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## A Note on Indecomposable Modules Over Valuation Domains.

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### 1. Introduction.

This note was motivated by the results of Luigi Salce and Paolo Zanardo on the existence and the structure of indecomposable finitely generated modules and indecomposable finite rank torsion-free modules over valuation domains.

Zanardo [Z] introduced the concept of  $u$ -independent sets of units in a maximal immediate extension of  $R$  and used them to construct finitely generated indecomposable modules  $M$  of the following two types:

- 1)  $M$  has arbitrary finite length greater than one and has Goldie dimension equal to one, and
- 2)  $M$  has arbitrary finite Goldie dimension greater than one and has length equal to one plus its Goldie dimension.

Until now, these were the only explicit examples of indecomposable finitely generated modules over valuation domains.

In order to obtain indecomposable modules without restrictions on the Goldie dimension, we introduce the notion of quadratic  $u$ -independence. We show that in most valuation domains there exist quadratically  $u$ -independent sets of arbitrary finite cardinality. Quadratically  $u$ -independent sets and compatible triples of ideals are used to construct

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new examples of finitely generated modules with local endomorphism ring. In fact, it follows from our construction that over most valuation domains, indecomposable finitely generated modules  $M$  exist such that the only condition on the two numerical invariants of  $M$  (its length and its Goldie dimension) is the necessary condition that the Goldie dimension of the module  $M$  must be less than its length.

Interestingly, in the finite rank torsion-free setting, the same concept of quadratic  $u$ -independence facilitates the construction of indecomposable modules of a new type. Analogous to the finitely generated case, our construction reveals that over most valuation domains, indecomposable torsion-free modules  $M$  of finite rank exist such that the only requirement on the rank of the module and the rank of its basic submodules is the obviously necessary condition that the basic submodules must have rank less than the rank of  $M$ .

For references on modules over valuation domains, we refer to Fuchs and Salce [FS].

## 2. Preliminaries.

Throughout,  $R$  will denote a valuation domain (i.e. a commutative integral domain whose ideals are totally ordered by inclusion).  $P$  will denote the unique maximal ideal of  $R$ ,  $S$  a maximal immediate extension of  $R$ , and  $\mathbb{Q}$  the usual field of quotients of  $R$ .  $U(T)$  will denote the multiplicative group of units of the domain  $T$ .

For an ideal  $I$  of  $R$ , let  $\pi_I: S \rightarrow S/IS$  be the canonical map from  $S$  onto  $S/IS$ . The image of  $R$  under this map is a subring of  $S/IS$  which is isomorphic to  $R/I$ . We will identify this subring  $\pi_I(R)$  of  $S/IS$  with  $R/I$ . Evidently,  $R/I \leq (\overline{R/I}) \leq S/IS$ . Of primary concern will be the case when  $R/I$  is properly contained in its completion.

We recall the definition of the breadth ideal of a unit  $u \in S$ . The *breadth ideal* of  $u$  is the ideal of  $R$  defined by

$$B(u) = \langle a \in R \mid u \notin aS + R \rangle.$$

We note that if  $u \in U(R)$ , then  $B(u)$  is generated by the empty set and thus is equal to the zero ideal. In the following discussions, we assume that  $u \in U(S) \setminus R$ .

The equality  $S = R + PS$  implies that  $B(u) \leq P$  and that  $B(u)$  is a  $\nu$ -ideal, i.e. it is the intersection of the ideals properly containing it. Hence, the valuation ring  $R/B(u)$  is Hausdorff in its  $R/B(u)$ -topology.

From the definition of the breadth ideal of  $u$ , it is clear that  $u \notin B(u)S + R$ . For every  $r \notin B(u)$ , there exists an element  $u_r \in U(R)$

such that  $u - u_r \in rS$ . The collection  $\{u_r \mid r \notin B(u)\}$  is called a *set of approximations* for the unit  $u$ . Furthermore,  $u + B(u)S$  is the limit in  $S/B(u)S$  of the Cauchy net  $\{u_r + B(u)S \mid r \notin B(u)\}$ . As  $u + B(u)S \in S/B(u)S$  but is not an element of  $R/B(u)$ , it follows that  $R/B(u)$  is not complete in the  $R/B(u)$ -topology.

### 3. Quadratic $u$ -independence.

DEFINITION 1. We say that the set  $\{u_1, \dots, u_n\} \subset U(S)$  is *quadratically  $u$ -independent* over an ideal  $I$  of  $R$  if

- 1)  $B(u_i) = I$  for all  $i$ , and
- 2) for every polynomial  $f \in R[X_1, \dots, X_n]$  of degree  $\leq 2$ ,  $f(u_1, \dots, u_n) \in IS$  implies  $f \in P[X_1, \dots, X_n]$ .

Some important properties of quadratic  $u$ -independence are given in the following lemma.

LEMMA 1. Let  $I$  be an ideal of  $R$ . If  $\{u_1, \dots, u_n\}$  is a set of quadratically  $u$ -independent units over  $I$ , then

- 1)  $\{u_1, \dots, u_n\}$  is  $u$ -independent over  $I$ ,
- 2) every non-empty subset of  $\{u_1, \dots, u_n\}$  is quadratically  $u$ -independent over  $I$ , and
- 3)  $I$  is a prime ideal of  $R$ .

PROOF. The first two statements are obvious from the above definitions. To prove the third statement, we suppose that  $I$  is not a prime ideal and show that no singleton  $\{u\} \subset U(S)$  is quadratically  $u$ -independent over  $I$ . Assuming that  $I$  is not prime, there exists an element  $t \in R \setminus I$  such that  $t^2 \in I$ .  $B(u) = I$  implies that  $u \in tS + R$ , i.e. there exists an  $a \in U(R)$  such that  $u - a \in tS$ . Hence,  $u^2 - 2ua + a^2 = (u - a)^2 \in t^2S \subset IS$ . Consequently, the set  $\{u\}$  is not quadratically  $u$ -independent  $I$ . ■

If  $\{u_1, \dots, u_n\}$  is a set of units in  $S$  which is  $u$ -independent over a *prime* ideal  $I$ , then one can conclude that  $\{u_1 + IS, \dots, u_n + IS\} \subset S/IS$  is linearly independent over the valuation domain  $R/I$  (see [SZ3]). Moreover, if  $\{u_1, \dots, u_n\}$  is quadratically  $u$ -independent over  $I$ , then we have the following conclusion:

LEMMA 2. Let  $\{u_1, \dots, u_n\} \subset U(S)$  be quadratically  $u$ -independent over  $I$ . For every polynomial  $f \in R[X_1, \dots, X_n]$  of degree  $\leq 2$ ,  $f(u_1, \dots, u_n) \in IS$  implies  $f \in I[X_1, \dots, X_n]$ .

PROOF. Let  $f \in R[X_1, \dots, X_n]$  be of degree  $\leq 2$  such that  $f(u_1, \dots, u_n) \in IS$ . The quadratic  $u$ -independence of  $\{u_1, \dots, u_n\}$  guarantees that  $f \in P[X_1, \dots, X_n]$ . By way of contradiction, assume that there exists at least one coefficient of  $f$  which is not in  $I$ . Pick such a coefficient  $a \in P \setminus I$  of  $f$  with minimal value in the value group. Multiplying  $f(u_1, \dots, u_n)$  by  $a^{-1}$  yields

$$a^{-1}f(u_1, \dots, u_n) \in a^{-1}IS.$$

By Lemma 1,  $I$  is a prime ideal of  $R$ , thus  $aI = I$  and hence  $a^{-1}IS = IS$ . Therefore,  $a^{-1}f \in R[X_1, \dots, X_n]$  with  $a^{-1}f(u_1, \dots, u_n) \in IS$ , but  $a^{-1}f \notin P[X_1, \dots, X_n]$ . This contradiction completes the proof.  $\blacksquare$

We now give a typical example to indicate that there exists a large class of valuation domains  $R$  such that  $S$  contains quadratically  $u$ -independent sets of arbitrary finite cardinality.

EXAMPLE. Let  $R$  be a valuation domain such that, for some prime ideal  $I$  of  $R$ ,  $R/I$  is Hausdorff and not complete in its  $R/I$ -topology. Furthermore, assume that the completion of  $R/I$  is transcendental over  $R/I$ . (For example, take  $R/I \cong \mathbb{Z}_p$ , where  $\mathbb{Z}_p$  is the localization of the ring of integers  $\mathbb{Z}$  at some non-zero prime ideal  $p$ . In this case,  $\widehat{(R/I)}$  is isomorphic to the ring of  $p$ -adic integers  $J_p$ , which is transcendental over  $\mathbb{Z}_p$ .) Choose  $u \in U(S) \setminus R$  such that  $u + IS \in \widehat{(R/I)} \setminus R/I$  and  $u + IS$  is transcendental over  $R/I$ . Then, it is readily seen that the set  $\{u^{2^i-1} \mid 1 \leq i \leq n\}$  is quadratically  $u$ -independent over  $I$  for every choice of  $n \in \mathbb{N}$ .

#### 4. Finitely generated modules.

For a finitely generated  $R$ -module  $M$ , we consider two numerical invariants for  $M$ . The *length* of  $M$ , denoted by  $\ell(M)$ , is defined as the cardinality of a minimal generating set for  $M$ ; this coincides with the dimension of the vector space  $M/PM$  over the field  $R/P$ . The maximal number of summands in a direct sum of cyclic submodules contained in  $M$  is called the *Goldie dimension* of  $M$  and is denoted by  $g(M)$ . It is well known (see [SZ1]) that  $g(M) \leq \ell(M)$  for a finitely generated  $R$ -module  $M$ . Furthermore,  $M$  is a direct sum of cyclics if and only if  $g(M) = \ell(M)$ .

We state the next lemma in the more general setting of commutative local rings. It will be useful when examining the endomorphism ring of a finitely generated module.

LEMMA 3. Let  $R$  be a commutative local ring with maximal ideal  $P$ , and  $M$  a finitely generated  $R$ -module with minimal generating set  $\{x_1, \dots, x_n\}$ . For  $f \in \text{End}_R M$ , we can write

$$f(x_i) = \sum_{j=1}^n c_{ij} x_j \quad (c_{ij} \in R, 1 \leq i \leq n).$$

$f$  is an automorphism of  $M$  if and only if the coefficient matrix  $C = [c_{ij}]$  is invertible in the matrix ring  $\text{Mat}_n(R)$ .

PROOF. See Matsumura [Mt, pp. 8-9]. ■

The matrix  $C$  appearing in the statement of the lemma is known as an *associated matrix* of the endomorphism  $f$ .

The following definition is due to Salce and Zanardo [SZ]. Recall that  $A: J = \{r \in R \mid rJ \leq A\}$ .

DEFINITION 2. Let  $A, J, I$  be ideals of  $R$ . We say that  $(A, J, I)$  is a *compatible triple of ideals* of  $R$  if the following are satisfied:

- 1)  $0 < A < J$ ,
- 2)  $I = A: J$  and  $r^{-1}A > I$ , for every  $0 \neq r \in J$ , and
- 3)  $R/I$  is not complete in the  $R/I$ -topology.

We are now ready to begin our construction.

Let  $h, k \in \mathbb{N}$ . Given a set  $\{u_{ij} \mid 1 \leq i \leq h; 1 \leq j \leq k\} \subset U(S) \setminus R$  with  $B(u_{ij}) = I$  for every  $i$  and  $j$ , let  $\{u_{ij}^r \mid 0 \neq r \in J\}$  be a set of approximations for the unit  $u_{ij} \in S$  for each  $i$  and for each  $j$ . We define a finitely generated  $R$ -module

$$M = M_{h,k} = \langle x_1, \dots, x_h, y_1, \dots, y_k \rangle,$$

where the generators are subject to the relations:

$$ax_i = 0 \quad (a \in A, 1 \leq i \leq h),$$

$$ry_i = \sum_{i=1}^h ru_{ij}^r x_i \quad (0 \neq r \in J, 1 \leq j \leq k).$$

The following observations are immediate consequences of the definition of  $M$ .

- (a) The length of  $M$  is  $h + k$ .
- (b) The ideals  $\text{Ann } x_i$  and  $\text{Ann } y_i$  are equal to  $A$  for every  $i$  and  $j$ .

(c) The submodule  $B_A = Rx_1 \oplus \dots \oplus Rx_h$  of  $M$  is isomorphic to a direct sum of  $k$  copies of  $R/J$ .

(d) The ideal  $\text{Ann}(y_j + B_A)$  is equal to  $J$  for every  $j$ .

(e)  $B_A$  is a pure submodule of  $M$ .

We conclude that  $M$  is a pure extension of  $B_A$  by  $M/B_A$ . In fact,  $M$  has a pure-composition series of the form

$$0 < Rx_1 < Rx_1 + Rx_2 < \dots < Rx_1 + \dots + Rx_h + Ry_1 < \dots < Rx_1 + \dots + Ry_k$$

which yields the following annihilator sequence:

$$\underbrace{A = \dots = A}_h < \underbrace{J = \dots = J}_k.$$

Next, we show that under an additional assumption on the set of units  $\{u_{ij}\}$  we can reach a stronger conclusion regarding the submodule  $B_A$ .

LEMMA 4. If  $\{u_{ij} \mid 1 \leq i \leq h, 1 \leq j \leq k\}$  is  $u$ -independent over  $I$ , then  $B_A$  is a basic submodule of  $M$ .

PROOF. It suffices to show that  $M$  has no cyclic pure submodule isomorphic to  $R/J$ . Let  $z \in M$  and write  $z = r_1x_1 + \dots + r_hx_h + s_1y_1 + \dots + s_ky_k$  with all  $r_i, s_j \in R$ . Suppose  $Rz \cong R/J$ . Then, at least one of the  $s_j$  is non-zero. The defining relations on  $M$  imply that for every non-zero  $r \in J$ ,

$$rz = r \left( r_i + \sum_{j=1}^k s_j u_{ij}^r \right) x_i \quad (1 \leq i \leq h).$$

Thus,  $0 = rz \in B_A = Rx_1 \oplus \dots \oplus Rx_h$  for every  $r \in J$ . Since the annihilator of  $x_i$  is equal to  $A$  for all  $i$ , it follows that

$$r_i + \sum_{j=1}^k s_j u_{ij}^r \in r^{-1}A \quad (0 \neq r \in J, 1 \leq i \leq h).$$

$u_{ij}$  is the limit modulo  $IS = \bigcap_{0 \neq r \in J} r^{-1}AS$  of its approximating set  $\{u_{ij}^r\}$ , so we can take limits to obtain

$$r_i + \sum_{j=1}^k s_j u_{ij} \in IS \quad (1 \leq i \leq h).$$

The  $u$ -independence of the set  $\{u_{ij}\}$  over  $I$  forces all  $r_i, s_j \in P$ . Therefore,  $z \in PM$  and hence  $Rz$  is not pure in  $M$ . ■

We turn our attention to the ring of  $R$ -endomorphism of the module  $M$  constructed above. Our goal is to show that under an appropriate condition on the units  $\{u_{ij}\}$ ,  $M$  will have local endomorphism ring (which guarantees the indecomposability of  $M$ ).

Let  $\eta \in \text{End}_R M$ . We can write

$$(1) \quad \eta x_i = \sum_{m=1}^h a_{im} x_m + \sum_{p=1}^k b_{ip} y_p \quad (1 \leq i \leq h),$$

$$(2) \quad \eta y_j = \sum_{m=1}^h c_{jm} x_m + \sum_{p=1}^k d_{jp} y_p \quad (1 \leq j \leq k),$$

where the set  $T = \{a_{im}, b_{ip}, c_{jm}, d_{jp} \mid 1 \leq i, m \leq h; 1 \leq j, p \leq k\}$  of coefficients is contained in  $R$ . Using the defining relations for  $M$ , the endomorphism  $\eta$  of  $M$  must satisfy  $a\eta x_i = \eta(ax_i) = 0$  for every  $a \in A$  (this is easily seen as the annihilator of  $M$  is equal to  $A$ ) and, more significantly, for every non-zero  $r \in J$ ,

$$r\eta y_j = \eta \left( \sum_{i=1}^h r u_{ij}^r x_i \right) \quad (1 \leq j \leq k).$$

Thus, for every  $0 \neq r \in J$  and for every  $m$  and  $j$  such that  $1 \leq m \leq h$ ;  $1 \leq j \leq k$ , we have an equation in  $Rx_m$  of the form:

$$\sum_{i=1}^h r u_{ij}^r \left( a_{im} x_m + \sum_{p=1}^k b_{ip} u_{mp}^r x_m \right) = r c_{jm} x_m + \sum_{p=1}^k r d_{jp} u_{mp}^r x_m.$$

Recalling that the annihilator of  $x_m$  is equal to  $A$ , we arrive at the following inclusion which holds for every  $0 \neq r \in J$  and for every  $m$  and  $j$ :

$$\sum_{i=1}^h u_{ij}^r \left( a_{im} + \sum_{p=1}^k b_{ip} u_{mp}^r \right) - c_{jm} - \sum_{p=1}^k d_{jp} u_{mp}^r \in r^{-1} A.$$

As  $\{u_{ij}^r\}$  is an approximating set for the unit  $u_{ij} \in S$  for every  $i$  and  $j$ , we can take limits to obtain the following condition:

$$(3) \quad \sum_{i=1}^h u_{ij} \left( a_{im} + \sum_{p=1}^k b_{ip} u_{mp} \right) - c_{jm} - \sum_{p=1}^k d_{jp} u_{mp} \in IS.$$

The proof of the next lemma is immediate from the above discussion.

LEMMA 5. Let  $\eta: \{x_1, \dots, x_h, y_1, \dots, y_k\} \rightarrow M$  be a map defined on the generating set of  $M$  by (1) and (2) where  $T = \{a_{im}, b_{ip}, c_{jm}, d_{jp} \mid 1 \leq i, m \leq h; 1 \leq j, p \leq k\} \subset R$ . The map  $\eta$  extends to an endomorphism of  $M$  if and only if the entries of its coefficient matrix

$$C = \begin{bmatrix} a_{11} & \dots & a_{1h} & b_{11} & \dots & b_{1k} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{h1} & \dots & a_{hh} & b_{h1} & \dots & b_{hk} \\ c_{11} & \dots & c_{1h} & d_{11} & \dots & d_{1k} \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{k1} & \dots & c_{kk} & d_{k1} & \dots & d_{kk} \end{bmatrix}$$

satisfy condition (3) above. ■

Hence, we have characterized the matrices in  $\text{Mat}_{h+k}(R)$  which, with respect to the minimal generating set  $\{x_1, \dots, x_h, y_1, \dots, y_k\}$  are associated to an endomorphism of  $M$ .

For fixed  $m$  and  $j$ , we view condition (3) as a quadratic equation modulo  $IS$  in the unknowns  $\{u_{ip} \mid 1 \leq i \leq h; 1 \leq p \leq k\}$  with coefficients from  $R$ . If we assume that the set  $\{u_{ip}\}$  is quadratically  $u$ -independent over  $I$ , then we can conclude (after collecting like terms) that

- a)  $c_{jm} \in P$ ,
- b)  $a_{mm} - d_{jj} \in P$ ,
- c)  $b_{ip} \in P$  ( $1 \leq i \leq h; 1 \leq p \leq k$ ),
- d)  $a_{im} \in P$  ( $i \neq m, 1 \leq i \leq h$ ),
- e)  $d_{jp} \in P$  ( $p \neq j, 1 \leq p \leq k$ ).

Since these conditions must hold for every  $m$  and for every  $j$ , we have that

$$T \setminus \{a_{11}, \dots, a_{ii}, \dots, a_{hh}, d_{11}, \dots, d_{jj}, \dots, d_{kk}\} \subset P,$$

and

$$a_{mm} - d_{jj} \in P \quad \text{for all } m \text{ and } j.$$

Thus we have proved the following lemma.

LEMMA 6. Let  $M$  be as above. If the set  $\{u_{ij} \mid 1 \leq i \leq h; 1 \leq j \leq k\}$  is quadratically  $u$ -independent over  $I$ , then every matrix

$C \in \text{Mat}_{h+k}(R)$  associated to an endomorphism of  $M$  is of the form  $C = rE + D$ , where  $r \in R$ ,  $E$  is the identity matrix of  $\text{Mat}_{h+k}(R)$ , and  $D \in \text{Mat}_{h+k}(P)$ . ■

**COROLLARY 1.** Under the assumptions of the above lemma, the endomorphism ring of  $M$  is a local ring.

**PROOF.** A matrix  $C = rE + D$  ( $r, E, D$  as in the statement of Lemma 6) is invertible in  $\text{Mat}_{h+k}(R)$  if and only if  $r \in U(R)$ . If  $\eta_1, \eta_2 \in \text{End}_R M$  are non-units, then by Lemmas 3 and 6, each has an associated matrix with entries in the maximal ideal  $P$ ; hence, the sum  $\eta_1 + \eta_2$  has an associated matrix of the same form and is thus also a non-unit. Therefore,  $\text{End}_R M$  is local. ■

We can now prove the main result of this section.

**THEOREM 1.** Let  $R$  be a valuation domain,  $(A, J, I)$  a compatible triple of ideals of  $R$ , and  $m < n$  a pair of positive integers. If there exists a set  $\mathcal{U}$  of units contained in a maximal immediate extension  $S$  of  $R$  which is of cardinality  $m(n - m)$  and is quadratically  $u$ -independent over  $I$ , then there exists a finitely generated  $R$ -module  $M$  with local endomorphism ring (and hence indecomposable) such that  $\ell(M) = n$  and  $g(M) = m$ .

**PROOF.** Choose  $h = m$  and  $k = n - m$ . Using the set  $\mathcal{U}$ , form the  $R$ -module  $M_{h,k}$ . Clearly  $M_{h,k}$  has length  $h + k = n$  and by Lemma 4, the Goldie dimension of  $M_{h,k}$  is  $m$ ;  $M_{h,k}$  has local endomorphism ring by Corollary 1. ■

## 5. Finite rank torsion-free modules.

The following generalization of the notion of a compatible triple of ideals is due to Salce and Zanardo [SZ4]. Recall that for two submodules  $L$  and  $H$  of  $Q$ ,  $L: H = \{q \in Q \mid qH \leq L\}$  is also a submodule of  $Q$ . If  $L < H$ , then  $L: H$  coincides with the ideal  $\{r \in R \mid rH \leq L\}$  of  $R$ .

**DEFINITION 3.** Let  $L, H, I$  be  $R$ -submodules of  $Q$ . We say that  $(L, H, I)$  is a *compatible triple of submodules* of  $Q$  if the following are satisfied:

- 1)  $0 < L < H$ ,
- 2)  $I = L: H$  and  $I > rL$  for every  $r \in R$  with  $r^{-1} \in H \setminus L$ , and
- 3)  $R/I$  is not complete in the  $R/I$ -topology.

Let  $(L, H, I)$  be a compatible triple of submodules of  $Q$ . Then  $I$  is a  $\nu$ -ideal of  $R$  and is necessarily the zero ideal when  $H = Q$ . In fact, if  $R$  is not complete in its  $R$ -topology and if  $0 < L < Q$ , then  $(L, Q, I = 0)$  is always a compatible triple of submodules. If  $L$  and  $H$  happen to be ideals of  $R$ , then  $(L, H, I)$  is a compatible triple of ideals.

For our purposes, it suffices to consider compatible triples of the form  $(R, R_I, I)$ , where  $I < P$  is a prime ideal of  $R$  and  $R_I$  is the valuation domain obtained by localizing  $R$  at  $I$ . The following lemma gives a necessary and sufficient condition for this kind of compatible triple to exist.

**LEMMA 7.** For a prime ideal  $I < P$  of  $R$ ,  $(R, R_I, I)$  is a compatible triple of submodules of  $Q$  exactly when  $R/I$  is not complete in its  $R/I$ -topology.

**PROOF.** We must show that the triple  $(R, R_I, I)$  satisfies conditions 1) and 2) of Definition 3 for a prime ideal  $I < P$ . Trivially,  $0 < R < R_I$ . Since  $I$  is also an ideal of  $R_I$ ,  $r \in I$  implies  $rR_I \leq I < R$ . Moreover, if  $r \notin I$ , then for every  $s \in P \setminus I$ ,  $(rs)^{-1} \in R_I$  but  $r(rs)^{-1} \notin R$ . Hence  $I = R : R_I$ . Finally, it is clear that  $I < Rr$  for every  $r \in P \setminus I$  (i.e. for every  $r \in R$  such that  $r^{-1} \in R_I \setminus R$ ). ■

From now on, we assume that  $I$  is a prime ideal of  $R$  such that  $I < P$  and  $(R, R_I, I)$  is a compatible triple of submodules of  $Q$ . We note that  $I = \bigcap Rr$ , where the intersection is taken over all  $R \in P \setminus I$ . As  $R/I$  is Hausdorff and not complete in its  $R/I$ -topology,  $I$  is the breadth ideal of some unit in  $S$ .

We begin our construction of torsion-free modules.

For each pair  $h, k$  of positive integers, let  $F_{h,k}$  be the free  $R_I$ -module with basis  $\{x_1, \dots, x_h, y_1, \dots, y_k\}$  and let  $\{u_{ij} \mid 1 \leq i \leq h; 1 \leq j \leq k\}$  be a set of units in  $S \setminus R$  with  $B(u_{ij})$  for every  $i$  and  $j$ . Choose a set  $\{u_{ij}^r \mid r \in P \setminus I\}$  of approximations for the unit  $u_{ij} \in S$ . Consider the  $R$ -submodule  $M_{h,k}$  of  $F_{h,k}$  defined as follows:

$$M_{h,k} = \langle x_1, \dots, x_h, y_1, \dots, y_k, z_j^r \mid r \in P \setminus I, 1 \leq j \leq k \rangle$$

where

$$z_j^r = r^{-1} \left( y_j + \sum_{i=1}^h u_{ij}^r x_i \right) \quad (r \in P \setminus I, 1 \leq j \leq k).$$

The following properties are easily seen to hold for the  $R$ -module  $M = M_{h,k}$  for every choice of  $h, k \in \mathbb{N}$ .

- a)  $M$  has rank  $h + k$ .
- b) For  $s \in R$ ,  $x_i \in sM$  for some  $i$  or  $y_j \in sM$  for some  $j$  exactly if  $s \in U(R)$ .
- c) For  $s \in R$ ,  $rz_j^r \in sM$  if and only if  $Rr \leq Rs$ .
- d) The submodule  $B = Rx_1 \oplus \dots \oplus Rx_h$  of  $M$  is a free  $R$ -module of rank  $h$  and the quotient module  $M/B$  is isomorphic to the direct of  $k$  copies of the ring  $R_I$ .
- e) The submodule  $B = Rx_1 \oplus \dots \oplus Rx_h$  is pure in  $M$ .

LEMMA 8. Let  $M$  be as above.  $B$  is a basic submodule of  $M$  whenever  $\{u_{ij} \mid 1 \leq i \leq h; 1 \leq j \leq k\}$  is  $u$ -independent over  $I$ .

PROOF. As the quotient module  $M/B$  is torsion-free, it is immediate that  $B$  is pure in  $M$ . Now, suppose that the set  $\{u_{ij}\}$  of units of  $S$  is  $u$ -independent over  $I$ . We show that in this case,  $M$  has no pure rank-one submodule isomorphic to  $R_I$  from which it follows that  $B$  is basic in  $M$ . Let  $z \in M$  and write  $z = r_1x_1 + \dots + r_hx_h + s_1y_1 + \dots + s_ky_k$  where, without loss of generality, we can assume that all  $r_i, s_j \in R$ . Further, suppose that  $R_Iz$  (the cyclic  $R_I$ -submodule generated by  $z$  in the free  $R_I$ -module  $F = F_{h,k}$ ) is an  $R$ -submodule of  $M$ . For every  $r \in P \setminus I$ , we can re-write  $z$  in the following form in which the generators  $y_j$  of  $M$  do not appear explicitly and where at least one of the  $s_j$  is non-zero:

$$z = r(s_1z_1^r + \dots + s_kz_k^r) + \left(r_1 - \sum_{j=1}^k s_ju_{ij}^r\right)x_1 + \dots + \left(r_h - \sum_{j=1}^k s_ju_{hj}^r\right)x_h.$$

As  $z \in rM$  and  $r(s_1z_1^r + \dots + s_kz_k^r) \in rM$ , for every  $r \in P \setminus I$ , the purity of the submodule  $B = Rx_1 \oplus \dots \oplus Rx_h$  in  $M$  implies that the coefficients of  $x_i$  satisfy:

$$r_i - \sum_{j=1}^k s_ju_{ij}^r \in Rr \quad (r \in P \setminus I, 1 \leq i \leq h).$$

Recalling that  $u_{ij}$  is the limit modulo  $IS = \bigcap_{r \in P \setminus I} rS$  of its approximating set  $\{u_{ij}^r\}$ , we can take limits to obtain

$$r_i - \sum_{j=1}^k s_ju_{ij} \in IS \quad (1 \leq i \leq h).$$

The  $u$ -independence of the set  $\{u_{ij}\}$  over  $I$  forces all  $r_i$  and  $s_j \in I$ . Therefore,  $z \in IM$  and hence  $R_Iz$  is not a pure submodule of  $M$ . ■

Let  $\eta$  be an endomorphism of  $M$ . For convenience, we assume that  $\eta$  satisfies

$$\begin{aligned} \eta(Rx_1 \oplus \dots \oplus Rx_h \oplus Ry_1 \oplus \dots \oplus Ry_k) &\leq \\ &\leq Rx_1 \oplus \dots \oplus Rx_h \oplus Ry_1 \oplus \dots \oplus Ry_k. \end{aligned}$$

This is no loss of generality, for there always exist an  $r \in P \setminus I$  such that  $r\eta$  is of this form, and then we replace  $\eta$  by  $r\eta$ . Under this assumption, we can write

$$\eta x_i = \sum_{m=1}^h a_{im} x_m + \sum_{p=1}^k b_{ip} y_p \quad (1 \leq i \leq h),$$

$$\eta y_j = \sum_{m=1}^h c_{jm} x_m + \sum_{p=1}^k d_{jp} y_p \quad (1 \leq j \leq k),$$

where  $T = \{a_{im}, b_{ip}, c_{jm}, d_{jp} \mid 1 \leq i, m \leq h; 1 \leq j, p \leq k\} \subset R$ . For every  $r \in P \setminus I$ , the image of  $z_j^r$  under the endomorphism  $\eta$  can be expressed in terms of the maximal linearly independent set of  $M$ :

$$r\eta(z_j^r) = \eta(rz_j^r) = \eta\left(y_j + \sum_{i=1}^h u_{ij}^r x_i\right).$$

Therefore, we obtain

$$\begin{aligned} \sum_{m=1}^h x_m \left( c_{jm} + \sum_{i=1}^h a_{im} u_{ij}^r \right) - \\ - \sum_{m=1}^h x_m \left( \sum_{p=1}^k u_{mp}^r \left( d_{jp} + \sum_{i=1}^h b_{ip} u_{ij}^r \right) \right) \in rM \cap B. \end{aligned}$$

Since  $B = Rx_1 \oplus \dots \oplus Rx_h$  is pure in  $M$ ,  $rM \cap B = Rrx_1 \oplus \dots \oplus Rrx_h$ . This observation yields an inclusion in  $Rrx_m$  of the form:

$$\left( c_{jm} + \sum_{i=1}^h a_{im} u_{ij}^r \right) x_m - \left( \sum_{p=1}^k d_{jp} u_{mp}^r \right) x_m - \sum_{p=1}^k u_{mp}^r \left( \sum_{i=1}^h b_{ip} u_{ij}^r \right) x_m \in Rrx_m.$$

Thus, the coefficient of  $x_m$  must satisfy:

$$c_{jm} + \sum_{i=1}^h a_{im} u_{ij}^r - \sum_{p=1}^k d_{jp} u_{mp}^r - \sum_{p=1}^k u_{mp}^r \left( \sum_{i=1}^h b_{ip} u_{ij}^r \right) \in Rr.$$

For each  $i$  and  $j$ , we take the limit modulo  $IS$  of the Cauchy net  $\{u_{ij}^r + IS \mid r \in P \setminus I\}$  to arrive at the following condition:

$$(4) \quad c_{jm} + \sum_{i=1}^h a_{im} u_{ij} - \sum_{p=1}^k d_{jp} u_{mp} - \sum_{p=1}^k u_{mp} \left( \sum_{i=1}^h b_{ip} u_{ij} \right) \in IS.$$

We are now able to state a very important lemma regarding the endomorphism ring of  $M$ .

LEMMA 9. Let  $C \in \text{Mat}_{h+k}(R_I)$  be given by:

$$C = \begin{bmatrix} a_{11} & \dots & a_{1h} & b_{11} & \dots & b_{1k} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{h1} & \dots & a_{hh} & b_{h1} & \dots & b_{hk} \\ c_{11} & \dots & c_{1h} & d_{11} & \dots & d_{1k} \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{k1} & \dots & c_{kk} & d_{k1} & \dots & d_{kk} \end{bmatrix}.$$

If  $C$  represents an endomorphism of  $M$ , then condition (4) above holds for the entries of  $C$ . Conversely,  $C \in \text{Mat}_{h+k}(R)$  represents an endomorphism of  $M$  exactly if its entries satisfy condition (4).

PROOF. From the above discussion, it follows immediately that condition (4) is necessary and sufficient for the matrix  $C \in \text{Mat}_{h+k}(R)$  to represent an endomorphism of  $M$ . Suppose  $C \in \text{Mat}_{h+k}(R_I) \setminus \text{Mat}_{h+k}(R)$ . There exists  $r \in P \setminus I$  such that the matrix  $(rE)C = rC$ , where  $E$  is the identity matrix of  $\text{Mat}_{h+k}(R_I)$ , has all of its entries in  $R$ . If  $C$  represents an endomorphism of  $M$ , then so does  $rC$  and hence condition (4) holds for the entries of  $rC$ . Using the fact that  $I$  is a prime ideal of  $R_I$ , it follows that condition (4) also must hold for the entries of  $C$ . ■

To characterize the endomorphism ring of  $M$ , we now assume that  $\{u_{ij} \mid 1 \leq i \leq h; 1 \leq p \leq k\}$  is quadratically  $u$ -independent over  $I$ . Noting that for each  $m$  and  $j$ , condition (4) yields a quadratic equation modulo  $IS$  in the unknowns  $\{u_{ip}\}$  with coefficients from  $R$ , we must have

- a)  $c_{jm} \in I$ ,
- b)  $a_{mm} - d_{jj} \in I$ ,
- c)  $b_{ip} \in I \quad (1 \leq i \leq h; 1 \leq p \leq k)$ ,
- d)  $a_{im} \in I \quad (i \neq m, 1 \leq i \leq h)$ ,
- e)  $d_{jp} \in I \quad (p \neq j, 1 \leq p \leq k)$ .

As the above inclusions must hold for every  $m$  hold for every  $j$ , we can further conclude (after collecting like terms) that

$$T \setminus \{a_{11}, \dots, a_{ii}, \dots, a_{hh}, d_{11}, \dots, d_{jj}, \dots, d_{kk}\} \subset I,$$

and

$$a_{mm} - d_{jj} \in I \quad \text{for all } m \text{ and } j.$$

We have the following lemma.

LEMMA 10. Let  $M$  be as above and  $\{u_{ij}\}$  quadratically  $u$ -independent over  $I$ . The endomorphism ring of  $M$  is a subring of  $\text{Mat}_{h+k}(R)$  and every  $C \in \text{Mat}_{h+k}(R)$  representing an endomorphism of  $M$  must be of the form  $C = rE + D$ , where  $r \in R$ ,  $E$  is the identity matrix of  $\text{Mat}_{h+k}(R)$ , and  $D \in \text{Mat}_{h+k}(I)$ .

PROOF. Let  $\eta \in \text{End}_R M$  have representative matrix  $C \in \text{Mat}_{h+k}(R_I)$ . The above remarks imply that  $C = rE + D$  for some  $r \in R_I$  and  $D \in \text{Mat}_{h+k}(I)$ . We claim that  $r$  must in fact be an element of  $R$ ; for if not, then—since  $\eta(x_1) - rx_1 \in IM$  implies that  $rx_1 \in M$ —we would have  $rx_1 \in M$  with  $r \in R_I \setminus R$ , which is impossible. Thus,  $C$  is in fact of the desired form. ■

Under the assumption that  $\{u_{ij} \mid 1 \leq i \leq h; 1 \leq p \leq k\}$  is quadratically  $u$ -independent over  $I$ , the above lemmas combine to give the following corollaries.

COROLLARY 2. An endomorphism  $\eta$  of  $M$  is an automorphism if and only if its representative matrix  $C$  is invertible in  $\text{Mat}_{h+k}(R)$ .

PROOF. Clearly, an automorphism of  $M$  is represented by an invertible matrix. Moreover, for an invertible matrix  $C = rE + D$  ( $r, E, D$  as in the statement of Lemma 10), condition (4) holds or fails simultaneously for the entries of  $C$  and for the entries of its inverse  $C^{-1}$ . This observation completes the proof. ■

COROLLARY 3. The endomorphism ring of  $M$  is a local ring.

PROOF. A matrix  $C = rE + D$  ( $r, E, D$  as above) is invertible in  $\text{Mat}_{h+k}(R)$  exactly when  $r \in U(R)$ . So if  $\eta_1, \eta_2 \in \text{End}_R M$  are non-units, then each is represented by a matrix having its entries in the maximal

ideal  $P$  (where, of course, the elements off the diagonal are from  $I$ ); hence, the sum  $\eta_1 + \eta_2$  also is represented by a matrix of this form and thus also a non-unit. Therefore,  $M$  has local endomorphism ring. ■

We now state the main result of this section.

**THEOREM 2.** Let  $R$  be a valuation domain and  $I$  a prime ideal of  $R$  different from  $P$  such that  $(R, R_I, I)$  is a compatible triple of submodules of  $Q$ . For a pair of positive integers  $m < n$ , if there exists a set  $\mathcal{U}$  of units of  $S \setminus R$  which is of cardinality  $m(n - m)$  and is quadratically  $u$ -independent over  $I$ , then there exists a torsion-free  $R$ -module  $M$  with local endomorphism ring (and hence indecomposable) such that  $\epsilon(M) = n$  and  $\epsilon(B) = m$ .

**PROOF.** Take  $h = m$  and  $k = n - m$ . Using the set  $\mathcal{U}$ , form the torsion-free  $R$ -module  $M_{h,k}$ . Then  $M_{h,k}$  has rank  $h + k = n$ , and by Lemma 8, the basic submodules of  $M_{h,k}$  are of rank  $m$ ;  $M_{h,k}$  has local endomorphism ring by Corollary 3. ■

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