consider the group $G = \Sigma_3 \operatorname{Wr} C_3$, the regular wreath product of the symmetric group of degree 3 with a cyclic group of order 3. Let $S_1 \times \times S_2 \times S_3$ be the basis group of G, where $S_1 \cong S_2 \cong S_3 \cong \Sigma_3$, and for each $i \in \{1, 2, 3\}$, let $\langle b_i \rangle \in \operatorname{Syl}_3(S_i)$ and $\langle c_i \rangle \in \operatorname{Syl}_2(S_i)$. Take $\pi = \{2\}$ and the groups $U = S_1 \times \langle c_2 \rangle \times \langle c_3 \rangle$ and $V = S_1 \times \langle c_2^{b_2} \rangle \times \langle c_3{}^{b_3} \rangle$. So considered, we have that U and V are π-normally embedded subgroups of G, $U \cap V$ is a subnormal subgroup of G and $U \cap V \cong \Sigma_3$ is not nilpotent. It is clear that $U \cap V$ is not π-normally embedded in G.

3) In Theorem 1.0 (iv) it is also necessary that $U \cap V$ is subnormal in G.

Consider the group $G = \Sigma_3 \operatorname{wr} C_2$, the regular wreath product of the symmetric group of degree 3 with a cyclic group of order 2. Let $S_1 \times S_2$ be the basis group of G, where $S_1 \cong S_2 \cong \Sigma_3$, and for each $i \in \{1, 2\}$, let $\langle b_i \rangle \in \operatorname{Syl}_3(S_i)$ and $\langle a_i \rangle \in \operatorname{Syl}_2(S_i)$ such that $a_1^c = a_2$ and $b_1^c = b_2$ where $C_2 = \langle c \rangle$. Take $\pi = \{2\}$, $U = \langle a_1, a_2 \rangle C_2$ and $V = \langle a_1^{b_1}, a_2^{b_2} \rangle C_2$. It is clear that U and V are π-normally embedded in G and $U \cap V = C_2$ is a nilpotent subgroup of G which is not subnormal in G. Moreover $U \cap V$ is not π-normally embedded in G.

PROOF OF THEOREM 2. (i) Since $UV = VU$, there exist a Hall π-subgroup U_π of U and a Hall π-subgroup V_π of V such that $U_\pi V_\pi = V_\pi U_\pi$ is a Hall π-subgroup of UV. But $U_\pi V_\pi$ is a π-normally embedded subgroup of G because of Lemma 2.0, and consequently UV is π-normally embedded in G.

(ii) Let $T = \langle U, V \rangle$ and let N be a minimal normal subgroup of G. Arguing by induction on $|G|$ and using Lemma 1.0, we may assume that TN is a π-normally embedded subgroup of G.

If $\operatorname{Core}_G(T) \neq 1$, we deduce the result taking $N \leq \operatorname{Core}_G(T)$.

If $O_{\pi'}(G) \neq 1$, it is enough to consider $N \leq O_{\pi'}(G)$. In this case T is a

π-normally embedded subgroup of G because a Hall π-subgroup of T is a Hall π-subgroup of TN.

Consequently we may assume that $\mathrm{Core}_G(T)=1$ and $O_{\pi'}(G)=1$.

Let Σ be a Hall system of G reducing into U and V and let G_π be the Hall π-subgroup of G in Σ. Then $G_\pi \cap U = U_\pi$ and $G_\pi \cap V = V_\pi$ are Hall π-subgroups of U and V respectively. Since U and V are π-normally embedded subgroups of G, it follows by Lemma 2.0 that $U_\pi V_\pi = V_\pi U_\pi$ is a π-normally embedded subgroup of G. Hence $U_\pi V_\pi$ is a Hall π-subgroup of a normal subgroup of G, say M.

If $M \neq 1$, take N a minimal normal subgroup of G contained in M. Since $O_{\pi'}(G) = 1$, it is clear that $N \leq U_\pi V_\pi \leq T$ which is a π-normally embedded subgroup of G.

If $M = 1$, then U and V are π'-subgroups of G. So U and V are contained in the Hall π'-subgroup of G in Σ and consequently $T = \langle U, V\rangle$ is a π'-subgroup of G. Clearly T is then a π-normally embedded subgroup of G.

(iii) Assume that U is a subnormal subgroup of $\langle U, V\rangle$. We know that there exists a Hall system of $\langle U, V\rangle$ reducing into V, say $\Sigma_{\langle U, V\rangle}$. Take a Hall system Σ of G such that $\Sigma \cap \langle U, V\rangle = \Sigma_{\langle U, V\rangle}$.

It is clear that Σ reduces into V, and since U is subnormal in $\langle U, V\rangle$, we have that Σ also reduces into U. Now $\langle U, V\rangle$ is a π-normally embedded subgroup of G because of (ii). ■

Remarks and examples.

1) Recall Example 2. There we consider a group G with two π-normally embedded subgroups U and V such that $\langle U, V\rangle$ is not π-normally embedded in G.

Notice that in this example $UV \neq VU$, neither U nor V are subnormal subgroups of $\langle U, V\rangle$ and there exists no Hall system of G reducing into both U and V. So none of the hypothesis in Theorem 2.0 can be dispensed with to obtain the result.

Moreover in this example each Sylow 3-subgroup of G reduces into U and V. So the result fails if in Theorem 2.0 (ii) we only consider a Hall π-subgroup reducing in U and V instead of a complete Hall system.

2) If U and V are π-normally embedded subgroups of G and $U \cap V$ is a nilpotent subnormal subgroup of G, it is not true in general that $\langle U, V\rangle$ is a π-normally embedded subgroup of G.

For instance let $X = \Sigma_3$ and let V be an irreducible and faithful X-module over $GF(7)$, the Galois field of 7 elements. Then $2 \leq \dim_{GF(7)} V$. Consider $G = [V]X$, the corresponding semidirect product, and $U \in$

$\in \mathrm{Syl}_3(X)$. Then it is clear that U is a normal subgroup of X. By a well known theorem of Clifford $V_U = \oplus V_i$, where each V_i is an irreducible and faithful U-module over $GF(7)$. So $\dim_{GF(7)} V_i = 1$. In particular, if $1 \neq v_1 \in V_1$, then $C_U(v_1) = 1$. Take $V = U^{v_1}$. If $\pi = \{7\}$, it is clear that U and V are π-normally embedded in G and $U \cap V = C_U(v_1) = 1$ is a nilpotent subnormal subgroup of G. However $\langle U, V \rangle$ is not a π-normally embedded subgroup of G.

3) If U and V are π-normally embedded subgroups of G and $\langle U, V \rangle$ is subnormal in G, it is not true in general that $\langle U, V \rangle$ is a π-normally embedded subgroup of G.

Let $X \cong \Sigma_3$ be the symmetric group of degree 3 and take $G = X \,\mathrm{wr}\, C_2$ the regular wreath product of X with a cyclic group of order 2. Let $X^{\#} = X_1 \times X_2$ be the basis group of G, where $X_1 \cong X_2 \cong X$. Take U a Sylow 2-subgroup of X_1 and $V = U^x$, where $1 \neq x \in X_1 - U$. If $\pi = \{3\}$, then U and V are π-normally embedded in G, $X_1 = \langle U, V \rangle$ is subnormal in G but X_1 is not π-normally embedded in G.

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