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A result on the well posedness of the Cauchy problem for a class of hyperbolic operators with double characteristics

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A Result on the Well Posedness of the Cauchy Problem for a Class of Hyperbolic Operators with Double Characteristics.

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ABSTRACT - Let p_2 be the principal symbol of a hyperbolic differential operator P of order two admitting characteristics roots of variable multiplicity. Suppose that the double characteristic manifold Σ of p_2 contains a submanifold $\tilde{\Sigma}$ such that at each point of $\tilde{\Sigma}$ the Hamiltonian matrix of p_2, F, has a Jordan block of dimension 4, whereas at each point of $\Sigma \backslash \tilde{\Sigma}$, F admits only Jordan blocks of size 2 and F is not effectively hyperbolic. We prove that under suitable conditions on the 3-jet of p_2 at $\tilde{\Sigma}$ the Cauchy problem for P is well posed provided the usual Levi conditions on the lower order terms are satisfied.

0. Introduction.

Let $T^* \mathbb{R}^{n+1}$ be the cotangent bundle of $\mathbb{R}^{n+1}$, with canonical coordinates $(x, \xi) = (x_0, x'; \xi_0, \xi')$, $x_0 \in \mathbb{R}$, $x' \in \mathbb{R}^n$; by $\sigma = \sum_{j=0}^{n} d\xi_j \wedge dx_j$ we denote the symplectic two-form on $T^* \mathbb{R}^{n+1}$.

Let $P(x, D)$ be a second order operator, differential in x_0 and pseudodifferential in x', $\left(D = (D_0, D_1, \dots, D_n), D_j = \frac{1}{i} \partial_{x_j}\right)$ with C^∞ coefficients defined in $\mathbb{R}^{n+1}$.

We denote by $p(x, \xi)$ its symbol,

$$p(x, \xi) = p_2(x, \xi) + p_1(x, \xi) + \dots,$$

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and suppose that:

(H) p_2 is hyperbolic with respect to ξ_0, i.e. $p_2(x, \xi_0, \xi') = 0 \Rightarrow \xi_0 \in \mathbb{R}$.

By using a canonical transformation preserving the planes x_0 = const., we can reduce p_2 to the form:

$$p_2(x, \xi) = -\xi_0^2 + a(x, \xi'), \tag{0.1}$$

with $a \geqslant 0$, $a \in S^2(\mathbb{R}_x^{n+1} \times \mathbb{R}_{\xi'}^n)$, where by $S^m(\mathbb{R}_x^{n+1} \times \mathbb{R}_{\xi'}^n)$ we denote the space of homogeneous symbols of degree m with respect to ξ' smoothly dependent on $x_0 \in \mathbb{R}$. Let

$$\Sigma = \{(x, \xi) \in T^* \mathbb{R}^{n+1} \backslash 0 \mid p_2(x, \xi) = dp_2(x, \xi) = 0\}$$

be the set of double points, $\Sigma \neq \emptyset$.

At every point $\rho \in \Sigma$ we consider the fundamental (or Hamiltonian) matrix $F(\rho)$, invariantly defined by

$$\sigma(X, F(\rho) Y) = \frac{1}{2} \langle \mathrm{Hess}\, p_2(\rho) X, Y \rangle, \qquad \forall X, Y \in T_\rho(T^* \mathbb{R}^{n+1}).$$

We shall suppose that the principal symbol p_2 satisfies the following hypotheses:

H_1) Σ is a smooth submanifold of $T^* \mathbb{R}^{n+1}$ of codimension $d + 1$ such that:

(i) $\mathrm{rg}\, \sigma|_\Sigma = \mathrm{const}$;

(ii) $T_\rho \Sigma = \mathrm{Ker}\, F(\rho)$, $\forall \rho \in \Sigma$;

(iii) $\mathrm{sp}\,(F(\rho)) \subseteq i\mathbb{R}$, $\forall \rho \in \Sigma$.

As a consequence of (iii), in the canonical form of F either Jordan blocks of dimension 2 or both Jordan blocks of dimension 2 and one block of dimension 4 are allowed (see [4]).

In the first case (symplectic case) the well posedness of the Cauchy problem has been established under condition (0.12) (see [4], [6]). In the second case (non-symplectic case) i.e. when there is a Jordan block of size 4 in the canonical form of $F(\rho)$ for every $\rho \in \Sigma$, a sufficient condition for the well posedness of the Cauchy problem has been recently established in [13].

In the present paper, by using the same approach as in [3], we study the case when there is a transition on Σ between the two cases of non effective hyperbolicity.

Precisely, we shall suppose that:

H_2) there exists a smooth submanifold $\emptyset \neq \widetilde{\Sigma} \subsetneq \Sigma$ such that:

(i) $\forall \rho \in \widetilde{\Sigma}$, $\operatorname{Ker} F^2(\rho) \cap \operatorname{Im} F^2(\rho) \neq (0)$;

(ii) $\forall \rho \in \Sigma \backslash \widetilde{\Sigma}$, $\operatorname{Ker} F^2(\rho) \cap \operatorname{Im} F^2(\rho) = (0)$.

H_3) For every $\rho \in \widetilde{\Sigma}$: $\operatorname{Ker} F(\rho) \cap \operatorname{Im} F^3(\rho) \subset T_\rho \widetilde{\Sigma}$.

Some remarks are in order.

REMARK. 1) Assumptions H_1) (i), (ii) yield dim $\operatorname{Ker} F^2 = \text{const}$ on Σ; hence $\operatorname{Ker} F$ and $\operatorname{Ker} F^2$ are smooth vector bundles on Σ.

2) Assumptions H_1) (ii), H_2) (i) imply that for any $\rho \in \widetilde{\Sigma}$ the Hamilton matrix $F(\rho)$ has, in its canonical form, a Jordan block of size 4, corresponding to the zero eigenvalue, moreover the associated eigenspace is a smooth vector bundle of rank 4, as ρ varies in $\widetilde{\Sigma}$.

In view of the Remark 2, the results of Proposition 2.2 in Bernardi, Bove [1] will hold on $\widetilde{\Sigma}$

PROPOSITION 0.1. *There exist two smooth sections of* $\{T_\rho(T^* \mathbb{R}^{n+1});\ \rho \in \widetilde{\Sigma}\}$, z_1, z_2, *such that,* $\forall \rho \in \widetilde{\Sigma}$:

(0.2) $$z_1(\rho) \in \operatorname{Ker} F(\rho) \cap \operatorname{Im} F^3(\rho);$$

(0.3) $$z_2(\rho) \in \operatorname{Ker} F^2(\rho) \cap \operatorname{Im} F^2(\rho);$$

(0.4) $$\forall w \in [z_1(\rho)]^\sigma \textit{ we have: } \sigma(w, F(\rho) w) \geqslant 0;$$

(0.5) *if* $w \in [z_1(\rho)]^\sigma$ *and* $\sigma(w, F(\rho) w) = 0$, *then* $w \in \operatorname{Ker} F(\rho) \oplus [z_2(\rho)]$.

In particular, from (0.2)-(0.5) it follows that $\forall \rho \in \widetilde{\Sigma}$:

$$\dim \operatorname{Ker} F^2(\rho) \cap \operatorname{Im} F^2(\rho) = 2,$$

$$\dim \operatorname{Ker} F(\rho) \cap \operatorname{Im} F^3(\rho) = 1.$$

We shall assume, without loss of generality, that

(0.6) $$F(\rho) z_2(\rho) = -z_1(\rho), \qquad \forall \rho \in \widetilde{\Sigma}.$$

A general method to obtain the C^∞ well posedness is to prove (micro)local energy estimates. V. Ja. Ivrii defined in [6] a class of hyperbolic operators and for such a class of operators proved an a priori energy estimate yielding the well-posedness of the Cauchy problem. We recall the following definition:

DEFINITION 0.1. *We say that p_2 admits an elementary decomposition (in the sense of Ivrii) in a conic neighborhood U of Σ, if there exist λ, μ, Q real valued symbol in (x', ξ') smoothly dependent on x_0, homogeneous of order* 1, 1, 2 *respectively, with $Q \geq 0$, such that:*

$$p_2(x, \xi) = -(\xi_0 - \lambda(x, \xi'))(\xi_0 - \mu(x, \xi')) + Q(x, \xi'), \tag{0.7}$$

$$|\{\xi_0 - \lambda(x, \xi'), \xi_0 - \mu(x, \xi')\}| \leq C[|\lambda(x, \xi')| + \sqrt{Q(x, \xi')}], \tag{0.8}$$

$$|\{\xi_0 - \lambda(x, \xi'), Q(x, \xi')\}| \leq C' Q(x, \xi'), \tag{0.9}$$

where C, C' are positive constants depending on the conical neighborhood U.

We shall write $\Lambda(x, \xi) = \xi_0 - \lambda(x, \xi')$, $M(x, \xi) = \xi_0 - \mu(x, \xi')$.
We can now state the main result of this paper.

THEOREM 0.1. *Let $p_2(x, \xi)$ as in* (0.1) *satisfying assumptions* H_1), H_2), H_3), *and let $S(x, \xi)$ be any smooth real function defined on $T^* \mathbb{R}^{n+1}$, homogeneous of degree* 0, *such that:*

$$S(x, \xi) = 0 \qquad \text{if } (x, \xi) \in \Sigma; \tag{0.10}$$

$$[H_S(\rho)] = [z_2(\rho)], \qquad \forall \rho \in \widetilde{\Sigma}. \tag{0.11}$$

Then the following assertions are equivalent:

(i) *$p_2(x, \xi)$ admits an elementary decomposition in a neighborhood of Σ.*

(ii) $H_S^3 p_2(\rho) = 0$, $\forall \rho \in \widetilde{\Sigma}$.

Condition (ii) in Theorem 0.1 is obviously canonically invariant with respect to the different choices of the function S (for a proof we refer to [3]).

We recall that at every point $\rho \in \Sigma$ we can invariantly define:

(*a*) the *subprincipal symbol* of p:

$$p_1^S(\rho) = p_1(\rho) - \frac{1}{2} \sum_{j=0}^{n} \frac{\partial^2 p_2(\rho)}{\partial x_j \partial \xi_j};$$

(*b*) $\mathrm{Tr}^+ F(\rho) = \sum_j \mu_j$, where $i\mu_j$ are the eigenvalues of $F(\rho)$ on the positive imaginary axis, repeated according to their multiplicities.

We now state the main result of C^∞-well posedness of the Cauchy problem.

THEOREM 0.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and let P be a differential operator with p_2 satisfying assumptions* H_1), H_2), H_3).
Assume furthermore that:

(0.12) $\exists \varepsilon > 0$ *such that on Σ we have*

$$\begin{cases} -(1-\varepsilon)\operatorname{Tr}^+ F \leq \operatorname{Re} p_1^S \leq (1-\varepsilon)\operatorname{Tr}^+ F, \\ \operatorname{Im} p_1^S = 0. \end{cases}$$

Then, if the condition (ii) *in Theorem* 0.1 *holds, the Cauchy problem for P is well posed in $C^\infty(\Omega)$.*

REMARK. We point out that nothing is known about the well posedness of the Cauchy problem when the condition H_3) is violated.

1. Some preparations.

Let p_2 be as in (0.1).
For any $\rho \in \Sigma$ we consider $p_{2,\rho}: T_\rho(T^*\mathbb{R}^{n+1}) \to \mathbb{R}$. the localization of p_2 in ρ, defined as

$$p_{2,\rho}(v) = \frac{1}{2}\langle \operatorname{Hess} p_2(\rho) v, v\rangle = \sigma(v, F(\rho) v).$$

It is well known that $p_{2,\rho}$ is a hyperbolic polynomial with respect to $\vartheta = (v_x = 0;\, v_\xi = (1, \dots, 0))$.
Moreover, from assumption H_1) (ii) it follows that $\forall \rho \in \Sigma$, $p_{2,\rho}$ is strictly hyperbolic on $N_\rho\Sigma = T_\rho(T^*\mathbb{R}^{n+1})/T_\rho\Sigma$ with respect to the image of ϑ.
We denote by Γ_ρ the hyperbolicity one of $p_{2,\rho}$ and let $C_\rho = \{z \in T_\rho(T^*\mathbb{R}^{n+1}) \mid \sigma(v, z) \geq 0, \forall v \in \Gamma_\rho\}$ (the propagation cone of $p_{2,\rho}$); we recall that, under the assumptions H_1), H_2) on p_2, we have (see [5], vol. III):

(1.1) $\forall \rho \in \Sigma \backslash \widetilde{\Sigma}$:

$$\{v \in \operatorname{Ker} F(\rho) \cap \operatorname{Im} F(\rho) \mid v \neq 0,\ p_{2,\rho}|_{[v]^\sigma} \geq 0,\ \operatorname{Ker} F(\rho)|_{[v]^\sigma} = \operatorname{Ker} F(\rho)\} =$$

$$= [\operatorname{Int}(C(\rho)) \cup \operatorname{Int}(-C(\rho))] \cap \operatorname{Ker} F(\rho),$$

where Int $(C(\rho))$, Int $(-C(\rho))$ are the interior parts in $\operatorname{Im} F(\rho)$ of the sets $C(\rho)$, $-C(\rho)$, respectively, whereas:

$$\text{(1.2)}\quad \forall\rho\in\widetilde{\Sigma}:\qquad \dim\{v\in\operatorname{Ker}F(\rho)\cap\operatorname{Im}(\rho)\,|\,v=0 \text{ or } p_{2,\rho}|_{[v]^\sigma}\geqslant 0\}=1$$

and

$$\text{(1.3)}\quad \forall\rho\in\widetilde{\Sigma}:\qquad [\bar{\Gamma}_\rho\cup(-\bar{\Gamma}_\rho)]\cap\operatorname{Ker}F^2(\rho)=\operatorname{Ker}F(\rho)\oplus([z_2(\rho)].$$

For the proof of Theorem 0.1 we will use the following geometrical result:

LEMMA 1.1. *Let p_2 be as in* (0.1) *and satisfy assumptions* $H_1)$ $H_2)$. *For every smooth vector field $\widetilde{\zeta}$ on $\widetilde{\Sigma}$ such that*

$$0\neq\widetilde{\zeta}(\rho)\in\operatorname{Ker}F^2(\rho)\cap\partial\Gamma_\rho,\qquad \forall\rho\in\widetilde{\Sigma},$$

there exists a smooth vector field on Σ, ζ, such that

$$\text{(1.4)}\qquad \forall\rho\in\widetilde{\Sigma}:\qquad \zeta(\rho)=\widetilde{\zeta}(\rho),$$

$$\text{(1.5)}\qquad \forall\rho\in\Sigma\backslash\widetilde{\Sigma},\qquad \zeta(\rho)\in\operatorname{Ker}F^2(\rho)\cap\Gamma_\rho.$$

PROOF. To construct ζ we patch together local extensions of the vectorial field $\widetilde{\zeta}$ hence we argue in a neighborhood of a fixed point $\bar{\rho}\in\widetilde{\Sigma}$. Since $\operatorname{Ker}F$ and $\operatorname{Ker}F^2$ are smooth vector bundles on Σ, we can locally identify Σ with $\mathbb{R}^\nu$, $\nu=\dim\Sigma=2n-d+1$, and $T(T^*\mathbb{R}^{n+1})|_\Sigma$ with $\mathbb{R}^\nu\times\mathbb{R}^N$, $N=2(n+1)$, in such a way that

$$\widetilde{\Sigma}=\{y=(y',y'')\in\mathbb{R}^l\times\mathbb{R}^{\nu-l}\,|\,y'=0\},\qquad l=\operatorname{codim}_\Sigma\widetilde{\Sigma},\qquad \bar{\rho}=(0,0),$$

$$\operatorname{Ker}F=\{(\eta,\tau,\sigma)\in\mathbb{R}^h\times\mathbb{R}^k\times\mathbb{R}^{N-(h+k)}\,|\,\eta=\tau=0\},$$

$$h+k=\operatorname{codim}\operatorname{Ker}F,$$

$$\operatorname{Ker}F^2=\{(\eta,\tau,\sigma)\in\mathbb{R}^h\times\mathbb{R}^k\times\mathbb{R}^{N-(h+k)}\,|\,\eta=0\},\qquad h=\operatorname{codim}\operatorname{Ker}F^2.$$

Through this identification the localized polynomial $p_\rho(v)$ becomes a function

$$q(y;\lambda)=\frac{1}{2}\langle A(y)\lambda,\lambda\rangle,\qquad \lambda=(\eta,\tau),$$

for some smooth non-singular symmetric matrix $A(y)$. The quadratic form q is strictly hyperbolic and we can suppose that the hyperbolicity cone is given by

$$\Gamma(y)=\{\lambda=(\eta,\tau)\,|\,\tau_1>0,\ q(y;\lambda)<0\}.$$

The vector field $\tilde{\zeta}$, defined near 0 in $\tilde{\Sigma}$, is now a smooth function $\tilde{\zeta}(y'') = (0, \tau(y''), \sigma(y''))$, for which

$$\begin{cases} q(0, y''; \tilde{\zeta}(y'')) = 0\,, \\ \nabla_\lambda q(0, y''; \tilde{\zeta}(y'')) \neq 0\,. \end{cases} \tag{1.6}$$

We try to extend $\tilde{\zeta}$ by defining

$$\zeta(y', y'') = \left(0,\ \tau(y'') + \alpha(y'')\,y' + \frac{1}{2}\langle \beta(y'')\,y', y'\rangle, \sigma(y'')\right)$$

where $\alpha(y'')$ is a smooth $k \times l$ matrix and $\beta(y'') = (\beta^{(1)}(y''), \dots, \beta^{(k)}(y''))$ is a k-vector of smooth symmetric matrices.

In order that $\zeta(y) \in \operatorname{Ker} F^2(y) \cap \Gamma(y)$ we are led to impose the condition

$$\nabla_{y'}[q(y; \zeta(y))]|_{y'=0} = 0\,, \tag{1.7}$$

which is equivalent to

$$(\nabla_{y'} q)(0, y''; \tilde{\zeta}(y'')) + {}^t\alpha(y'')\nabla_\lambda q(0, y''; \tilde{\zeta}(y'')) = 0\,. \tag{1.8}$$

Since $\nabla_\lambda q(0, y''; \tilde{\xi}(y'')) \neq 0$, we can obviously find a smooth matrix $\alpha(y'')$ such that (1.8) holds in a neighborhood of $y'' = 0$; this purpose it is enough to fix any $\alpha(0)$ such that (1.8) holds true when $y'' = 0$ and then use Dini's theorem.

Having already selected $\alpha(y'')$, we require that the matrix

$$C(y'') = \operatorname{Hess}_{y'}[q(y; \zeta(y))]|_{y'=0}$$

is negative definite. It is easily seen that

$$\begin{cases} C(y'') = (C_{rs}(y''))_{r, s = 1, \dots, l}\,, \\ C_{rs}(y'') = \sum_{j=1}^{k} \beta_{rs}^{(j)}(y'')\left(\dfrac{\partial q}{\partial \tau_j}\right)(0, y''; \tilde{\zeta}(y'')) + \gamma_{rs}(y''), \quad r, s = 1, \dots, l, \end{cases} \tag{1.9}$$

for some smooth symmetric matrix $(\gamma_{rs}(y''))$.

For $y'' = 0$, we choose $\beta(0)$ so that $C(0) < 0$, which is possible because $\nabla_\lambda q(0, y''; \tilde{\zeta}(y'')) \neq 0$, and then smoothly extend β in a neighborhood of $y'' = 0$ by Dini's theorem. It is then obvious that $\zeta(y) \in \Gamma(y)$ for y close to 0, hence the result. ∎

Lemma 1.1 will be applied when $\tilde{\zeta}$ is a vector field with $[\tilde{\zeta}(\rho)] = [z_2(\rho)]$, $\forall \rho \in \tilde{\Sigma}$. Before we prove Theorem 0.1 two remarks are in order.

First of all condition (ii) in Theorem 0.1 is independent of the function S, provided S satisfy conditions (0.10), (0.11) as can be seen using the same arguments as in [3]. Moreover, as in [3] we can always suppose that S is independent of ξ_0.

2. Proof of the Theorems.

PROOF OF THEOREM 0.1. Implication (i) $\Rightarrow$ (ii) is proved by the same argument as in [3], Theorem 2.2, taking into account that condition H_3) yields $H_\Lambda(\rho) \in T_\rho \tilde{\Sigma}$, $\forall \rho \in \tilde{\Sigma}$. We will now prove that (ii) $\Rightarrow$ (i).

Let p_2 as in (0.1). In a conic neighborhood of a given point in Σ we can write

$$p_2(x, \xi) = -\xi_0^2 + \sum_{j=1}^{d} \psi_j^2(x, \xi') \tag{2.1}$$

for some smooth real functions $\psi_j(x, \xi')$, $j = 1, \dots, d$, homogeneous of degree 1 with respect to ξ', for which $H_{\psi_1}, \dots, H_{\psi_d}$ are independent on the manifold

$$\Sigma' = \{(x, \xi') \,|\, \psi_j(x, \xi') = 0, \ \ j = 1, \dots, d\}.$$

Note that $\Sigma = \Sigma' \cap \{\xi_0 = 0\}$.

Moreover, let $\alpha_j(x, \xi')$, $j = 1, \dots, d'$, be a set of smooth real functions, homogeneous of degree 1 with respect to ξ' such that we have $\tilde{\Sigma}' = \Sigma' \cap \Gamma'$, where

$$\Gamma' = \{(x, \xi') \,|\, \alpha_1(x, \xi') = \dots = \alpha_{d'}(x, \xi') = 0\}, \tag{2.2}$$

and $H_{\psi_1}, \dots, H_{\psi_d}$, $H_{\alpha_1}, \dots, H_{\alpha_{d'}}$ linearly independent on $\tilde{\Sigma}'$ (hence $\tilde{\Sigma} = \tilde{\Sigma}' \cap \{\xi_0 = 0\}$).

From now on we shall work in the neighborhood of $\tilde{\Sigma}'$ where

$$|\alpha(x, \xi'/|\xi'|)|^2 < 1, \qquad \alpha = (\alpha_1, \dots, \alpha_{d'}).$$

Let now $S(x, \xi')$ satisfy conditions (0.10), (0.11) and according to Lemma 1.1 denote by ζ a smooth vector field on Σ such that $\zeta|_{\tilde{\Sigma}} = -H_S|_{\tilde{\Sigma}}$ and, when $\rho' = (\bar{x}, \bar{\xi}') \in \Sigma'$, $\rho = (\xi_0 = 0, \rho') \in \Sigma$:

$$\sigma(\zeta(\rho'), F(\rho)\zeta(\rho')) = -|\alpha(\bar{x}, \bar{\xi}'/|\bar{\xi}'|)|^2 \sigma(\zeta(\rho'), H_{\xi_0})^2 \tag{2.3}$$

(observe that $\sigma(\zeta, H_{\xi_0})|_{\Sigma'} \neq 0$).

For every $\rho' \in \Sigma'$ we define

$$(2.4)\quad \tilde{\gamma}_j(\rho') = \sigma(\zeta(\rho'), H_{\psi_j}(\rho'))/\sigma(\zeta(\rho'), H_{\xi_0}), \qquad j = 1, \dots, d.$$

If γ_j is a smooth continuation of $\tilde{\gamma}_j$ outside Σ', $j = 1, \dots, d$, chosen such that:

$$(2.5)\quad |\gamma(x, \xi')| = (1 - |\alpha(x, \xi'/|\xi'|)|^2)^{1/2}, \qquad \gamma = (\gamma_1, \dots, \gamma_d),$$

then $|\gamma| < 1$ outside Γ' and $|\gamma| = 1$ only on Γ'.

Thus near Σ' the principal symbol can be factored as:

$$(2.6)\quad p_2(x, \xi) = -(\xi_0 - \langle \gamma, \psi \rangle)(\xi_0 + \langle \gamma, \eta \rangle) + |\psi|^2 - \langle \gamma, \psi \rangle^2,$$

where $|\psi|^2 - \langle \gamma, \psi \rangle^2 \geqslant |\psi|^2(1 - |\psi|^2)$; as a consequence $|\psi|^2 - \langle \gamma, \psi \rangle^2$ is positive outside $\Sigma' \cup \Gamma'$ vanishes to the second order on Σ' and it is transversally elliptic with respect to $\Sigma' \backslash \tilde{\Sigma}'$.

We now twist our coordinates $\psi_1, \dots, \psi_d$ into a new set of coordinates $\varphi_1, \dots, \varphi_d$, in such a way that in a neighborhood of $\tilde{\Sigma}'$ we have:

$$(2.7)\quad \langle \gamma, \psi \rangle = |\gamma| \varphi_d, \qquad |\psi|^2 = |\varphi|^2 \qquad (\varphi = (\varphi_1, \dots, \varphi_d)).$$

Hence:

$$(2.8)\quad p_2 = -(\xi_0 - |\gamma| \varphi_d)(\xi_0 + |\gamma| \varphi_d) + |\varphi'|^2 + (1 - |\gamma|^2)\varphi_d^2 =$$
$$= -(\xi_0 - (1 - |\alpha|^2)^{1/2} \varphi_d)(\xi_0 + (1 - |\alpha|^2)^{1/2} \varphi_d) + |\varphi'|^2 + |\alpha|^2 \varphi_d^2,$$

where $\varphi' = (\varphi_1, \dots, \varphi_{d-1})$.

Let now m, β_j, $j = 1, \dots, d-1$, be smooth real functions of (x, ξ'), homogeneous of degree -2, -1 respectively with respect to ξ'.

We write (2.8) as:

$$(2.9)\quad p_2 = -(\xi_0 - (1 - |\alpha|^2)^{1/2}(1 + \langle \beta, \varphi' \rangle - m\varphi_d^2)\varphi_d) \cdot$$
$$\cdot (\xi_0 + (1 - |\alpha|^2)^{1/2}(1 + \langle \beta, \varphi' \rangle - m\varphi_d^2)\varphi_d) +$$
$$+ |\varphi'|^2 + |\alpha|^2 \varphi_d^2 + 2m(1 - |\alpha|^2)\left(1 + \langle \beta, \varphi' \rangle - \frac{1}{2} m\varphi_d^2\right)\varphi_d^4 -$$
$$-(1 - |\alpha|^2)(2 + \langle \beta, \varphi' \rangle)\langle \beta, \varphi' \rangle \varphi_d^2 = -\Lambda M + Q.$$

We now observe that whatever is the choice of the $\beta_j' s$, we can choose $m(x, \xi'/|\xi'|)$ large enough so that:

$$Q \gtrsim |\varphi''|^2 + |\alpha|^2 \varphi_d^2 + \varphi_d^4/|\xi'|^2 . \tag{2.10}$$

We now show how to choose the $\beta_j' s$, in order to satisfy condition (0.9).

In order to estimate the Poisson bracket $\{\Lambda, Q\}$, we point out that from the definition of γ on Σ' we have

$$F(\rho)\zeta(\rho') = -\sigma(\zeta(\rho'), H_{\xi_0})H_{\xi_0 - |\gamma|\varphi_d} \text{ on } \Sigma, (\rho = (\xi_0 = 0, \rho'), \rho' \in \Sigma'),$$

so in view of (1.5) we have $\{\xi_0 - (1-|\alpha|^2)^{1/2}\varphi_d, \varphi_j\}|_{\Sigma'} = 0$, $\forall j = 1, \dots, d$; moreover, assumption H$_3$) yields $\{\xi_0 - (1-|\alpha|^2)^{1/2}\varphi_d, \alpha_k\}|_{\bar{\Sigma}'} = 0$, $\forall k = 1, \dots, d'$.

More precisely, we can write:

$$\{\xi_0 - (1 - |\alpha|^2)^{1/2}\varphi_d, \varphi_j\} = \sum_{l=1}^{d} a_{j.l}\varphi_l, \qquad j = 1, \dots, d; \tag{2.11}$$

$$\{\xi_0 - (1 - |\alpha|^2)^{1/2}\varphi_d, \alpha_k\} = \sum_{l=1}^{d} b_{k,l}\varphi_l + \sum_{l=1}^{d'} c_{k,l}\alpha_l, \qquad k = 1, \dots, d', \tag{2.12}$$

for suitable smooth functions $a_{j,l}(x, \xi')$, $b_{k,l}(x, \xi')$, $c_{k,l}(x, \xi')$, homogeneous of degree 0 with respect to ξ'.

Using (2.11) we have that:

$$\begin{aligned} \{\Lambda, Q\} = \{\xi_0 - (1 - |\alpha|^2)^{1/2}\varphi_d, |\varphi'|^2 + |\alpha|^2\varphi_d^2 - \\ -(1 - |\alpha|^2)(2 + \langle\beta, \varphi'\rangle)\langle\beta, \varphi'\rangle\varphi_d^2\} - \\ -(1 - |\alpha|^2)^{1/2}\{\langle\beta, \varphi'\rangle\varphi_d, |\varphi'|^2\} + O(Q). \end{aligned} \tag{2.13}$$

We can estimate these terms by means of (2.11) and (2.12).

Thus we find:

$$\begin{aligned} \{\xi_0 - (1 - |\alpha|^2)^{1/2}\varphi_d, |\varphi'|^2\} - (1 - |\alpha|^2)^{1/2}\{\langle\beta, \varphi'\rangle\varphi_d, |\varphi'|^2\} = \\ = 2\sum_{j=1}^{d-1}\varphi_j\sum_{l=1}^{d} a_{j.l}\varphi_l - 2(1 - |\alpha|^2)^{1/2}\sum_{j=1}^{d-1}\varphi_j\sum_{k=1}^{d-1}\beta_k\{\varphi_k, \varphi_j\}\varphi_d + O(Q); \end{aligned} \tag{2.14}$$

$$\{\xi_0 - (1 - |\alpha|^2)^{1/2}\varphi_d, |\alpha|^2\varphi_d^2\} = O(Q); \tag{2.15}$$

$$\begin{aligned} \{\xi_0 - (1 - |\alpha|^2)^{1/2}\varphi_d, (1 - |\alpha|^2)(2 + \langle\beta, \varphi'\rangle)\langle\beta, \varphi'\rangle\varphi_d^2\} = \\ = 2(1 - |\alpha|^2)\sum_{k=1}^{d-1}\beta_k\sum_{j=1}^{d} a_{k,j}\varphi_j\varphi_d^2 + O(Q). \end{aligned} \tag{2.16}$$

In conclusion, distinguishing the role of φ_d from that of φ', we have

$$(2.17)\quad \{\Lambda, Q\} = 2\sum_{j=1}^{d-1} a_{j,d}\varphi_j\varphi_d - 2(1-|\alpha|^2)^{1/2}\sum_{j=1}^{d-1}\sum_{k=1}^{d-1}\beta_k\{\varphi_k, \varphi_j\}\varphi_j\varphi_d -$$

$$-2(1-|\alpha|^2)\sum_{k=1}^{d-1}\beta_k a_{k,d}\varphi_d^3 + O(Q) =$$

$$= 2(\langle a_d', \varphi'\rangle\varphi_d + (1-|\alpha|^2)^{1/2}\langle\{\varphi', \varphi'\}\beta, \varphi'\rangle\varphi_d - (1-|\alpha|^2)\varphi_d^3\langle a_d', \beta\rangle) + O(Q),$$

where we put $a_d' = (a_{1,d}, \dots, a_{d-1,d})$, $\{\varphi', \varphi'\} = [\{\varphi_h, \varphi_k\}]_{h,k=1,\dots,d-1}$.

At this point we need to express the assumption $H_S^3 p_2|_{\tilde{\Sigma}} = 0$ with respect to the new set of coordinates.

First of all, since S vanishes on Σ and does not depend on ξ_0,

$$(2.18)\qquad S(x, \xi') = \sum_{j=1}^{d} c_j(x, \xi')\varphi_j(x, \xi'),$$

for suitable smooth real functions c_j, homogeneous of degree -1 with respect to ξ', defined near Σ'.

Then

$$(2.19)\quad F(\rho)H_S(\rho) =$$

$$= -\frac{1}{2}\sigma(H_S, H_M)H_\Lambda + |\alpha|^2\sigma(H_S, H_{\varphi_d})H_{\varphi_d} + \sum_{k=1}^{d-1}\sigma(H_S, H_{\varphi_k})H_{\varphi_k} =$$

$$= -(1-|\alpha|^2)^{1/2}\sum_{j=1}^{d} c_j\sigma(H_{\varphi_j}, H_{\varphi_d})H_\Lambda + |\alpha|^2\sum_{j=1}^{d} c_j\sigma(H_{\varphi_j}, H_{\varphi_d})H_{\varphi_d} +$$

$$+ \sum_{k=1}^{d-1}\sum_{j=1}^{d} c_j\sigma(H_{\varphi_j}, H_{\varphi_k})H_{\varphi_k}.$$

On $\tilde{\Sigma}'$, in view of the definition of γ, we have $F(\rho)H_S(\rho') = -\sigma(H_S(\rho'), H_{\xi_0})H_\Lambda(\rho)$, hence

$$\sum_{j=1}^{d} c_j\{\varphi_j, \varphi_k\}|_{\tilde{\Sigma}'} = 0, \qquad \forall k = 1, \dots, d-1,$$

$$\sum_{j=1}^{d} c_j\{\varphi_j, \varphi_d\}(\rho') = \sigma(H_S, H_{\xi_0})(\rho'), \qquad (\rho' \in \tilde{\Sigma}'),$$

so that on $\tilde{\Sigma}'$ we have:

$$(2.20)\qquad \begin{cases} c_d = 0\,, \\ \sum_{j=1}^{d} c_j\{\varphi_j, \varphi_k\} = 0\,, \qquad \forall k = 1, \dots, d-1\,, \\ \sum_{j=1}^{d} c_j\{\varphi_j, \varphi_d\} = \{S, \xi_0\}\,, \end{cases}$$

On the other hand, arguing as in [3], it is easily seen that condition $H_S^3 p_2|_{\tilde{\Sigma}} = 0$ is equivalent to $H_S^2 \Lambda|_{\tilde{\Sigma}} = 0$.

By using (2.18), (2.11), we have on Σ:

$$(2.21)\qquad H_S^2\Lambda = \{S, \{S, \Lambda\}\} = \left\{\sum_{j=1}^{d} c_j\varphi_j, \left\{\sum_{k=1}^{d} c_k\varphi_k, \Lambda\right\}\right\} =$$

$$= \left\{\sum_{j=1}^{d} c_j\varphi_j, \sum_{k=1}^{d}(c_k\{\varphi_k, \Lambda\} + \{c_k, \Lambda\}\varphi_k)\right\} =$$

$$= \sum_{k,j=1}^{d} c_j c_k\{\varphi_j, \{\varphi_k, \Lambda\}\} + \sum_{k,j=1}^{d} c_j\{\varphi_j, \varphi_k\}\{c_k, \Lambda\}\,.$$

In view of (2.11), on $\tilde{\Sigma}'$ we have:

$$(2.22)\qquad \{\varphi_j, \{\varphi_k, \Lambda\}\} =$$

$$= \{\varphi_j, \{\varphi_k, \xi_0 - (1-|\alpha|^2)^{1/2}\varphi_d\}\} - \{\varphi_j, \{\varphi_k, (1-|\alpha|^2)^{1/2}\langle\beta, \varphi'\rangle\varphi_d\}\} =$$

$$= \sum_{l=1}^{d} a_{k,l}\{\varphi_l, \varphi_j\} + (1-|\alpha|^2)^{1/2}\{\varphi_j, \varphi_d\}\sum_{l=1}^{d-1}\beta_l\{\varphi_l, \varphi_k\} +$$

$$+ (1-|\alpha|^2)^{1/2}\{\varphi_k, \varphi_d\}\sum_{l=1}^{d-1}\beta_l\{\varphi_l, \varphi_j\}\,.$$

Moreover, from the first condition in (2.20), we can write

$$(2.23)\qquad c_d(x, \xi') = \sum_{l=1}^{d}\tilde{c}_{d,l}\varphi_l + \sum_{l=1}^{d'}\tilde{\tilde{c}}_{d,l}\alpha_l\,,$$

for suitable $\tilde{c}_{d,l}(x, \xi')$, $\tilde{\tilde{c}}_{d,l}(x, \xi')$ homogeneous of degree -2 with respect to ξ' near Σ'. Hence, from (2.11), (2.12), we obtain

$$(2.24)\qquad \{c_d, \Lambda\}|_{\tilde{\Sigma}} = 0\,.$$

Thus, by replacing (2.22) and (2.20), (2.21) becomes on $\tilde{\Sigma}'$:

$$(2.25)\quad H_S^2 \Lambda|_{\tilde{\Sigma}} = -\left(\sum_{j=1}^{d-1} c_j \{\varphi_j, \varphi_d\} \right)\left(\sum_{k=1}^{d-1} c_k a_{k,d} \{S, \xi_0\} \right) = -\sum_{k=1}^{d-1} c_k a_{k,d} \{S, \xi_0\}.$$

In conclusion

$$(2.26)\quad H_S^3 p_2|_{\tilde{\Sigma}} = 0 \Leftrightarrow \langle a_d'(\rho), c'(\rho)\rangle = 0 \quad \forall \rho \in \tilde{\Sigma}', \qquad c'(c_1, \dots, c_{d-1}).$$

Turning back to (2.17), we choose β in such a way that on $\tilde{\Sigma}'$:

$$(2.27)\quad \{\varphi', \varphi'\}\beta = -a_d'$$

which in particular guarantees that $\langle a_d', \beta\rangle = 0$ on $\tilde{\Sigma}'$.
From the first equation in (2.20) we have that on $\tilde{\Sigma}'$:

$$(2.28)\quad c' \in \operatorname{Ker}\{\varphi', \varphi'\} \backslash \{v \in \mathbb{R}^{d-1} \,|\langle \{\varphi', \varphi_d\}, v\rangle = 0\}.$$

Therefore (2.16) and (2.28) give that a_d' is orthogonal to $\operatorname{Ker}\{\varphi', \varphi'\}$ on $\tilde{\Sigma}'$; this condition allows us to solve the system in (2.27) at each point $\rho' \in \tilde{\Sigma}'$, choosing β as a smooth function on $\tilde{\Sigma}'$, due to H_1) and H_2). In fact we can use the same arguments as in [3] to show that the matrix $\{\varphi', \varphi'\}$ has constant rank at every point of $\tilde{\Sigma}'$. Then we can consider any smooth extension of the β_j's on Σ'. ■

Proof of Theorem 0.2. Let

$$(2.29)\quad P(x, D) = p_2(x, D) + p_1(x, D)$$

be a linear differential operator whose principal symbol p_2 satisfies H_1), H_2), H_3). Define, for $\tau > 0$, $u \in C_0^\infty(\mathbb{R}^{n+1})$:

$$(2.30)\quad \|u\|_{s,\tau}^2 = \int_{-\infty}^{0} e^{-2\tau x_0} \|u\|_s^2(x_0)\, dx_0,$$

where $\|u\|_s^2(x_0) = \int_{\mathbb{R}^n} |\hat{u}(x_0, \xi')|^2 (1 + |\xi'|^2)^s d\xi'$.

Then the proof of Theorem 0.2 will follow by well known arguments (see [4]) from the following a priori inequality.

Lemma 2.1. *Suppose P satisfies* H_1), H_2), H_3), (ii) *of Theorem* 0.1 *and* (0.12) *on* Σ. *Then, if K is any compact subset of* $\mathbb{R}^{n+1}$, *there exists* $C_K > 0$ *such that* $\forall u \in C_0^\infty(K)$ *we have for a sufficiently* *large* τ

(2.31) $$\tau^4 \|u\|^2_{0,\tau} \leqslant C_K \|Pu\|^2_{0,\tau}.$$

The proof goes exactly as in [3].

3. An example.

We consider an operator P whose principal symbol is given by

(3.1) $$p_2(x, \xi) = -\xi_0^2 + (x_0 - \langle a, x' \rangle)^2 \xi_n^2 + \\ + \left(\frac{1}{|a|^2} + r(x)^2 \right) |\xi'|^2, \qquad a \in \mathbb{R}^{n-1}, \ a \neq 0,$$

$(x, \xi) = (x_0, x', x_n; \xi_0, \xi', \xi_n) \in T^*(\mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R})$, such that:

(3.2) $$r(x) \in C^\infty(\mathbb{R}^{n+1}).$$

In this case we have:

$$\Sigma = \{(x, \xi) \,|\, \xi_0 = 0, \ x_0 = \langle a, x' \rangle, \ \xi_j = 0, \ j = 1, \dots, n-1\},$$

and for $\rho = (\bar{x}; 0, 0, \bar{\xi}_n) \in \Sigma (\bar{\xi}_n \neq 0)$:

$$p_{2,\rho}(\delta x, \delta \xi) = -(\delta \xi_0)^2 + (\delta x_0 - \langle a, \delta x' \rangle)^2 \bar{\xi}_n^2 + \left(\frac{1}{|a|^2} + r(\bar{x})^2 \right) |\delta \xi'|^2,$$

$\forall (\delta x, \delta \xi) \in T_\rho T^* \mathbb{R}^{n+1}$;

$$\Gamma(\rho) = \Bigg\{ (\delta x, \delta \xi) \in T_\rho T^* \mathbb{R}^{n-1} \,|\, \delta \xi_0 > 0, \\ \delta \xi_0 > \left((\delta x_0 - \langle a, \delta x' \rangle)^2 \bar{\xi}_n^2 + \left(\frac{1}{|a|^2} + r(\bar{x})^2 \right) |\delta \xi'|^2 \right)^{1/2} \Bigg\},$$

$$C(\rho) = \Bigg\{ (\delta x, \delta \xi) \in T_\rho T^* \mathbb{R}^{n+1} \,|\, \delta x_0 \geqslant 0, \\ -(\delta x_0)^2 + \left(\left\langle \frac{a}{|a|^2}, \delta \xi' \right\rangle \right)^2 (\bar{\xi}_n)^{-2} + \left(\frac{1}{|a|^2} + r(\bar{x})^2 \right)^{-1} |\delta \xi'|^2 \leqslant 0, \\ \delta \xi' = -\delta \xi_0 a, \ \delta x_n = 0 = \delta \xi_n \Bigg\}.$$

Moreover:

$$\operatorname{Ker} F^2(\rho)\cap \operatorname{Im} F^2(\rho)=$$

$$=\left\{(\delta x, \delta\xi)\in T_\rho T^*\mathbb{R}^{n+1}\,|\,\delta x_0=\langle a, \delta x'\rangle,\ \delta\xi_0+\left(\frac{1}{|a|^2}+r(\bar{x})^2\right)\langle a, \delta\xi\rangle=0\right\}\cap$$

$$\cap\left\{\left(\delta x_0,\ \delta x_0\left(\frac{1}{|a|^2}+r(\bar{x})^2\right)a,\ 0;\ \delta\xi_0,\ -\delta\xi_0 a,\ 0\right)\right\}.$$

Then we have

$$\operatorname{Ker} F^2(\rho)\cap \operatorname{Im} F^2(\rho)\neq(0)\quad \text{if}\quad r(\bar{x})=0,$$

i.e.

(3.3) $$\tilde{\Sigma}=\Sigma\cap\{(x, \xi)\,|\,r(x)=0\}.$$

On $\tilde{\Sigma}$ it will be

$$\operatorname{Im} F^3(\rho)=\left\{(\delta x, \delta\xi)\in T_\rho T^*\mathbb{R}^{n+1}\,|\left(\delta x_0,\ \delta x_0\frac{a}{|a|^2},\ 0;\ 0\right)\right\},$$

$$\operatorname{Ker} F(\rho)\cap \operatorname{Im} F^3(\rho)=\operatorname{Im} F^3(\rho).$$

Let now $S(x, \xi)$ the following function on $T^*(\mathbb{R}^{n+1})$

(3.4) $$S(x, \xi)=(x_0-\langle a, x'\rangle)\,\xi_n\,.$$

Clearly $S(x, \xi)$ verifies (0.10), (0.11) and for every $\rho\in\tilde{\Sigma}$, $\operatorname{Ker} F(\rho)\cap \operatorname{Im} F^3(\rho)$ is the one dimensional subspace of the vectors collinear to $F(\rho)H_S(\rho)$.

In order to have condition H_3) satisfied, we require that

(3.5) $$\sigma(F(\rho)H_S(\rho), H_r(\rho))=\frac{\partial r}{\partial x_0}(\rho)+\left\langle\frac{1}{|a|^2},\ \frac{\partial r}{\partial x'}(\rho)\right\rangle=0\quad \forall\rho\in\tilde{\Sigma}.$$

From the calculation of $H_S^2 p_2$ we find, if $\rho = (\bar{x}; 0, 0, \bar{\xi}_n) \in \Sigma$, $H_S^3 p_2(\rho) = 0$, then condition (ii) in Theorem 0.1 holds.

Thus the principal symbol p_2 admits an elementary decomposition in the sense of Ivrii (0.7)-(0.9) and for such a decomposition we have that:

$$\text{for every } \rho = (\bar{x}, \bar{\xi}_n) \in \tilde{\Sigma},\ H_A(\rho) \text{ is collinear to } \left(\bar{\xi}_n \frac{a}{|a|^2} \bar{\xi}_n, 0; 0\right) = F(\rho) H_S(\rho), \text{ whereas for } \rho = (\bar{x}, \bar{\xi}_n) \in \Sigma \backslash \tilde{\Sigma},\ H_A(\rho) \in \operatorname{Ker} F(\rho) \cap [\operatorname{Int}(C(\rho)) \cup \operatorname{Int}(-C(\rho))] = \left\{ (\delta x_0, \delta x', 0; 0) \in T_\rho T^* \mathbb{R}^{n+1} \mid \delta x_0 = \langle a, \delta x' \rangle,\ -(\delta x_0)^2 + \left(\frac{1}{|a|^2} + r(\bar{x})^2\right)^{-1} |\delta x'|^2 < 0 \right\}.$$

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