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Even canonical surfaces with small K^2 - II

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Even Canonical Surfaces with Small K^2 - II.

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Introduction.

This is the second part of a study of even canonical surfaces which we began in [12] (referred to as Part I). Let S be a canonical surface, and let $\Phi_K: S \to P^{p_g - 1}$ denote the canonical map. We put $X = \Phi_K(S)$ and call it the canonical image. In Part I, we considered even canonical surfaces S with $K^2 < 4\chi(\mathcal{O}_S) - 16$ and showed that X cannot be cut out by hyperquadrics. More precisely, the irreducible component of the quadric hull $Q(X)$ of X containing it is of dimension 3, answering affirmatively to a conjecture of Reid [14, p. 541] in the case of regular even surfaces.

The purpose of this part is to list up even canonical surfaces with $K^2 = 4\chi(\mathcal{O}_S) - 16$ whose canonical image is cut out by hyperquadrics. Hence, we need not worry about surfaces with a pencil of trigonal curves or plane quintic curves by [Part I, Theorem 8.3]. By the nature of the problem, we study the semi-canonical ring and write down explicitly the defining equation of a birational model.

We show that, when $q > 0$, S has a pencil of trigonal curves and the irreducible component of $Q(X)$ containing X is of dimension 3. This in particular implies that Reid's conjecture is also true for even surfaces with $q = 1$. When $q = 0$, we find the following types of surfaces whose canonical image is cut out by hyperquadrics:

— Some weighted complete intersections (Theorems 3.1, 4.1, 8.1).

— Hyperquartic sections of normal Del Pezzo threefolds (Theorem 3.1).

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— Surfaces with a pencil of non-hyperelliptic curves of genus 7 whose general member has a g_6^2 (Theorem 5.1).

— Bi-K3 surfaces, i.e., double coverings of K3 surfaces (Theorem 7.3).

— Surfaces with a pencil of non-hyperelliptic, non-trigonal curves of genus 5 (Theorem 8.1).

Among them, the last two are the majority in the sense that p_g is unbounded. Another point to be noticed is that the canonical image is projectively normal.

We hope that our experiments in this series of works give an evidence for the validity of Reid's conjecture [14] and illustrate what happens on Reid's line $K^2 = 4p_g - 12$. The author would like to thank Professor T. Ashikaga for stimulating discussions.

1, Classification by the semi-canonical map.

Let S be a nonsingular projective surface defined over the complex number field $\boldsymbol{C}$. It is called an *even surface* if the second Stiefel-Whitney class vanishes [9]. It is called a *canonical surface* if it is minimal and the rational map associated with the canonical linear system $|K|$ induces a birational map of S onto the image [9]. Throughout the paper, we denote by S an even canonical surface with $K^2 = 4\chi(\mathcal{O}_S) - 16$. Let L be a semi-canonical bundle, i.e., a line bundle on S satisfying $2L = K$. We call the rational map Φ_L associated with $|L|$ the semi-canonical map of S. Put $n = h^0(L) - 1$.

PROPOSITION 1.1. *Let S be an even canonical surface with $K^2 = = 4\chi(\mathcal{O}_S - 16$. Then $L^2 \leqslant 4n - 4$ and the semi-canonical map $\Phi_L: S \to \boldsymbol{P}^n$ satisfies one of the following*:

(I) *Φ_L induces a birational map of S onto its image.*

(II) *Φ_L induces a holomorphic map of degree 2 onto a surface of degree $2n-2$ in $\boldsymbol{P}^n$ which is not birationally equivalent to a ruled surface.*

(III) *Φ_L induces a rational map of degree 3 onto a ruled surface.*

(IV) *Φ_L induces a holomorphic map of degree 4 onto a surface of degree $n-1$ in $\boldsymbol{P}^n$.*

(V) *Φ_L is composed of a nonhyperelliptic pencil of genus 3 or 4.*

Furthermore, $L^2 = 4n - 4$ *and* $H^1(L) = 0$ *hold when* S *is of type* (I), (II) *or* (IV). *When* S *is of type* (II) *or* (IV), $|L|$ *is free from base points.*

PROOF. Since S is an even surface, L^2 is a positive even integer. Hence, there exists an integer k which satisfies $L^2 = 4n - 2k$. By the Riemann-Roch theorem, we have

$$2h^0(L) - h^1(L) = -L^2/2 + \chi(\mathcal{O}_S). \tag{1.1}$$

Since $4L^2 = K^2 = 4\chi - 16$, we have $\chi = 4n - 2k + 4$. It follows from (1.1) that $k = h^1(L) + 2 \geqslant 2$. Hence we have $L^2 \leqslant 4n - 4$. Notice that we have $h^1(L) = 0$ if $L^2 = 4n - 4$.

First, assume that Φ_L is not composed of a pencil. Put $V = \Phi_L(S)$ and consider Φ_L as a rational map of S onto V. Since V is a nondegenerate surface in $\boldsymbol{P}^n$, we have $\deg V \geqslant n - 1$ and

$$L^2 \geqslant (\deg \Phi_L)(\deg V) \geqslant (n-1)\deg \Phi_L .$$

Since $L^2 \leqslant 4n - 4$, we get $\deg \Phi_L \leqslant 4$. When $\deg \Phi_L = 1$, we have $L^2 \geqslant$ $\geqslant 4n - 6$ by [Part I, Lemma 2.1]. If $L^2 = 4n - 6$, then, as we showed in [Part I, Lemma 2.3], the numerical characters of S must satisfy $K^2 =$ $= 4p_g - 16$ and $q = 0$, which is absurd. Hence $L^2 = 4n - 4$. When $\deg \Phi_L = 2$, we have $\deg V \leqslant 2n - 2$. If $\deg V < 2n - 2$, it is well-known that V is birationally equivalent to a ruled surface (see, e.g. [1]), and it follows that S has a pencil of hyperelliptic curves, contradicting that S is a canonical surface. Hence $\deg V = 2n - 2$ and V is not a ruled surface. Since we have $L^2 = 2\deg V$, $|L|$ is free from base points. When $\deg \Phi_L = 3$, we have $\deg V < 2n - 2$. Therefore, V is birationally equivalent to a ruled surface. When $\deg \Phi_L = 4$, we have $\deg V = n - 1$ and $L^2 = 4n - 4$. Since $L^2 = 4\deg V$, $|L|$ is free from base points.

Next, assume that Φ_L is composed of a pencil. Then there are an irreducible pencil $\{D\}$ and an effective divisor Z such that L is numerically equivalent to $mD + Z$, where m is an integer satisfying $m \geqslant n$. Then $4n - 4 \geqslant L^2 = mLD + LZ \geqslant nLD$. It follows that $LD \leqslant 3$, and we have $3 \geqslant LD = mD^2 + DZ \geqslant nD^2$. Since S is an even surface, D^2 is nonnegative even integer. Since $n \geqslant 2$, we get $D^2 = 0$. Therefore, $\{D\}$ is a pencil of curves of genus at most 4 without base points. Since S is a canonical surface, $\{D\}$ must be of nonhyperelliptic type. Q.E.D.

We call a pencil on a surface *Petri special,* if a general member is trigonal or plane quintic. Otherwise, it is said to be Petri general. For any nondegenerate variety W in $\boldsymbol{P}^r$, we denote by $Q(W)$ the intersection of all hyperquadrics through W and call it the quadric hull of W.

The following is a special case of [Part I, Theorem 8.3].

LEMMA 1.2. *Let S be as above and X its canonical image. If S has a Petri special pencil, then the irreducible component of $Q(X)$ containing X is of dimension* 3. *In particular, if S is a surface of type* (III) *or* (V), *then X is not cut out by hyperquadrics.*

Since we are interested only in a surface whose canonical inage is cut out by hyperquadrics, we can exclude surfaces of type (III) or (V) from our consideration. Then it follows from Proposition 1.1 that $\chi = 4n$ and $L^2 = 4n - 4$. Furthermore, we can assume that S has no Petri special pencils in what follows.

2. Quadric hull of a semi-canonical surface.

From this section up to Sect. 6, we let S be a surface of type (I) in the sense of Proposition 1.1.

LEMMA 2.1. *Let S be a surface of type* (I). *Then either*

(1) $|L|$ *is free from base points, or*

(2) $|L|$ *has a unique transversal base point.*

PROOF. Let $\sigma\colon \widetilde{S} \to S$ be a composite of blowing-ups such that the variable part $|M|$ of $|\sigma^* L|$ is free from base points. We can assume that σ is the shortest among those with such a property. Since $L^2 = 4n - 4$, we have $M^2 \geqslant 4n - 5$ by [Part I, Lemma 2.1]. If $M^2 = 4n - 4$, then $|L|$ is free from base points. If $M^2 = 4n - 5$, then σ is a simple blowing-up and $|L|$ has a unique base point. Q.E.D.

LEMMA 2.2. *Every surface of type* (I) *is regular.*

PROOF. Let $\sigma\colon \widetilde{S} \to S$ and M be as in the proof of Lemma 2.1. Let C be a general member of $|M|$, and consider the cohomology long exact sequence for

$$0 \to \mathcal{O} \to \mathcal{O}(M) \to \mathcal{O}_C(M) \to 0\,.$$

From this, we get $h^0(M|_C) \geqslant n$. On the other hand, we have $M^2 \geqslant 4h^0(M|_C) - 6$ by the proof of [Part I, Lemma 2.1]. Since $M^2 = 4n - 4$ or $4n - 5$, we have $h^0(M|_C) = n$. In particular, the restriction map $H^0(M) \to H^0(M|_C)$ is surjective. Hence $q(S) = h^1(\mathcal{O}_S) \leqslant h^1(M)$.

If $|L|$ is free from base points, then $M = L$ and we have $h^1(L) = 0$ by Proposition 1.1. Thus $q = 0$. If $|L|$ has a base point, we have $h^0(M) - h^1(M) + h^0(M + 3E) = 2n + 1$ by the Riemann-Roch theorem, where E is the exceptional (-1)-curve. Since $\sigma^* L = [M + E]$, we get $h^0(M + 3E) = n + 1$ from

$$0 \to \mathcal{O}(M + iE) \to \mathcal{O}(M + (i + 1)E) \to \mathcal{O}_E(-i) \to 0$$

for $1 \leqslant i \leqslant 2$. Hence we get $h^1(M) = 1$ and $q \leqslant 1$.

Assume that $q = 1$. Then we have $h^1(E) = 1$. From the cohomology long exact sequence for

$$0 \to \mathcal{O}(E) \to \mathcal{O}(\sigma^* L) \to \mathcal{O}_C(\sigma^* L) \to 0\,,$$

we get $h^0(\sigma^* L|_C) = n + 1$. Since $h^0(M|_C) = n$, this implies that $|\sigma^* L|_C|$ is free from base points. Hence it induces a birational holomorphic map of C onto its image which is of degree $4n - 4$ in $\boldsymbol{P}^n$. Then Castelnuovo's bound [7] implies $g(C) \leqslant 6n - 9$. This is absurd, since $g(C) = 6n - 5$. Q.E.D.

For any nondegenerate subvariety $W \subset \boldsymbol{P}^r$, the Hilbert function h_W of W is defined by

$$h_W(m) = \operatorname{rank}\{H^0(\boldsymbol{P}^r, \mathcal{O}(m)) \to H^0(W, \mathcal{O}_W(m))\}\,, \qquad m \in \boldsymbol{N}\,.$$

For the properties, consult [7].

LEMMA 2.3. *Let S be a surface of type* (I). *Then the quadric hull $Q(V)$ of the semi-canonical image V is an irreducible 3-fold of degree $n - 2$ or $n - 1$ unless $n = 4$ and $Q(V) = \boldsymbol{P}^4$.*

PROOF. Let M be as in the proof of Lemma 2.1. We denote by C a general member of $|M|$. Then it is an irreducible nonsingular curve of genus $6n - 5$. The image $C_0 = \Phi_M(C)$ is considered as a general hyperplane section of V. We denote by $Z_0 \subset \boldsymbol{P}^{n-2}$ a set of points obtained by cutting C_0 by a general hyperplane. Then it consists of M^2 distinct points in uniform position.

Assume first that $|L|$ is free from base points. The canonical bundle of C is induced by $3L$ and we have $h^0(L|_C) = n$. Then $h^0(2L|_C) = 3n - 2$ by the Riemann-Roch theorem. We have

$$3n - 2 \geqslant h_{C_0}(2) \geqslant h_{C_0}(1) + h_{Z_0}(2) = n + h_{Z_0}(2)\,.$$

Hence we get $h_{Z_0}(2) = 2n - 3$ or $2n - 2$. When $h_{Z_0}(2) = 2n - 3$, $Q(Z_0)$ is a rational normal curve by Castelnuovo's lemma (see, e.g. [7]). Since V, C_0 and Z_0 are linearly normal, $Q(V)$ is an irreducible 3-fold of degree

$n-2$. When $h_{Z_0}(2)=2n-2$ and $n \geqslant 5$, $Q(Z_0)$ is an elliptic normal curve by a result of Harris-Eisenbud [7]. Hence, if $n \geqslant 5$, $Q(V)$ is an irreducible 3-fold of degree $n-1$. When $n=4$, we have $Q(V)=\boldsymbol{P}^4$.

We next assume that $|L|$ has a base point. Then the canonical bundle of C is induced by $3M+3E$, where E denotes the exceptional (-1)-curve for $\sigma\colon \widetilde{S}\to S$. Since $h^0(2M|_C) \leqslant h^0(2\sigma^*L|_C)=3n-2$, we have $h_{Z_0}(2)=2n-3$ or $2n-2$ as in the previous case. We show that $h_{Z_0}(2)=2n-3$. Assume that $h_{Z_0}(2)=2n-2$. Then we have $h^0(2M|_C)=h_{C_0}(2)=3n-2$. Since C is nonhyperelliptic, we have $h^0(2E|_C)=1$. Hence we get $h^0((3M+E)|_C)=6n-7$ by the Riemann-Roch theorem. Since $h^0(3M|_C) \leqslant h^0((3M+E)|_C)$ and since

$$h^0(3M|_C) \geqslant h_{C_0}(3) \geqslant h_{C_0}(2)+h_{Z_0}(3),$$

we get $h_{Z_0}(3) \leqslant 3n-5$. On the other hand, we have $h_{Z_0}(3) \geqslant n-2+ h_{Z_0}(2)=3n-4$, which is absurd. Hence $h_{Z_0}(2)=2n-3$, and we see that $Q(V)$ is an irreducible 3-fold of degree $n-2$ in $\boldsymbol{P}^n$ as in the previous case. Q.E.D.

For a further study of surfaces of type (I), we recall an explicit description of an irreducible, nondegenerate threefold of degree $n-2$, $n-1$ in $\boldsymbol{P}^n$.

LEMMA 2.4. (See, e.g. [3]). *An irreducible nondegenerate 3-fold of degree $n-2$ in $\boldsymbol{P}^n$ is one of the following varieties*:

(1) $\boldsymbol{P}^3$ $(n=3)$.

(2) *A hyperquadric in* $\boldsymbol{P}^4$ $(n=4)$.

(3) *A cone over $\boldsymbol{P}^2$ embedded into $\boldsymbol{P}^5$ by the holomorphic map associated with $|\mathcal{O}_{\boldsymbol{P}^2}(2)|$, i.e., the weighted projective space $\boldsymbol{P}(1,1,1,2)$* $(n=6)$.

(4) *A rational normal scroll, that is, the image of the total space of the $\boldsymbol{P}^2$-bundle*

$$\pi\colon \boldsymbol{P}_{a,b,c}=\boldsymbol{P}(\mathcal{O}(a)\oplus\mathcal{O}(b)\oplus\mathcal{O}(c))\to\boldsymbol{P}^1,$$

by the holomorphic map associated with $|T|$, where T denotes a tautological divisor on $\boldsymbol{P}_{a,b,c}$ and a, b, c are integers satisfying

$$0\leqslant a\leqslant b\leqslant c, \qquad a+b+c=n-2 \qquad (n\geqslant 5).$$

To state the next result, we use the following notation: We denote by Σ_d the Hirzebruch surface of degree d. Let Δ_0 and Γ be a section with $\Delta_0^2=-d$ and a fibre of the projection $\Sigma_d\to\boldsymbol{P}^1$, respectively.

LEMMA 2.5 ([4], [5], [6]). *An irreducible nondegenerate threefold W of degree $n-1$ in $\boldsymbol{P}^n$ is one of the following varieties:*

(1) *A hypercubic* ($n = 4$).

(2) *A complete intersection of two hyperquadrics* ($n = 5$).

(3) *A cone over a surface Σ of degree $n-1$ in $\boldsymbol{P}^{n-1}$, where Σ is one of the following* (*see, e.g.* [2], [13]):

(3*a*) *The Veronese embedding into $\boldsymbol{P}^8$ of a quadric in $\boldsymbol{P}^3$* ($n = 9$).

(3*b*) *The image of $\boldsymbol{P}^2$ by the rational map associated with the linear system $|3l - \sum_{i=1}^{k} x_i|$, where l is a line on $\boldsymbol{P}^2$ and the x_i are points on $\boldsymbol{P}^2$ which are possibly infinitely near* ($n = 10 - k$, $0 \leqslant k \leqslant 6$).

(3*c*) *A cone over a nonsingular elliptic curve.*

(3*d*) *A projection of a surface of degree $n-1$ in $\boldsymbol{P}^n$ from a point.*

(4) *A non-conic normal Del Pezzo threefold* ($6 \leqslant n \leqslant 9$).

(4*a*.1) *Let $Gr(2,5)$ be the Grassmannian of two planes in $\boldsymbol{C}^5$ embedded into $\boldsymbol{P}^9$ by the Plücker coordinates. Then W is a nonsingular threefold obtained by cutting $Gr(2,5)$ three times by hyperplanes* ($n = 6$).

(4*a*.2) *Consider the $\boldsymbol{P}^2$-bundle ϖ: $\boldsymbol{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(\Delta_0 + 2\Gamma)) \to \Sigma_1$, and let $\widetilde{T}$ denote a tautological divisor. Then W is the image of a member of $|\widetilde{T} + \varpi^*(\Delta_0 + \Gamma)|$ under the holomorphic map associated with $|\widetilde{T}|$* ($n = 6$).

(4*a*.3) *Consider the $\boldsymbol{P}^1$-bundle ϖ: $\boldsymbol{P}(\mathcal{O} \oplus \mathcal{O}(T)) \to \boldsymbol{P}_{1,1,1}$, and let $\widetilde{T}$ denote a tautological divisor. Then W_0 is the image of a member of $|\widetilde{T} + \varpi^*(T - F)|$ under the holomorphic map associated with $|\widetilde{T}|$, where F is a fiber of $\boldsymbol{P}_{1,1,1} \to \boldsymbol{P}^1$* ($n = 6$).

(4*b*.1) *$\boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$ embedded by $|H_1 + H_2 + H_3|$, where H_i is the pull-back of a point of the i-th factor* ($n = 7$).

(4*b*.2) *$\boldsymbol{P}(\Theta_{\boldsymbol{P}^2})$ embedded by $|H|$, where $\Theta_{\boldsymbol{P}^2}$ denotes the tangent sheaf of $\boldsymbol{P}^2$ and H is a tautological divisor on $\boldsymbol{P}(\Theta_{\boldsymbol{P}^2})$*($n = 7$).

(4*b*.3) *Consider the $\boldsymbol{P}^2$-bundle $\boldsymbol{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2))$ over $\boldsymbol{P}^2$, and let $\widetilde{T}$ and $\widetilde{F}$ denote a tautological divisor and the pull-back of a line in $\boldsymbol{P}^2$, respectively. Then W is the image of a member of $|\widetilde{T} + \widetilde{F}|$ under the holomorphic map associated with $|\widetilde{T}|$*($n = 7$).

(4b.4) *Consider the* $\boldsymbol{P}^2$*-bundle* ϖ: $\boldsymbol{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(\Delta_0 + 3\Gamma)) \to \Sigma_2$, *and let* $\tilde{T}$ *denote a tautological divisor. Then* W *is the image of a non-singular member of* $|\tilde{T} + \varpi^*(\Delta_0 + \Gamma)|$ *under the holomorphic map associated with* $|\tilde{T}|$ $(n = 7)$.

(4b.5) *Consider the* $\boldsymbol{P}^1$*-bundle* ϖ: $\boldsymbol{P}(\mathcal{O} \oplus \mathcal{O}(T)) \to \boldsymbol{P}_{1,1,2}$, *and let* $\tilde{T}$ *denote a tautological divisor. Then* W *is the image of a member of* $|\tilde{T} + \varpi^*(T - 2F)|$ *under the holomorphic map associated with* $|\tilde{T}|$ $(n = 7)$.

(4b.6) *Consider the* $\boldsymbol{P}^2$*-bundle* ϖ: $\boldsymbol{P}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(\Delta_0 + 2\Gamma)) \to \Sigma_0$, *and let* $\tilde{T}$ *denote a tautological divisor. Then* W *is the image of a non-singular member of* $|\tilde{T} + \varpi^* \Delta_0|$ *under the holomorphic map associated with* $|\tilde{T}|$ $(n = 7)$.

(4c) W *is* $\boldsymbol{P}^3$ *blown up at one point* x. *If we denote by* H *and* E *the pull-back of a plane in* $\boldsymbol{P}^3$ *and the inverse image of* x, *respectively, then* W *is embedded by* $|2H - E|$ $(n = 8)$.

(4d) $\boldsymbol{P}^3$ *embedded by* $|\mathcal{O}(2)|$ $(n = 9)$.

(5) *A projection of a threefold of degree* $n - 1$ *in* $\boldsymbol{P}^{n+1}$ *from a point.*

DEFINITION 2.6. We divide surfaces of type (I) into the following three classes:

(Ia) $Q(V)$ is a threefold of degree $n - 1$ $(n \geqslant 5)$, or $Q(V) = \boldsymbol{P}^4$ $(n = 4)$.

(Ib) $Q(V)$ is a threefold of degree $n - 2$, and $|L|$ has no base points.

(Ic) $Q(V)$ is a threefold of degree $n - 2$, and $|L|$ has one base point.

We shall study each type separately in the following sections.

3. Surfaces of type (Ia).

In this section, we consider surfaces of type (Ia) and show the following:

THEOREM 3.1. *Let* S *be a surface of type* (Ia). *Then the semi-canonical image* V *is isomorphic to the canonical model. When* $n = 4$, V *is a complete intersection of a hypercubic and a hyperquartic. When*

$n = 5$, V is a complete intersection of two hyperquadrics and a hyperquartic. When $n \geqslant 6$, V can be obtained as a hyperquartic section of a normal Del Pezzo threefold $Q(V)$. In particular, $4 \leqslant n \leqslant 10$. Furthermore, the canonival image is cut out by hyperquadrics.

LEMMA 3.2. *Let S be a surface of type* (Ia) *with $n = 4$. Then the semi-caconical image V is a complete intersection of a hypercubic and a hyperquartic, and it is isomorphic to the canonical model of S.*

PROOF. Let x_i, $0 \leqslant i \leqslant 4$,be a basis for $H^0(L)$. Then, by the assumption, the 15 products $x_i x_j$ are linearly independent. Since $h^0(2L) = p_g = 15$, they form a basis for $H^0(2L)$. The products $x_i x_j x_k$ give 35 elements in $H^0(3L)$. On the other hand, we have $h^0(3L) = 34$ by the Riemann-Roch theorem and Ramanujam's vanishing theorem. Hence we have a cubic relation $A_3 = 0$ in the x_i. The products $x_i x_j x_k x_l$ give 65 elements in $H^0(4L)$ modulo $A_3 = 0$. Since $h^0(4L) = h^0(2K) = = 64$, we have a quartic relation $A_4 = 0$ in the x_i. It is easy to see that the semi-canonical ring $\bigoplus_{m \geqslant 0} H^0(mL)$ is generated by the x_i and that there are no further relations. Since $K = 2L$, V is isomorphic to the canonical model. Q.E.D.

LEMMA 3.3. *Let S be a surface of type* (Ia) *with $n \geqslant 5$. Then V is projectively normal and is isomorphic to the canonical model.*

PROOF. This follows from a similar observation as in [10, § 4]. We have $h^0(L) = n + 1$ and $h^0(2L) = p_g = 4n - 1$. By the Riemann-Roch theorem and Ramanujam's vanishing theorem, we have

$$h^0(mL) = 2m(m-2)(n-1) + 4n \tag{3.1}$$

for $m \geqslant 3$. Since $Q(Z_0)$ is an elliptic normal curve, a calculation shows that

$$h^0(mL_C) = h_{C_0}(m) = h_{C_0}(m-1) + h_{Z_0}(m),$$

$$h^0(mL) = h_V(m) = h_V(m-1) + h_C(m)$$

hold for any $m \geqslant 1$. It follows that V is projectively normal and, thus, the semi-canonical ring is generated in degree 1. Since $K = 2L$, we see that V is isomorphic to the canonical model. Q.E.D.

LEMMA 3.4. *Let S be as in Lemma 3.3. When $n \geqslant 6$, $Q(V)$ is a normal Del Pezzo threefold. In particular, $Q(V)$ is projectively normal.*

PROOF. It suffices to show that (3c), (3d) and (5) in Lemma 2.5 are inadequate. Since S is regular, $Q(V)$ cannot be a scroll over an elliptic curve. Furthermore, as in [10, § 4, Claim], we can show that $Q(V)$ can be neither a projection of a 3-fold of degree $n-1$ in $\boldsymbol{P}^{n+1}$ nor a cone over a projection of a surface of degree $n-1$ in $\boldsymbol{P}^n$. Hence $Q(V)$ is a normal Del Pezzo 3-fold when $n \geqslant 6$. It is projectively normal by [5]. Q.E.D.

We complete the proof of Theorem 3.1 with the following:

LEMMA 3.5. *Let S be as in Lemma* 3.3. *Then V is a hyperquartic section of $Q(V)$. Hence the canonical image of S is cut out by hyperquadrics.*

PROOF. Recall that V and $Q(V)$ are both projectively normal. Since we have $\deg Z_0 = 4n-4$, $h_{Z_0}(4) = 4n-5$ and $h_{Q(Z_0)}(4) = 4n-4$, we see that V is a hyperquartic section of $Q(V)$. Since the canonical image X is the Veronese transform of V, it is a hyperquadric section of the Veronese transform of $Q(V)$. Since the homogeneous ideal of $Q(V)$ is generated in degree 2, it follows that X is cut out by hyperquadrics and hence $X = Q(X)$. Q.E.D.

Conversely, we check the existence of surfaces of type (Ia). When $Q(V)$ is a non-conic Del Pezzo threefold, this is straightforward by using a description of $Q(V)$ in Lemma 2.5, (4): one can check that a generic hyperquartic section of a non-conic normal Del Pezzo 3- fold has at most rational double points, and that its minimal resolution is an even canonical surface with $K^2 = 4p_g - 12$, $q = 0$. In particular, we have an octic surface when $Q(V)$ is the Veronese transform of $\boldsymbol{P}^3$.

In the rest of the section, we consider (3a) and (3b) in Lemma 2.5. Let v be the vertex of $Q(V)$. We denote by Λ_0 the pull-back to S by Φ_L of the linear system of hyperplanes through v. We let G be the fixed part of Λ_0 and put $\Lambda = \Lambda_0 - G$. Then Λ defines a rational map μ: $S \to \Sigma$ and we have $L = [H + G]$, $H \in \Lambda$. Since $|L|$ has no base points, we have $LG = 0$. Then $4n - 4 = L^2 = LH = H^2 + HG \geqslant H^2 \geqslant (\deg \mu)(\deg \Sigma) = (\deg \mu)(n-1)$, and it follows $\deg \mu \leqslant 4$. Since S is a canonical surface, $\deg \mu$ is not less than 3. If $\deg \mu = 3$, then S would have a pencil of trigonal curves. Since the canonical image is cut out by hyperquadrics by Lemma 3.5, this contradicts Lemma 1.2. Therefore, we have $\deg \mu = 4$, $HG = 0$ and $H^2 = 4n - 4$. Since $HG = 0$ and $G^2 = 0$, we get $G = 0$ by Hodge's index theorem. We remark that μ is holomorphic. In fact, if Λ has a base point P, blowing S up at P and considering the strict transform $\widetilde{\Lambda}$ of Λ, we would have $\widetilde{H}^2 < H^2$ for $\widetilde{H} \in \widetilde{\Lambda}$ which implies that the map induced by $\widetilde{\Lambda}$ is of degree less than 4. This is impossible by Lemmas 1.2 and 3.5.

LEMMA 3.6. *Let S be a surface of type* (Ia) *with $n = 9$ such that $Q(V)$ is a cone over the Veronese transform of a quadric surface as in Lemma* 2.5, (3a). *Then the canonical model of S is a weighted complete intersection of type* (2, 8) *in the weighted projective space $\boldsymbol{P}(1,1,1,1,2)$ defined by*

$$A_2 = u^4 + B_2 u^3 + B_4 u^2 + B_6 u + B_8 = 0\,, \tag{3.2}$$

where (x_0, x_1, x_2, x_3, u) is a system of coordinates with $\deg x_i = 1$, $\deg u = 2$, *and the A_j, B_j are homogeneous forms of degree j in the x_i.*

PROOF. Since Σ is the Veronese transform of a quadric surface, we can find a divisor L_0 on S such that $L = [H] = 2[L_0]$ and L_0 induces a holomorphic map of degree 4 onto the quadric surface. Let (x_0, x_1, x_2, x_3) be a basis for $H^0(L_0)$. We have a quadric relation $A_2 = 0$ among them. Modulo $A_2 = 0$, the products $x_i x_j$ give us 9 elements. On the other hand, since $h^0(2L_0) = h^0(L) = 10$, we have a new element $\xi \in H^0(L)$. We look at $H^0(8L_0) = H^0(2K)$ which is of dimension 164. Here we have the following elements:

octics in the x_i	(sextics in the $x_i)\xi$,
(quartics in the $x_i)\xi^2$	(quadratics in the $x_i)\xi^3$.

Modulo $A_2 = 0$, these present 164 elements which are clearly linearly independent. Hence ξ^4 can be expressed as a linear combination of them, and we get

$$\xi^4 + B_2 \xi^3 + B_4 \xi^2 + B_6 \xi + B_8 = 0\,,$$

where the B_i are homogeneous forms of degree i in the x_j. Putting $u = \xi$, we get a holomorphic map of S into $\boldsymbol{P}(1,1,1,1,2)$ whose image is defined by (3.2). Conversely, if the coefficients are sufficiently general, (3.2) defines a nonsingular surface whose canonical bundle is induced by $\mathcal{O}(4)$ Hence it is an even canonical surface with the desired numerical characters. Q.E.D.

REMARK 3.7. Let t be a complex parameter ranging in a neighbourhood of the origin. Replacing $A_2 = 0$ by $tu - A_2 = 0$ in (3.2), we get a family of deformations of surfaces of type (Ia). When $t \neq 0$, we get an octic surface by substituting $u = A_2/t$ to the second equation of (3.2). Therefore, a surface in Lemma 3.6 is a specialization of octic surfaces.

LEMMA 3.8. *Let S be a surface of type* (Ia) *and assume that $Q(V)$ is a cone over a weak Del Pezzo surface Σ as in Lemma* 2.5, (3b). *Let $\tilde{\Sigma}$ denote the minimal resolution of Σ, and let $\tilde{H}$ be the pull-back of a hyperplane section of Σ. Then S is birationally equivalent to a 4-sheeted covering defined in the total space of $[\tilde{H}]$ by*

$$u^4 + A_1 u^3 + A_2 u^2 + A_3 u + A_4 = 0\,, \tag{3.3}$$

where u denotes a fiber coordinate on $[\tilde{H}]$ and the A_i are sections in $H^0(i\tilde{H})$.

PROOF. Let $\tilde{\Sigma}$ be $\boldsymbol{P}^2$ blown up at $k = 10 - n$ points $x_1, \dots, x_k$ and let $\lambda: \tilde{\Sigma} \to \boldsymbol{P}^2$ be the natural map. Let $\tilde{H} = 3\lambda^* l - \sum \lambda^{-1}(x_i)$ be the pull-back of a hyperplane of $\boldsymbol{P}^{n-1}$.

Note that $H^0(\tilde{\Sigma}, \mathcal{O}(m\tilde{H})) \simeq H^0(\Sigma, \mathcal{O}(m))$ holds for any $m > 0$. Since $[-\tilde{H}]$ is the canonical bundle of $\tilde{\Sigma}$, we have

$$h^0(m\tilde{H}) = \frac{1}{2}\, m(m+1)(n-1) + 1 \qquad \text{for } m \in \boldsymbol{N}$$

by the Riemann-Roch theorem and Ramamujam's vanishing theorem. Let $x_0, \dots, x_{n-1}$ and ξ be a basis for $H^0(L)$ such that the x_i span $\mu^* H^0(\Sigma, \mathcal{O}(1))$. Since $h^0(2K) = 20n - 16$, the following $20n - 15$ products

$$\xi^4, \qquad x_i \xi^3, \qquad x_i x_j \xi^2, \qquad x_i x_j x_k \xi, \qquad x_i x_j x_k x_l$$

in $H^0(2K)$ are linearly dependent. Thus S is birationally equivalent to a quadruple covering of Σ defined by

$$\eta^4 + \alpha(z)\eta^3 + \beta(z)\eta^2 + \gamma(z)\eta + \delta(z) = 0$$

where α, β, γ and δ are homogeneous forms of respective degree 1, 2, 3 and 4 in the homogeneous coordinates $(z_0, \dots, z_{n-1})$ of $\boldsymbol{P}^{n-1}$ and $\xi = \Phi_L^* \eta$. This induces via $\tilde{\Sigma} \to \Sigma$ a quadruple covering S^* of $\tilde{\Sigma}$. Then the equation of S^* is as in the statement.

Conversely, if we choose the A_i generic, (3.3) defines a nonsingular surface whose canonical bundle is induced by $2\tilde{H}$, and we get an even canonical surface with the desired numerical characters. Q.E.D.

4. Surfaces of type (Ib): $Q(V)$ is not a scroll.

In this section, the following result will be proven by using several lemmas.

THEOREM 4.1. *Let S be a surface of type (Ib) whose canonical image is cut out by hyperquadrics. Assume further that the quadric hull of the semi-canonical image is either $\boldsymbol{P}^3$, a hyperquadric or a cone over the Veronese surface. Then S is birationally equivalent to one of the following surfaces:*

(1) *A weighted complete intersection of type* (4, 4) *in the weighted projective space* $\boldsymbol{P}(1, 1, 1, 1, 2)$ *defined by*

$$u^2 + A_2 u + A_4 = B_2 u + B_4 = 0 .$$

where (x_0, x_1, x_2, x_3, u) *is a system of coordinates with* $\deg x_i = 1$, $\deg u = 2$, *and the* A_i *and* B_i *are homogeneous forms of degree* i *in the* x_j $(n = 3)$.

(2) *A weighted complete intersection of type* $(2,3,4)$ *in the weighted projective space* $\boldsymbol{P}(1, 1, 1, 1, 1, 2)$ *defined by*

$$A_2 = B_1 u + B_3 = u^2 + C_2 u + C_4 = 0 .$$

where $(x_0, x_1, x_2, x_3, x_4, u)$ *is a system of coordinates with* $\deg x_i = 1$ $(0 \leqslant i \leqslant 4)$, $\deg u = 2$, *and the* A_i, B_i *and* C_i *are homogeneous forms of degree* i *in the* $x_j (n = 4)$.

(3a) *A weighted complete intersection of type* $(4,5,8)$ *in the weighted projective space* $\boldsymbol{P}(1, 1, 1, 2, 4, 4)$ *defined by*

$$\begin{cases} u^2 + v + w + A_2 u + A_4 = 0 , \\ x_1 v + x_2 w + B_3 u + B_5 = 0 , \\ vw + (C_{21} v + C_{22} w) u + C_{41} v + C_{42} w + C_6 u + C_8 = 0 , \end{cases}$$

where (x_0, x_1, x_2, u, v, w) *is a system of coordinates with* $\deg x_i = 1$, $\deg u = 2$, $\deg v = \deg w = 4$, *the* A_j, B_j, C_j, B_{jk} *and* C_{jk} *are homogeneous forms of degree* j *in the* x_i $(n = 6)$.

(3b) *A weighted complete intersection of type* $(5,8)$ *in the weighted projective space* $\boldsymbol{P}(1,1,1,2,4)$ *defined by*

$$x_0 u^2 + A_1 v + A_3 u + A_5 = v^2 + B_2 uv + B_4 u^2 + B_6 u + B_8 = 0 .$$

where (x_0, x_1, x_2, u, v) *is a system of coordinates with* $\deg x_i = 1$, $\deg u = 2$ *and* $v = 4$, *and the* A_j, B_j *are homogeneous forms of degree* j *in the* $x_i (n = 6)$.

The following is useful in the study of surfaces of types (Ib).

LEMMA 4.2. *Let S be a surface of type* (Ib). *Then the image of the multiplication map*

$$\mu\colon \mathrm{Sym}^2 H^0(L) \to H^0(2L) = H^0(K)$$

is of codimensional 1.

PROOF. Since $Q(V)$ is an irreducible threefold of degree $n-2$ in $\boldsymbol{P}^n$, it is projectively normal and $h^0(Q(V), \mathcal{O}(2)) = 4n-2$. Clearly, we have $h_V(2) = h^0(Q(V), \mathcal{O}(2))$. Since $p_g = 4n-1$, the image of μ is of codimensional 1. Q.E.D.

LEMMA 4.3. *Let S be a surface of type* (Ib) *with $n=3$. Then the canonical model is a weighted complete intersection in Theorem* 4.1, (1).

PROOF. Let x_0, x_1, x_2, x_3 be a basis for $H^0(L)$. By Lemma 4.2, we can find a section $\xi \in H^0(2L)$ which is linearly independent from the products $x_i x_j$. We have $h^0(4L) = h^0(2K) = 44$. In $H^0(4L)$, we have the following elements:

$$\text{quartics in the } x_i, \qquad (\text{quadratic in the } x_i)\,\xi, \qquad \xi^2.$$

These represent 46 sections in total. Therefore, we have two relations

$$A_0\xi^2 + A_2\xi + A_4 = 0, \qquad B_0\xi^2 + B_2\xi + B_4 = 0,$$

where the A_i and B_i are homogeneous forms of degree i in the x_k. If $A_0 = B_0 = 0$, we would have a sextic relation $A_2B_4 = A_4B_2$ by eliminating ξ. This is impossible, since V is of degree 8 in $\boldsymbol{P}^3$. Therefore, we can assume that $A_0 = 1$ and $B_0 = 0$. We have shown that Φ_L can be lifted to a holomorphic map of S into $\boldsymbol{P}(1, 1, 1, 1, 2)$ by setting $u = \xi$. The image is a weighted complete intersection in Theorem 4.1, (1). It is easy to see that it is the canonical model of S. Q.E.D.

LEMMA 4.4. *Let S be a surface of type* (Ib) *with $n=4$. Then the canonical model is a weighted complete intersection in Theorem* 4.1, (2).

PROOF. Let x_i, $0 \leq i \leq 4$, be a basis for $H^0(L)$. Since V is contained in a hyperquadric, we have a quadratic relation $A_2 = 0$ in the x_i. In $H^0(2L) \simeq \boldsymbol{C}^{15}$, we have 15 products $x_i x_j$. Modulo A_2, these present 14 linearly independent elements. Therefore, we can find a new element $\xi \in H^0(2L)$. In $H^0(3L) \simeq \boldsymbol{C}^{34}$, we have $x_i x_j x_j$ and $x_i\xi$. Modulo A_2, they represent 35 sections. Hence, we have a relation of the form $B_1 \xi + B_3 = 0$, where the B_k are homogeneous forms of degree k in the x_i. Note that B_1 cannot be zero, since, otherwise, we have a cubic relation satisfied on V which is not induced by A_2. We look at $H^0(4L) = H^0(2K)$ which is of dimension 64. In $H^0(4L)$, we have

$$\text{(quartics in the } x_i), \quad \text{(quadrics in the } x_i)\xi, \quad \xi^2 .$$

Modulo $(A_2, B_1 \xi + B_3)$, these give 65 sections. Therefore, we have a non-trivial relation of the form

$$C_0 \xi^2 + C_2 \xi + C_4 = 0,$$

where the C_i are homogeneous forms of degree i in the x_k. Note that C_0 is a non-zero constant, since, otherwise, we have a quintic relation $B_1 C_4 = B_3 C_2$ which together with $A_2 = 0$ implies that V is a complete intersection of type (2, 5) contradicting $\deg V = 12$. Hence we can assume that $C_0 = 1$.

We have shown that Φ_L can be lifted to a holomorphic map of S into $\boldsymbol{P}(1,1,1,1,1,2)$ and the images is as in Theorem 4.1, (2). It is easy to see that it is the canonical model of S. Q.E.D.

Let S be a surface of type (Ib) with $n = 6$, and assume that $Q(V)$ is a cone over the Veronese surface. Let Λ_0 be the pull-back to S of the linear system of hyperplanes through the vertex of $Q(V)$, and let G be its fixed part. Since $Q(V)$ is a cone over the Veronese surface, we have a net Λ such that $2H + G \in \Lambda_0$ for $H \in \Lambda$. We have $L = [2H + G]$. Since $|L|$ is free from base points, we have $LG = 0$. We have $20 = L^2 = 2LH = 4H^2 + 2HG \geqslant 4H^2$. It follows that $H^2 \leqslant 5$. Since S is an even surface, H^2 must be a positive even integer. Hence $H^2 = 2, 4$. Since S is a canonical surface, $H^2 = 2$ is inadequate. Thus $H^2 = 4$, $HG = 2$ and $G^2 = -4$.

LEMMA 4.5. *G consists of (-2)-curves. More precisely, either*

(1) $G = G_1 + G_2$ with two disjoint (-2) curves G_1, G_2 such that $HG_i = 1$, or

(2) $G = 2(G_1 + \ldots + G_{m-2}) + G_{m-1} + G_m$, where the G_i are (-2)-curves with $HG_1 = 1$, $G_i G_{i+1} = 1$ for $1 \leqslant i \leqslant m-3$, $G_{m-2}G_{m-1} = G_{m-2}G_m = 1$ and $G_{m-1}G_m = 0$ $(m \geqslant 3)$.

PROOF. Since $KG = 2LG = 0$, G consists of (-2)-curves. Recall that $HG = 2$.

Assume that there exists only one irreducible component G_1 of G with $HG_1 = 2$. Then G_1 is of multiplicity one in G and $H(G - G_1) = 0$. Since $0 = LG_1 = 2HG_1 + G_1^2 + G_1(G - G_1)$, we get $G_1(G - G_1) = -2$ which is absurd. Hence we have either

(i) there exist two irreducible components G_1, G_2 of G such that $HG_i = 1$, or

(ii) there exists only one irreducible component G_1 of multiplicity 2 in G such that $HG_1 = 1$.

Assume that (i) is the case. Since $HG = 2$, each G_i is of multiplicity one in G. Since we have $0 = LG_1 = 2HG_1 + G_1^2 + G_1G_2 + G_1(G - G_1 - G_2)$, we get $G_1G_2 = G_1(G - G_1 - G_2) = 0$. Similarly, we get $G_2(G - G_1 - G_2) = 0$. Then we have $(G - G_1 - G_2)^2 = 0$ by $L(G - G_1 - G_2) = 0$. Since $H(G - G_1 - G_2) = 0$, Hodge's index theorem shows $G - G_1 - G_2 = 0$. Hence G consists of two disjoint (-2)-curves G_1, G_2 with $HG_i = 1$. This gives (1).

Assume that (ii) is the case. Then $H(G - 2G_1) = 0$. It follows from $LG_1 = 0$ that $G_1(G - 2G_1) = 2$. Since $L(G - 2G_1) = 0$, we get $(G - 2G_1)^2 = -4$. Hence we can repeat the above argument with the pair $(G_1, G - 2G_1)$ instead of (H, G). We have either $G - 2G_1 = G_2 + G_3$ with disjoint (-2)-curves G_2, G_3 satisfying $G_1G_2 = G_1G_3 = 1$, or there exists a component G_2 of multiplicity 2 in $G - 2G_1$ such that $G_1G_2 = 1$, $(G - 2G_1 - 2G_2)^2 = -4$. In the latter case, we repeat the above argument with $G_2, G - 2G_1 - 2G_2)$ instead of $(G_1, G - 2G_1)$. Since such a procedure must terminate, we see that G is as in (2). Q.E.D.

In particular, the support of G is connected unless G consists of two disjoint (-2)-curves.

LEMMA 4.6. *Let S be a surface of type (Ib) and assume that $Q(V)$ is a cone over the Veronese surface. Then S is birationally equivalent to one of the weighted complete intersections as in (3a), (3b) of Theorem 4.1.*

PROOF. Let z_0, z_1, z_2 be a basis for the module of Λ, and let $\zeta \in H^0([G])$ define G. Then the products $z_iz_j\zeta$ generate a subspace of dimension 6 in $H^0(L)$. Since $h^0(L) = 7$, we have a new element $\xi \in H^0(L)$. Since $|L|$ is free from base points and since $LG = 0$, we can assume that ξ does not vanish on G.

In $H^0(2L) = H^0(K) \simeq C^{23}$, we have the following elements:

$$(\text{quartics in the } z_i)\zeta^2, \quad (\text{quadrics in the } z_i)\zeta\xi,\ \xi^2.$$

These give 22 linearly independent elements.

Case 1: Supp(G) is connected.

We have a new section $\eta \in H^0(2L)$. Since ξ^2 is a constant on G, we can assume that η is identically zero on G by replacing η by $\eta - c\xi^2$ if necessary, where c is some constant. This implies that there exists a section $\psi \in H^0(K-G)$ with $\eta = \psi\zeta$ which is linearly independent from the 21 elements: (quartics in the z_i)ζ and (quadrics in the z_i)ξ. Since $HG_1 = 1$, G_1 is mapped biholomorphically onto a line. Hence we can assume that z_0 is identically zero on G_1 and that (z_1, z_2) induces a basis for $H^0(\mathcal{O}_{G_1}(1))$.

We have $h^0(K+H) = 36$ by the Riemann-Roch theorem and Ramanujam's vanishing theorem. In $H^0(K+H)$, we have

$$(\text{quintics in the } z_i)\zeta^2, \quad (\text{cubics in the } z_i)\zeta\xi, \quad z_i\xi^2, z_i\eta.$$

These are 37 elements in total. Therefore, we have a non-trivial relation of the form

$$\alpha_1\xi^2 + \alpha_1'\eta + \alpha_3\zeta\xi + \alpha_5\zeta^2 = 0, \tag{4.1}$$

where the α_1 are homogeneous forms of degree i in the z_k. Note that $\alpha_1 \neq 0$, since, otherwise, we have $\alpha_1'\psi + \alpha_3\xi + \alpha_5\zeta = 0$, which is absurd. However (4.1) tells us that α_1 vanishes identically on G. Hence, we can assume that $\alpha_1 = z_0$.

In $H^0(K+4H) \simeq \boldsymbol{C}^{96}$, we have the following elements:

$$\begin{array}{ll} (\text{octics in the } z_i)\zeta^2, & (\text{sexrics in the } z_i)\zeta\xi, \\ (\text{quartics in the } z_i)\xi^2, & (\text{quartics in the } z_i)\eta, \\ (\text{quadrics in the } z_i)\xi\psi, & \psi^2. \end{array}$$

Modulo (4.1), these represent 97 sections. Hence we have a non- trivial relation of the form

$$\beta_0\psi^2 + \beta_2\xi\psi + \beta_4\xi^2 + \beta_4'\eta + \beta_6\zeta\xi + \beta_8\zeta^2 = 0,$$

where the β_i are homogeneous forms of degree i in the z_k. Note that β_0 is a nonzero constant. Hence we can assume that $\beta_0 = 1$ and $\beta_4' = 0$ by a suitable linear change of ψ not involving ξ. Multipliying ζ^2, we get

$$\eta^2 + \beta_2(\zeta\xi)\eta + \beta_4(\zeta\xi)^2 + \beta_6\zeta^2(\zeta\xi) + \beta_8\zeta^4 = 0. \tag{4.2}$$

Recall that G consists of (-2)-curves which is contracted to a rational double point. Put $u = \xi/\zeta$ and $v = \eta/\zeta^2$. Then (4.1) and (4.2) shows that S is birationally equivalent to a weighted complete intersection of type (5, 8) in $\boldsymbol{P}(1, 1, 1, 2, 4)$. Hence we get Theorem 4.1, (3*b*).

Case 2: $G = G_1 + G_2$.

We let $\zeta_i \in H^0([G_i])$ define $G_i (i = 1, 2)$. Then we can assume that $\zeta = \zeta_1 \zeta_2$. Since ξ^3 is a nonzero constant on G_i, we have an exact sequence

$$0 \to H^0(K - G_i) \to H^0(K) \to H^0(\mathcal{O}_{G_i}) \simeq \boldsymbol{C} \to 0 .$$

Hence $h^0(K - G_i) = 22$. we have 21 linearly independent elements: (quartics in the z_j) ζ^2/ζ_i, (quadrics in the z_j) $(\zeta/\zeta_i)\xi$. Thus we have a new element $\eta_i \in H^0(K - G_i)$. We can assume that η_i is a nonzero constant on $G_j(\{i, j\} = \{1, 2\})$, since, otherwise, we have $h^0(K - G) = 22$ and can proceed as in Case 1. Since $HG_i = 1$, we see that G_i is mapped biholomorphically onto a line. Hence we can assume that z_j vanishes identically on G_i and that (z_0, z_i) induces a basis for $H^0(\mathcal{O}_{G_i}(1))$ $(\{i, j\} = \{1, 2\})$.

In $H^0(K)$, we have $\zeta_1\eta_1$ and $\zeta_2\eta_2$ in addition to the 22 elements: (quartics in the z_j) ζ^2 (quadrics in the z_j) $\zeta\xi$ and ξ^2. Hence we get a relation of the form

$$l(\zeta_1\eta_1, \zeta_2\eta_2, \xi^2) = \alpha_2\zeta\xi + \alpha_4\zeta^2 ,$$

where l is a linear form in $\zeta_1\eta_1, \zeta_2\eta_2, \xi^2$ and the α_k are homogeneous forms of degree k in the z_i. By restricting this to G_1 or G_2, we see that the coefficients of $\zeta_i\eta_i (i = 1, 2)$ and ξ^2 in l are all nonzero constants. Thus we can rewrite it as

$$\xi^2 = \zeta_1\eta_1 + \zeta_2\eta_2 + \alpha_2\zeta\xi + \alpha_4\zeta^2 . \tag{4.3}$$

In $H^0(K + H) \simeq \boldsymbol{C}^{36}$, we have

$$\text{(quintics in the } z_i)\,\zeta^2 , \qquad \text{(cubics in the } z_i)\,\zeta\xi , \qquad z_i\zeta_j\eta_j$$

modulo (4.3). These are 37 sections in total. Hence we have a relation of the form

$$\beta_{11}\zeta_1\eta_1 + \beta_{12}\zeta_2\eta_2 = \beta_3\zeta\xi + \beta_5\zeta^2 , \tag{4.4}$$

where the β_{kj} and β_k are homogeneous forms of degree k in the z_i. We can assume that $\beta_{11} = z_1$ and $\beta_{12} = z_2$.

In $H^0(2K) = H^0(8H + 4G) \simeq \boldsymbol{C}^{104}$, we have $(\zeta_1\eta_1)^2$ and $(\zeta_2\eta_2)^2$. Therefore, the restriction map $H^0(2K) \to H^0(\mathcal{O}_G)$ is surjective, and we have $h^0(8H + 3G) = 102$. In $H^0(8H + 3G)$, we have the following elements modulo (4.3):

(octics in the $z_i)\zeta^3$, (sextics in the $z_i)\zeta^2\xi$,
(quartics in the $z_i)\zeta(\zeta_j\eta_j)$, (quadrics in the $z_i)\xi(\zeta_j\eta_j)$,
$\eta_1\eta_2$.

Modulo (4.4), these give 103 sections. Hence we have a relation of the form

$$\gamma_0\eta_1\eta_2 + \gamma_{21}\xi(\zeta_1\eta_1) + \gamma_{22}\xi(\zeta_2\eta_2) + \\ + \gamma_{41}\zeta(\zeta_1\eta_1) + \gamma_{42}\zeta(\zeta_2\eta_2) + \gamma_6\zeta^2\xi + \gamma_8\zeta^3 = 0,$$

where the $\gamma_{ij}(j = 1, 2)$ and γ_i are homogeneous forms of degree i in (z_0, z_1, z_2). By restricting this to G_1 or G_2, we see that γ_0 is a nonzero constant. Multiplying ζ, we get

$$(4.5)\quad (\zeta_1\eta_1)(\zeta_2\eta_2) + \gamma_{21}\zeta\xi(\zeta_1\eta_1) + \gamma_{22}\zeta\xi(\zeta_2\eta_2) + \\ + \gamma_{41}\zeta^2(\zeta_1\eta_1) + \gamma_{42}\zeta^2(\zeta_2\eta_2) + \gamma_6\zeta^3\xi + \gamma_8\zeta^4 = 0.$$

Put $u = \xi/\zeta$, $v = \zeta_1\eta_1/\zeta^2$ and $w = \zeta_2\eta_2/\zeta^2$. Then, by (4.3), (4.4) and (4.5), S is birationally equivalent to a weighted complete intersection of type (4,5,8) in $\boldsymbol{P}(1,1,1,2,4,4)$. Hence we get Theorem 4.1, (3*a*).

Conversely, the weighted complete intersections as in (3*a*) and (3*b*) of Theorem 4.1 are surfaces with at most rational double points provided that the coefficients are sufficiently general. Furthermore, since the dualizing sheaf of such a surface is induced by $\mathcal{O}(4)$, the minimal resolution is an even canonical surface with $p_g = 23$, $q = 0$ and $K^2 = 80$. Q.E.D.

In sum, Theorem 4.1 has been shown.

REMARK 4.7. A surface in Theorem 4.1, (3*b*) can be obtained as a specialization of surfaces in Theorem 4.1, (3*a*).

5. Surfaces of type (I*b*): $Q(V)$ is a scroll.

Let π: $\boldsymbol{P}_{a,b,c} \to \boldsymbol{P}^1$ be as in Lemma 2.4. Let X_0, X_1, X_2 be sections of $[T - aF]$, $[T - bF]$, $[T - cF]$, respectively, such that they form a system of homogeneous coordinates on each fibre F of π. We let $\{z_0, z_1\}$ denote a system of homogeneous coordinates on $\boldsymbol{P}^1$, which we identify with a basis for $H^0(F)$. Let W be the total space of the bundle $[2T]$ over $\boldsymbol{P}_{a,b,c}$, and let w denote its fiber coordinate.

In this section, we show the following:

THEOREM 5.1. *Let S be a surface of type* (Ib) *and assume that the quadric hull of the semi-canonical image is a rational normal scroll of dimension* 3. *If the canonical image is cut out by hyperquadrics, then S is birationally equivalent to one of the following surfaces*:

(1) *A surface defined in W by*

$$X_2 w = A(z_0, z_1, X_0, X_1, X_2), \qquad w^2 = B(z_0, z_1, X_0, X_1, X_2),$$

where $A \in H^0(\boldsymbol{P}_{a,b,c}, \mathcal{O}(3T-(n-4)F))$, $B \in H^0(\boldsymbol{P}_{a,b,c}, \mathcal{O}((4T))$, *and* $(a, b, c) = (1, 1, n-4)$, $5 \leqslant n \leqslant 7$, *or* $(a, b, c) = (0, 2, n-4)$, $6 \leqslant n \leqslant$ $\leqslant 10$.

(2) *A surface defined in W on $\boldsymbol{P}_{0,1,n-3}$ by*

$$z_0 X_2 w = A(z_0, z_1, X_0, X_1, X_2), \qquad w^2 = B(z_0, z_1, X_0, X_1, X_2),$$

or, when $n = 5$,

$$X_1 w = A(z_0, z_1, X_0, X_1, X_2), \qquad w^2 = B(z_0, z_1, X_0, X_1, X_2),$$

where $A \in H^0(3T-(n-4)F)$ *and* $B \in H^0(4T)$, $5 \leqslant n \leqslant 7$.

Let S be a surface of type (Ib) with $n \geqslant 5$. Assume that the quadric hull $Q(V)$ of the semi-canonical image V is a rational normal scroll of dimension 3. Let Λ be the pencil of irreducible curves on S indiced by the ruling of $Q(V)$ via Φ_L. Let $\rho\colon \widehat{S} \to S$ be a composite of blowing-ups which eliminates BsΛ. We assume that ρ is the shortest. Let $\lambda\colon \widehat{S} \to \boldsymbol{P}^1$ be the corresponding fibration. We denote by D and $\widehat{D}$ a general member of Λ and a general fibre of λ, respectively.

Let $\mathcal{E}$ be the locally free subshead of $\lambda_* \rho^* L$ generically generated by elements in $H^0(\boldsymbol{P}^1, \lambda_* \rho^* L)$. Since D is mapped onto a plane curve via the semi-canonical map, we have rank$\{H^0(\rho^* L) \to H^0(\rho^* L|_{\widehat{D}})\} =$ $= 3$. It follows that $\mathcal{E}$ is of rank 3 and it is of the form

$$\mathcal{E} \simeq \mathcal{O}_{\boldsymbol{P}^1}(a) \oplus \mathcal{O}_{\boldsymbol{P}^1}(b) \oplus \mathcal{O}_{\boldsymbol{P}^1}(c), \qquad 0 \leqslant a \leqslant b \leqslant c,\ a+b+c = n-2.$$

We have the natural sheaf homomorphism $\lambda^* \mathcal{E} \subset \lambda^* \lambda_* \rho^* L \to \rho^* L$ which induces a rational map $h\colon \widehat{S} \to \boldsymbol{P}(\mathcal{E}) = \boldsymbol{P}_{a,b,c}$. Since $|\rho^* L|$ is free from base points, we can assume that h is holomorphic and $\rho^* L = h^* T$. Then, $Q(V)$ is nothing but the image of $\boldsymbol{P}_{a,b,c}$ under the holomorphic map defined by $|T|$. We have shown the following:

LEMMA 5.2. *There exists a lifting* $h\colon \widehat{S} \to \boldsymbol{P}_{a,b,c}$ *of the semi-canonical map with* $\rho^* L = h^* T$.

Assume that $h(\widehat{S})$ is linearly equivalent to $\alpha T + \beta F$ on $\boldsymbol{P}_{a,b,c}$. We remark that $\alpha = (\rho^* L)\widehat{D} = LD$. Since h maps $\widehat{D}$ birationally onto a plane curve of degree α, we have $\alpha \geqslant 6$ if $|\widehat{D}|$ is a Petri general pencil. Since $(\rho^* L)^2 = L^2 = 4n - 4$, we have $T^2(\alpha T + \beta F) = 4n - 4$. Hence

$$(5.1) \qquad \beta = -(\alpha - 4)(n - 1) + \alpha .$$

Let C be a general member of $|\rho^* L|$. Then it is an irreducible nonsingular curve of genus $6n - 5$. Hence the arithmetic genus of $h(C)$ can be written as $6n - 5 + \delta$ with a nonnegative integer δ. Since the dualizing sheaf of $h(C)$ is induced by $(\alpha - 2)T + (\beta + n - 4)F$, we have $12n - 12 + 2\delta = T(\alpha T + \beta F)((\alpha - 2)T + (\beta + n - 4)F)$. It follows from this and (5.1) that

$$(5.2) \qquad \delta = 3\alpha - 10 - \frac{1}{2}(\alpha - 4)(\alpha - 5)(n - 2).$$

LEMMA 5.3. *Assume that S has no Petri special pencils. Then $h(\widehat{S})$ is linearly equivalent to either*

(1) $6T - 2(n - 4)F$, $5 \leqslant n \leqslant 10$, *or*

(2) $7T - 5F$, $n = 5$.

In particular, b is positive. Furthermore, $|D|$ is free from base points when $n \geqslant 6$. When $n = 5$, $|D|$ is free from base points or $D^2 = 2$.

PROOF. Since $n \geqslant 5$, $\alpha \geqslant 6$ and $\delta \geqslant 0$, we see from (5.1) and (5.2) that only (1) and (2) are possible. If $a = b = 0$, then, it is easy to see that $|6T - 2(n - 4)F|$ and $|7T - 5F|$ have a fixed component linearly equivalent to $T - (n - 2)F$. It would follow that $h(\widehat{S})$ is reducible. Hence b must be positive. In order to show the last assertion, we recall that $(LD)^2 \geqslant L^2 D^2 = 4(n - 1)D^2$ holds by Hodge's index theorem. Since S is an even surface, D^2 is a nonnegative even integer. Hence D^2 when $n \geqslant 6$, and $D^2 = 0, 2$ when $n = 5$. Q.E.D.

LEMMA 5.4. *Assume that $LD = 6$. If the canonical image X is cut out by hyperquadrics, then $D^2 = 0$ and the restriction map $H^0(K) \to \to H^0(K_D)$ is surjective.*

PROOF. Since D is a nonhyperelliptic curve, we have $h^0(L|_D) \leqslant 3$ by Clifford's theorem. Since D is mapped birationally onto a plane sextic curve via the semi-canonical map, we have $h^0(L|_D) \geqslant 3$. Hence $h^0(L|_D) = 3$ and the restriction map $H^0(L) \to H^0(L|_D)$ is surjective. Since $2L = K$ and since it induces a special divisor on D, Clifford's theorem gives $h^0(2L|_D) \leqslant 7$ with equality holding if and only if $2L|_D$ is a canonical divisor, that is, $D^2 = 0$.

We consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Sym}^2 H^0(L) & \longrightarrow & H^0(2L) \\ \downarrow & & \downarrow \\ \mathrm{Sym}^2 H^0(L|_D) & \longrightarrow & H^0(2L|_D). \end{array}$$

Since $\Phi_L(D)$ is a plane sextic, the image of the multiplication map $\mathrm{Sym}^2 H^0(L|_D) \to H^0(2L|_D)$ is a subspace of dimension 6. If $H^0(2L) \to \to H^0(2L|_D)$ were of rank 6, then $\Phi_k(D)$ would be the Veronese transform of $\Phi_L(D)$ and, hence, it is not cut out by quadrics. Therefore, $D^2 = 0$ and $H^0(K) \to H^0(K|_D) = H^0(K_D)$ is surjective when X is cut out by quadrics. Q.E.D.

LEMMA 5.5. *Assume that $n = 5$ and $LD = 7$. Then the canonical image of S is not cut out by hyperquadrics.*

PROOF. Since $b > 0$ by Lemma 5.3, $(a, b, c) = (1, 1, 1)$ or $(0,1,2)$.

Assume that $(a, b, c,) = (1, 1, 1)$. Then ρ is the identity map and V is linearly equivalent to $7T - 5F$ on $\boldsymbol{P}_{1,1,1}$. We have $\boldsymbol{P}_{1,1,1} \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^2$. Let H_i be the pull-back of a hyperplane in $\boldsymbol{P}^i (i = 1, 2)$. Then T and F are linearly equivalent to $H_1 + H_2$ and H_1, respectively. Hence V is linearly equivalent to $2H_1 + 7H_2$, and it is a double covering of $\boldsymbol{P}^2$ via the projection $\boldsymbol{P}^1 \times \boldsymbol{P}^2 \to \boldsymbol{P}^2$. Since V is birational to a canonical surface S, this is impossible. Hence $(a, b, c) \neq (1, 1, 1)$.

Assume that $(a, b, c) = (0, 1, 2)$. Let C be, as before, a general member of $|L|$. Since $H^1(L) = 0$, $H^0(K) \to H^0(K|_C)$ is surjective. It suffices to show that $\Phi_K(C)$ is not cut out by hyperquadrics. Note that $\Phi_L(C)$ is linearly equivalent to $7\Delta_0 + 9\Gamma$ on Σ_1 and that its dualizing sheaf is induced by $5\Delta_0 + 6\Gamma$. Since $\delta = 2$ by (5.2), we see that $\Phi_L(C)$ has two singular points x_1, x_2 which are possibly infinitely near. The canonical bundle of C is induced by $3L$. Since $L|_C$ comes from $\Delta_0 + 2\Gamma$, the conductor of $C \to \Phi_L(C)$ is induced by $2\Delta_0$. It follows that x_1 and x_2 are on the minimal section Δ_0. We have $\Delta_0(7\Delta_0 + 9\Gamma) = 2$. Hence x_2 is infinitely near to x_1.

Let $\nu \colon \Sigma \to \Sigma_1$ denote the blowing-up at x_1, x_2, and let Δ be the proper transform of Δ_0 by ν. Put $E_i = \nu^{-1}(x_i)$ for $i = 1, 2$. The proper transform of $\Phi_L(C)$ by ν is linearly equivalent to $\nu^*(7\Delta_0 + 2\Gamma) - 2E_1 - 2E_2$, and we can identify it with C. Since $C\Delta = 0$, the divisors $H := 2\nu^*(\Delta_0 +$

$+ 2\Gamma) + \Delta$ and K are equivalent on C. Consider the following exact sequence:

$$0 \to \mathcal{O}(H - C) \to \mathcal{O}(H) \to \mathcal{O}_C(K|_C) \to 0\,.$$

Since $H - C \sim -2\nu^*(\Delta_0 + \Gamma) + K_{\tilde{\Sigma}} + E_2$, we have $H^0(H - C) = H^1(H - C) = 0$. It follows that $H^0(H)$ is restricted onto $H^0(K|_C)$ isomorphically. Note that $|H|$ induces a birational map of Σ onto its image. Since $H^0(2H - C) = 0$, $\Phi_K(C)$ cannot be cut out by hyperquadrics. Q.E.D.

LEMMA 5.6. *Assume that $LD = 6$ and $D^2 = 0$. Assume further that the canonical image is cut out by hyperquadrics. Then*

$$(a, b, c) = \begin{cases} (1, 1, n-4), & 5 \leqslant n \leqslant 7\,, \\ (0, 2, n-4), & 6 \leqslant n \leqslant 10\,, \\ (0, 1, n-3), & 5 \leqslant n \leqslant 7\,. \end{cases}$$

PROOF. We have a holomorphic map $h: S \to \boldsymbol{P}_{a,b,c}$ with $L = h^*T$ and $D = h^*F$ which is birational onto its image. Since $h(S)$ is linearly equivalent to $6T - 2(n-4)F$, the natural homomorphism $h^*: H^0(iT + jF) \to H^0(iL + jD)$ is injective for any j whenever $i \leqslant 5$. We can choose sections X_0, X_1 and X_2 of $[T - aF]$, $[T - bF]$ and $[T - cF]$, respectively, such that they from a system of homogeneous coordinates on fibers of $\boldsymbol{P}_{a,b,c} \to \boldsymbol{P}^1$. Put $\xi_i = h^*X_i$ $(0 \leqslant i \leqslant 2)$ and let z_0, z_1 be a basis for $H^0(D)$.

We have $h^0(2T - F) = 4n - 8$. Since the restriction map $H^0(K) \to H^0(K_D)$ is surjective by Lemma 5.4, we have $h^0(K - D) = p_g - 7 = 4n - 8$. Hence $h^*: H^0(2T - F) \to H^0(K - D)$ is an isomorphism. Since the image of $\mathrm{Sym}^2 H^0(L) \to H^0(K)$ is of codimension 1 by Lemma 4.2, we can find a new element $\eta \in H^0(K)$. Then, by Lemma 5.4 again, we see that $\xi_i\xi_j$ and η induce a basis for $H^0(K_D)$.

We have $h^0(3L) = 10n - 6$ and $h^0(3T) = 10n - 10$. In $H^0(3L)$, we also have the following $n + 1$ elements:

$$z_0^i z_1^{a-i}\xi_0\eta\,, \qquad z_0^i z_1^{b-i}\xi_1\eta, \qquad z_0^i z_1^{c-i}\xi_2\eta\,. \tag{5.3}$$

Since $h^*H^0(3T)$ is of codimension 4, there are at most 4 additional linearly independent elements in (5.3). Hence we have (at least) $n - 3$ relations of the form

$$l(z_0, z_1, \xi_0, \xi_1, \xi_2)\eta = \phi(z_0, z_1, \xi_0, \xi_1, \xi_2)\,,$$

where $l \in h^*H^0(T)$ and $\phi \in h^*H^0(3T)$. If there were two independent relations as above, by eliminating η, we would get a non-trivial relation which is of degree 4 in the ξ_i. This is absurd, since $LD = 6$. Hence all the $n-3$ relations are derived from a single relation in $H^0(3L-(n-4)D)$. It follows that $c \geqslant n-4$, since $H^0(3L-(n-4)D)$ must contain an element of the form $\psi(z_0, z_1)\xi_i\eta$. Since $a+b+c=n-2$ and $b>0$, we have $(a, b, c) = (1, 1, n-4), (0, 2, n-4)$ or $(0, 1, n-3)$.

Assume that $c = n-4$. We have shown that there is a non-trivial relation of the form

$$\xi_2\eta = A(z_0, z_1, \xi_0, \xi_1, \xi_2), \tag{5.4}$$

where $A \in h^*H^0(3T-(n-4)F)$. From this, we see that A cannot be divided by ξ_2. Hence $n \leqslant 7$ if $a=1$, and $n \leqslant 10$ if $a=0$.

When $c = n-3$, we can assume that there is a non-trivial relation of the form

$$z_0\xi_2\eta = A(z_0, z_1, \xi_0, \xi_1, \xi_2) \tag{5.5a}$$

or, when $n=5$.

$$\xi_1\eta = A(z_0, z_1, \xi_0, \xi_1, \xi_2), \tag{5.5b}$$

where $A \in h^*H^0(3T-(n-4)F)$. Recall that $\xi_i\xi_j$, η induce a basis for $H^0(K_D)$. Hence, in (5.5a), A cannot be divided by ξ_2. It follows $n \leqslant 7$. Q.E.D.

LEMMA 5.7. *In (5.5a), A cannot be divided by z_0.*

PROOF. Assume contrarily that z_0 divides A and that we have a relation of the form

$$\xi_2\eta = A_0(z_0, z_1, \xi_0, \xi_1, \xi_2), \tag{5.6}$$

where $A_0 \in h^*H^0(3T-(n-3)F)$. Then, in $H^0(3L)$, we find only three elements $\xi_0\eta$, $z_0\xi_1\eta$, $z_1\xi_1\eta$ modulo $h^*H^0(3T)$ and (5.6). Hence there is a new element $\psi \in H^0(3L)$. We have $h^0(3T+F) = 10n$ and $h^0(3L+D) = 10n+6$ by the Riemann-Roch theorem and Ramanujam's vanishing theorem. In $H^0(3L+D)$, we have the following 7 elements modulo $h^*H^0(3T+F)$ and (5.6):

$$z_0\xi_0\eta, \quad z_1\xi_0\eta, \quad z_0^iz_1^{2-i}\xi_1\eta \ (0 \leqslant i \leqslant 2), \quad z_0\psi, \ z_1\psi.$$

Hence we may assume that there is a relation of the form

$$z_0\psi = A_1\eta + A_2, \tag{5.7}$$

where $A_1 \in h^*H^0(T+F)$ and $A_2 \in h^*H^0(3T+F)$, since, if the relation does not involve ψ, we get a relation in $(z_0, z_1, \xi_0, \xi_1, \xi_2)$ which is of degree 4 in the ξ_i by eliminating η using (5.6).

We have $h^0(4L) = 20n - 16$ and $h^0(4T) = 20n - 25$. In $H^0(4L)$, we have the following 10 elements modulo $h^*H^0(4T)$ (5.6) and (5.7):

$$\eta^2,\ \xi_0^2\eta,\ z_0\xi_0\xi_1\eta,\ z_1\xi_0\xi_1\eta,\ z_0^i z_1^{2-i}\xi_1^2\eta,\ \xi_0\psi,\ z_1\xi_1\psi,\ z_1^{n-3}\xi_2\psi.$$

Hence we have a relation here. In this relation, the coefficient of η^2 cannot be zero, since, otherwise, we get a relation in $(z_0, z_1, \xi_0, \xi_1, \xi_2)$ which is of degree 5 in the ξ_i by eliminating η and ψ using (5.6) and (5.7). Hence we can assume that it is of the form

$$\eta^2 = A_3\psi + A_4\,, \tag{5.8}$$

where $A_3 \in h + H^0(T)$ and $A_4 \in h^*H^0(4T)$.

Now, eliminating η and ψ from (5.8) by using (5.6) and (5.7), we get

$$z_0A_0^2 = \xi_2A_0A_1A_3 + \xi_2^2A_2A_3 + z_0\xi_2^2A_4$$

which is a non-trivial relation in $H^0(6L - (2n-7)D)$. Since $h(S)$ is linearly equivalent to $6T - 2(n-4)$ this is impossible. Q.E.D.

We complete the proof of Theorem 5.1. We look at $H^0(4L) = H^0(2K)$ which is of dimension $20n - 16$. Here, we have η^2, $\phi\eta$, ψ with $\phi \in h^*H^0(2T)$, $\psi \in h^*H^0(4T)$. Since $h^0(2T) = 4n - 2$ and $h^0(4T) = 20n - 25$, these give $20n - 15$ elements modulo the relation (5.4), (5.5*a*) or (5.5*b*) in $H^0(3L - (n-4)D)$. Hence we have a non-trivial relation among them. If the coefficient of η^2 in the new relation were zero, then, by eliminating η using the first relation, we would get a non-trivial relation in $(z_0, z_1, \xi_0, \xi_1, \xi_2)$ which is of degree 5 in the ξ_j. This is absurd, since $LD = 6$. Therefore, by a suitable linear change of η if necessary, we get a relation of the form

$$\eta^2 = B(z_0, z_1, \xi_0, \xi_1, \xi_2), \tag{5.9}$$

where $B \in h^*H^0(4T)$.

Let W and w be as in Theorem 5.1. We have shown that h can be lifted to a holomorphic map of S into W by setting $w = \eta$. The image is as in Theorem 5.1. Conversely, if the coefficients are sufficiently general, a surface in Theorem 5.1 has at most rational double points as its singularities, and the minimal resolution is an even canonical surface with the desired numerical characters.

6. Surfaces of type (I*c*).

The purpose of this section is to show the following:

THEOREM 6.1. *The canonical image of a surface of type* (I*c*) *is not cut out by hyperquadrics.*

Let S be a surface of type (Ic). We denote by $\sigma: \widetilde{S} \to S$ the blowing-up at the base point of $|L|$ as in the proof of Lemma 2.1. We have $|\sigma^* L| = |M| + E$, where E is the expectional (-1)-curve. Then $M^2 = 4n - 5$, $ME = 1$.

LEMMA 6.2. *Assume that S is a surface of type* (I*c*) *with $n = 3$. Then the canonical image of S is not cut only by quadrics.*

PROOF. Let C be a general member of $|L|$. Since $|L|$ has only one base point, we can assume that it is a nonsingular irreducible curve of genus 13. Since $H^1(L) = H^1(3L) = 0$, we see that the restriction maps $H^0(K) \to H^0(K|_C)$ and $H^0(2K) \to H^0(2K|_C)$ are both surjective. Hence it suffices to show that $\Phi_K(C)$ is not cut only by hyperquadrics.

For simplicity, we denote the proper transform of C by $\sigma: \widetilde{S} \to S$ by the same symbol. Since $M^2 = 7$, $C_0 := \Phi_M(C)$ can be identified with a plane curve of degree 7. Since the arithmetic genus of C_0 is 15, C_0 has two singular points x_1, x_2 of multiplicity 2 which are possibly infinitely near. Since the dualizing sheaf of C_0 is induced by $\mathcal{O}(4)$, the conductor of $C \to C_0$ comes from $M - 3E$. It follows that C_0 contacts the line l jointing x_1 and x_2 to the third order at the point (corresponding to) $x = C \cap E$.

Let $\nu: \Sigma \to \boldsymbol{P}^2$ denote the blowing-up at x_1, x_2, and put $E_i = \nu^{-1}(x_i)$. Then the proper transform of C_0 by ν is isomorphic to C, and it is linearly equivalent to $7\nu^* l - 2E_1 - 2E_2$. Let $\widehat{\nu}: \widehat{\Sigma} \to \Sigma$ denote the blowing-up at x, and put $E_x = \widehat{\nu}^{-1}(x)$. Then $K|_C$ is induced by $H: = 2(\widehat{\nu} \circ \nu)^* l + \widehat{l}$, where $\widehat{l}$ is the proper transform of l by $\widehat{\nu} \circ \nu$. It is easy to see that $H^0(H)$ restrict to $H^0(K|_C)$ isomorphically. Since $2H - C$ is linearly equivalent to $-(\widehat{\nu} \circ \nu)^* l - E_x$, any hyperquadric through $\Phi_K(C)$ contains the image of $\widehat{\Sigma}$ via $|H|$. It follows that $\Phi_K(C)$ is not cut out by hyperquadrics. Q.E.D.

LEMMA 6.3. *If $n = 4$, then $Q(V)$ is a singular hyperquadric. If $n = 6$, then $Q(V)$ cannot be a cone over the Veronese surface.*

PROOF. We have $M^2 = 4n - 5$, Assume that $n = 4$. Then V is of degree 11 in $\boldsymbol{P}^4$. If $Q(V)$ were a nonsingular hyperquadratic, then its Picard group is isomorphic to $\boldsymbol{Z}$, and it follows that deg V must be even. This is impossible. Hence $Q(V)$ is singular. Assume that $n = 6$ and $Q(V)$ is a cone over the Veronese surface. Then, considering the pull-back of the linear system of hyperplanes through the vertex of $Q(V)$, we get a decomposition $M = 2H + G$, where G denotes the divisorial part of the inverse image of the vertex. Since $|M|$ is free from base points, we have $MG = 0$. Then $19 = M^2 = 2MH$, which is impossible. Q.E.D.

LEMMA 6.4. *A surface of type* (Ic) *with* $n = 4$ *has a Petri special pencil. In particular, the canonical image is not cut out by hyperquadrics.*

PROOF. By Lemma 6.3, $Q(V)$ is of rank 3 or 4. Hence $Q(V)$ is ruled by planes, and we have (not necessarily distinct) irreducible pencils Λ_1, Λ_2 on S respectively induced by a ruling of $Q(V)$ such that M is linearly equivalent to $D_1 + D_2 + G$ for $D_i \in \Lambda_i$, where G denotes the divisorial part of the inverse image of the vertex of $Q(V)$ via Φ_M. Since M is nef, we have $11 = M^2 = MD_1 + MD_2 + MG \geqslant MD_1 + MD_2$. Hence we can assume that $MD_1 \leqslant 5$. Since Φ_M maps D_1 birationally onto a plane curve of degree MD_1, we see that Λ_1 is a Petri special pencil. Q.E.D.

By Lemmas 6.3 and 6.4, we can assume that $Q(V)$ is a rational normal scroll and $n \geqslant 5$. Let Λ be a pencil of irreducible curves on $\widetilde{S}$ induced by a ruling of $Q(V)$. Let $\rho\colon \widehat{S} \to \widetilde{S}$ be a composite of blowing-ups which eliminates $Bs\Lambda$. We assume that ρ is the shortest. Let $\lambda\colon \widehat{S} \to \boldsymbol{P}^1$ be the corresponding fibration. We denote by $\widehat{D}$ a general fibre of λ. Using M instead of L, we can show the following similarly as in Lemma 5.2.

LEMMA 6.5. *There exists a lifting* $h\colon \widehat{S} \to \boldsymbol{P}_{a,b,c}$ *of the natural holomorphic map* $\Phi_M\colon \widetilde{S} \to Q(V)$ *satisfying* $\rho^* M = h^* T$, *where* a, b, c *are integers with* $0 \leqslant a \leqslant b \leqslant c$, $a + b + c = n - 2$.

Assume that $h(\widehat{S})$ is linearly equivalent to $\alpha T + \beta F$ on $\boldsymbol{P}_{a,b,c}$. Then $\alpha = (\rho^* M)\widehat{D}$. Since h maps $\widehat{D}$ birationally onto a plane curve of degree α, we have $\alpha \geqslant 6$ if $|\widehat{D}|$ is a Petri general pencil. Since $(\rho^* M)^2 = M^2 = 4n - 5$, we have $T^2(\alpha T + \beta F) = 4n - 5$. Hence

$$\beta = -(\alpha - 4)(n - 1) + \alpha - 1. \tag{6.1}$$

Let C be a general member of $|\rho^* M|$. Then it is an irreducible nonsingular curve of genus $6n-5$. Hence the arithmetic genus of $h(C)$ can be written as $6n-5+\delta$ with a nonnegative integer δ. Since the dualizing sheaf of $h(C)$ is induced by $(\alpha-2)T+(\beta+n-4)F$, we have $12n-12+2\delta = T(\alpha T+\beta F)((\alpha-2)T+(\beta+n-4)F)$. It follows from this and (6.1) that

$$\delta = 2\alpha - 9 - \frac{1}{2}(\alpha-4)(\alpha-5)(n-2). \tag{6.2}$$

LEMMA 6.6. *Assume that S has no Petri special pencils. Then $n=5$ and $h(\widehat{S})$ is linearly equivalent to $6T-3F$ on $\boldsymbol{P}_{0,1,2}$.*

PROOF. Since $n \geqslant 5$, $\alpha \geqslant 6$ and $\delta \geqslant 0$, it follows from (6.1) and (6.2) that $n=5$, $\alpha=6$ and $\beta=-3$. We have $(a, b, c) = (1, 1, 1)$ $(0, 1, 2)$ or $(0, 0, 3)$. If $(a, b, c) = (0, 0, 3)$, then it is easy to see that any member of $|6T-3F|$ is reducible. If $(a, b, c) = (1, 1, 1)$, then we can show similarly as in the proof of Lemma 5.5 that S is birationally equivalent to a triple covering of $\boldsymbol{P}^2$. Hence S has a pencil of trigonal curves. Q.E.D.

We complete the proof of Theorem 6.1 with the following:

LEMMA 6.7. *Let S be a surface of type* (Ic) *with $n=5$. Then the canonical image of S is not cut out by hyperquadrics.*

PROOF. We can assume that S is a surface as in Lemma 6.6. We let C denote a general member of $|L|$. It suffices for our purpose to show that $\Phi_K(C)$ is not cut out by quadrics. We can identify C with its proper transform in $\widetilde{S}$ by σ. Then $\Phi_M(C)$ is linearly equivalent to $6\Delta_0 + 9\Gamma$ on Σ_1. Note that we have $\delta = 0$ by (6.2). It follows that $\Phi_M(C)$ is nonsingular and we can identify $\Phi_M(C)$ with C. The canonical bundle of C is induced by $4\Delta_0 + 6\Gamma$. On the other hand, we already know that it is induced by $3M + 3E$. Since M and $\Delta_0 + 2\Gamma$ are equivalent on C, we see that Δ_0 induces $3E|_C$. Hence C contacts Δ_0 to the third order at $x = E \cap C$. Let $\nu\colon \Sigma \to \Sigma_1$ be the blowing up at x, and put $E_x = \nu^{-1}(x)$. We denote by Δ the proper transform of Δ_0 by ν. Then C is linearly equivalent to $\nu^*(5\Delta + 9\Gamma) + \Delta$ on Σ, and it contacts Δ to the second order at x. Furthermore, $K|_C$ is induced by $H := 2\nu^*(\Delta_0 + 2\Gamma) + \Delta$. We remark that $|H|$ is free from base points and induces a birational map of Σ onto the image. Since we have an exact sequence

$$0 \to \mathcal{O}(-\nu^*(3\Delta_0 + 5\Gamma)) \to \mathcal{O}(H) \to \mathcal{O}_C(K|_C) \to 0,$$

we have $H^0(\Sigma, \mathcal{O}(H)) \simeq H^0(K|_C)$. It follows that $\Phi_K(C)$ is isomorphic to $\Phi_H(C)$. Since $2H - C$ is linearly equivalent to $-\nu^*\Gamma - E_x$, any hyperquadric through $\Phi_H(C)$ contains $\Phi_H(\Sigma)$. Hence $\Phi_K(C)$ is not cut out by hyperquadrics. Q.E.D.

7. Surfaces of type (II).

Let S be a surface of type (II). Then the semi-caconical image V is of degree $2n - 2$ in $\boldsymbol{P}^n$ which is not a ruled surface. The following may be well-known (due to M. Reid?):

LEMMA 7.1. *Let V be an irreducible nondegenerate surface of degree $2n - 2$ in $\boldsymbol{P}^n$. Then the minimal resolution $\tilde{V}$ of V is either a ruled surface or a K3 surface. In the latter case, V is projectively normal and has only rational double points as the singularity.*

PROOF. We assume that V is not a ruled surface. Let τ: $\tilde{V} \to V$ be the natural holomorphic map. We denote by $\tilde{H}$ the pull- back by τ of a hyperplane section of V. We can assume that $|\tilde{H}|$ is free from base points. We take a general member C of $|\tilde{H}|$ which is an irreducible nonsingular curve. Since $\tilde{H}^2 = 2n - 2$, C is of genus $g(C) = \tilde{H}(\tilde{K} + \tilde{H})/2 + 1 = n + \tilde{K}\tilde{H}/2$, where $\tilde{K}$ denotes the canonical bundle of $\tilde{V}$. Since $\tau(C)$ is a nondegenerate curve in $\boldsymbol{P}^{n-1}$ which is birational to C, Castelnuovo's bound (e.g. [7]) implies that $g(C) \leqslant n$. It follows that $\tilde{K}\tilde{H} \leqslant 0$. When $\tilde{K}\tilde{H} < 0$, since $\tilde{H}$ is nef and big, all the pluri-genera of $\tilde{V}$ must vanish. Then, by Castelnuovo-Enriques criterion, $\tilde{V}$ is a ruled surface. Hence $\tilde{K}\tilde{H} = 0$ and $g(C) = n$. In particular, since C attains Castelnuovo's upper bound, $\tau(C) \simeq C$ and it is a canonical curve. Since a general hyperplane section is projectively normal, so is V.

We next show that $\tilde{V}$ is minimal. Let V_0 be the minimal model of $\tilde{V}$, and let σ: $\tilde{V} \to V_0$ be the natural map. Letting K_0 denote the canonical bundle of V_0, we have $\tilde{K} = \sigma^* K_0 + [E]$ with an exceptional divisor E for σ. Since τ is the minimal resolution, we have $\tilde{H}E > 0$ whenever $E \neq 0$. Hence, if $E \neq 0$, we would have $\sigma^* K_0 \tilde{H} < 0$ and it would follow that V_0 is a ruled surface. Therefore, we have $E = 0$ and $\tilde{V}$ itself is the minimal model.

Since $\tilde{K}\tilde{H} = 0$, $\tilde{V}$ is of Kodaira dimension 0. By the Riemann-Roch theorem, $\chi(\tilde{H}) = n - 1 + \chi(\mathcal{O}_{\tilde{V}})$. Since $m\tilde{K}$ is trivial for some positive integer m, it follows from Ramanujam's vanishing theorem that $H^1(\tilde{H}) = 0$, and we clearly have $H^2(\tilde{H}) = 0$. Since $h^0(\tilde{H}) = n + 1$, we get $\chi(\mathcal{O}_{\tilde{V}}) = 2$. By the Enriques-Kodaira classification, we see that $\tilde{V}$ is a $K3$ surface.

Since V is projectively normal, V is isomorphic to the image of $\tilde{V}$ under the holomorphic map associated with $|k\tilde{H}|$ for any positive integer k. It follows that any curve D with $\tilde{H}D > 0$ cannot be contracted to a point via τ. Assume that $\tilde{H}D = 0$ holds for an irreducible curve D on $\tilde{V}$. By Hodge's index theorem, we have $D^2 < 0$. Since $\tilde{K}$ is trivial, the arithmetic genus of D is zero and $D^2 = -2$. Hence D is a (-2)-curve. In sum, we see that V has at most rational double points. Q.E.D.

LEMMA 7.2. *Every surface of type* (II) *is regular.*

PROOF. Let C be a general member of $|L|$, and put $B = \Phi_L(C)$. We can identify B as a canonical curve of genus n in $\boldsymbol{P}^{n-1}$, and the natural map $\nu\colon C \to B$ is of degree 2. Since $L|_C = \nu^* K_B$, we have $K_C = 3\nu^* K_B$. Hence tre ramification divisor and the branch locus of ν are linearly equivalent to $2\nu^* K_B$ and $4K_B$, respectively. Then we have $\nu_* \mathcal{O}_C \simeq \mathcal{O}_B \oplus \mathcal{O}_B(-2K_B)$. It follows $H^0(L|_C) = H^0(\nu^* K_B) \simeq H^0(K_B) \oplus \oplus H^0(-K_B)$. Hence $h^0(L|_C) = n$. Since $h^1(L) = 0$ by Proposition 1.1, we get $h^1(\mathcal{O}_S) = 0$ as in the proof of Lemma 2.2. Q.E.D.

THEOREM 7.3. *Let S be a surface of type* (II). *Let $\nu\colon \tilde{V} \to V$ be the minimal resolution of its semi-canonical image V, and let $\tilde{H}$ be the pull-back on $\tilde{V}$ of a hyperplane section of V. Then S is birationally equivalent to a double covering of the K3 surface $\tilde{V}$ defined in the total space of $[2\tilde{H}]$ by*

$$w^2 + A_2 w + A_4 = 0 \tag{7.1}$$

where w is a fiber coordinate of $[2\tilde{H}]$, and the A_{2i} are in $H^0(\tilde{V}, \mathcal{O}(2i\tilde{H}))$. In particular, the canonical image is cut out by hyperquadrics.

PROOF. Note that we have $H^0(\tilde{V}, \mathcal{O}(k\tilde{H})) \simeq H^0(V, \mathcal{O}(k))$ for any integer k. By the Riemann-Roch theorem, we have $h^0(2\tilde{H}) = 4n - 2$. Since $p_g = 4n - 1 = h^0(2L)$, there is a section $\xi \in H^0(2L)$ which is not induced by $H^0(V, \mathcal{O}_V(2))$. In $H^0(4L)$ we have the following elements:

$$\begin{aligned} &\xi^2, \\ &\alpha\xi, \quad \text{with } \alpha \in f^* H^0(V, \mathcal{O}(2)), \\ &\beta, \quad \text{with } \beta \in f^* H^0(V, \mathcal{O}(4)), \end{aligned}$$

where $f\colon S \to V$ is the natural holomorphic map. Since $h^0(V, \mathcal{O}(4)) = 16n - 14$, these represent $20n - 15$ elements in total. On the other hand, we have $h^0(4L) = h^0(2K) = 20n - 16$. It follows that there is a non-trivial relation among them. In this relation, the coefficient of ξ^2 is not zero. Therefore, we have a relation of the form

$$\xi^2 + \alpha\xi + \beta = 0\,. \tag{7.2}$$

It is easy to see that there are no further relations in the canonical ring of S (modulo those already satisfied on V). Therefore, (7.2) defines a double covering of V which is the canonical model of S. Via ν, it induces a double covering of $\widetilde{V}$ which is defined by eq. (7.1). Conversely, consider a surface defined by (7.1). Since the dualizing sheaf ω is induced by $2\widetilde{H}$, we have $h^0(\omega) = 4n-1$, $h^1(\omega) = 0$ and $\omega^2 = 16n-16$. Therefore, if the coefficients are sufficiently general, we have a nonsingular surface with the desired numerical characters.

The last assertion may be clear. Q.E.D.

8. Surfaces of type (IV): special case.

Let S be a surface of type (IV). We denote by $f\colon S \to V$ the holomorphic map of degree 4 induced by $|L|$, where V is a surface of degree $n-1$ in $\boldsymbol{P}^n$. It is known that V is one of the following (see, e.g. [13], [3]):

(1) $\boldsymbol{P}^2$, $(n=2)$.

(2) A quadric surface $(n=3)$.

(3) The Veronese surface $(n=5)$.

(4) A rational normal scroll of dimension 2, that is, the image of Σ_d by the holomorphic map associated with $|\Delta_0 + ((n-1+d)/2)\Gamma|$, where $n-1-d$ is a nonnegative even integer $(n \geqslant 4)$.

Our result can be summarized in the following:

THEOREM 8.1. *Let S be a surface of type* (IV). *Then it is a regular surface.*

(1) *Assume that $n=2$. Then the canonical model of S is a weighted complete intersection of type* (4, 6) *in the weighted projective space* $\boldsymbol{P}(1, 1, 1, 2, 3)$ *defined by*

$$u^2 + A_1 v + A_4 = v^2 + B_4 u + B_6 = 0\,,$$

where (x_0, x_1, x_2, u, v) *is a system of coordinates with* $\deg x_i = 1$, $\deg u = 2$, $\deg v = 3$ *and the* A_i, B_i *are homogeneous forms of degree* i *in the* x_j.

(2) *Assume that* $n = 3$. *Then the canonical model of* S *is a weighted complete intersection of type* (2, 4, 4) *in the weighted projective space* $\boldsymbol{P}(1, 1, 1, 1, 2, 2)$ *defined by*

$$A_2 = u^2 + B_2 v + B_4 = v^2 + C^2 + C_2 u + C_4 = 0\,,$$

where $(x_0, x_1, x_2, x_3, u, v)$ *is a system of coordinates with* $\deg x_i = 1$, $\deg u = \deg v = 2$, *and the* A_i, B_i, C_i *are homogeneous forms of degree* i *in the* x_j.

(3) *Assume that* $n = 5$ *and* V *is the Veronese surface. Then the canonical model of* S *is a weighted complete intersection of type* (6, 8) *in* $\boldsymbol{P}(1, 1, 1, 3, 4)$ *defined by*

$$u^2 + A_2 v + A_6 = v^2 + B_5 u + B_8 = 0\,,$$

where (x_0, x_1, x_2, u, v) *is a system of coordinates with* $\deg x_i = 1$, $\deg u = 2$, $\deg v = 3$, *and the* A_i, B_i *are homogeneus forms of degree* i *in the* x_j.

(4) *Assume that* $n \geqslant 4$ *and* V *is a rational normal scroll of dimension* 2. *Then* L *induces a holomorphic map* $f\colon S \to \Sigma_d$ *of degree* 4 *such that* $K = f^*[2\Delta_0 + (n-1+d)\Gamma]$, *where* $n-1-d$ *is a nonegative even integer. Furthermore, there exists an integer* k *with* $\max(d, 2) \leqslant$ $\leqslant k \leqslant (n+1)/2$ *such that* S *is birationally equivalent to a surface* S^* *described as follows: Let* $W_{d,k}$ *be the total space of the* 2-*bundle*

$$[2\Delta_0 + (d+k)\Gamma] \oplus [2\Delta_0 + (n+1+d-k)\Gamma] \to \Sigma_d$$

and let (u, v) *denote a system of fiber coordinates.* S^* *is a* 4-*sheeted covering of* Σ_d *defined in* $W_{d,k}$ *by*

$$u^2 + A_1 u + A_2 v + A_3 = v^2 + B_1 uv + B_2 u + B_3 v + B^4 = 0\,,$$

where

$A_1 \in H^0(2\Delta_0 + (d+k)\Gamma)$, $A_2 \in H^0(2\Delta_0 + (d+3k-n-1)\Gamma)$,
$A_3 \in H^0(4\Delta_0 + 2(d+k)\Gamma)$,
$B_1 \in H^0((n+1-2k)\Gamma)$, $B_2 \in H^0(2\Delta_0 + (2n+2+d-3k)\Gamma)$,
$B_3 \in H^0(2\Delta_0 + (n+1+d-k)\Gamma)$, $B_4 \in H^0(4\Delta_0 + 2(n+1+d-k)\Gamma)$.

The canonical image is cut out by hyperquadrics when $n \geqslant 3$.

In this section, we restrict ourselves to the case that V is $\boldsymbol{P}^2$, a quadric surface or the Veronese surface.

LEMMA 8.2. *For a surface of type* (IV), $3q \leqslant n-1$ *holds.*

PROOF. Let C be a general member of $|L|$. It is a nonsingular curve of genus $6n-5$. We have $h^0(K|_C) = 3n+q-2$ and $\deg K|_C = 8n-8$. Since S is a canonical surface, $|K|_C|$ induces a birational map of C onto its image. Then it follows from Castelnuovo's bound that $g(C) \leqslant 7n-3q-6$. Hence $3q \leqslant n-1$. Q.E.D.

LEMMA 8.3. *Let C be a non-hyperelliptic curve of genus* 11. *Assume that $K_C = 5M$ with a line bundle M satisfying $Bs|M| = \emptyset$, $h^0(M) = 2$ and $h^0(2M) = 4$. Then C is a weighted complete intersection of type* (4, 10) *in $\boldsymbol{P}(1, 1, 2, 5)$. In particular, the rational map associated with $|4M|$ is not birational onto its image.*

PROOF. By the free pencil trick, we have the following exact sequence for any $i \geqslant 1$:

$$0 \to H^0((i-1)M) \to H^0(iM) \otimes H^0(M) \xrightarrow{\mu_i} H^0((i+1)M).$$

Let x_0, x_1 be a basis for $H^0(M)$. Since $h^0(2M) = 4$, there is an element $\xi \in H^0(2M)$ linearly independent from the products $x_i x_j$. We have $h^0(4M) = 8$ and $h^0(3M) = 6$ by the Riemann-Roch theorem. Hence μ_3 is surjective and we have a relation of the form

$$\xi^2 = \alpha_2 \xi + \alpha_4,$$

where the α_i are homogeneous forms of degree i in (x_0, x_1). Since μ_4 is of rank 10, there is a new element $\eta \in H^0(5M)$. It is easy to see that μ_i is surjective for $i \geqslant 5$. In $H^0(10M)$, we have $x_0^i x_1^{10-i}$, $x_0^j x_1^{8-j}\xi$ and $x_0^k x_1^{5-k}\eta$ modulo the above relation. Hence there is another relation of the form

$$\eta^2 = \alpha_5 \eta + \alpha_8 \xi + \alpha_{10}.$$

Hence C is isomorphic to a weighted complete intersection of type (4, 10) in $\boldsymbol{P}(1, 1, 2, 5)$. The last assertion is now clear. Q.E.D.

LEMMA 8.4. *Let S be a surface of type* (IV), *and assume that the semi-canonical images is $\boldsymbol{P}^2$, a quadric surface or the Veronese surface. Then S is a regular surface.*

PROOF. By Lemma 8.2, we can assume that V is the Veronese surface. Then, we have a line bundle L_0 such that $L = 2L_0$ and $|L_0|$ induces a holomorphic map of degree 4 onto $\boldsymbol{P}^2$. Let C be a general member of $|L_0|$. Since S is a canonical surface, C is a nonhyperelliptic curve of genus 11. Consider the cohomology long exact sequence for

$$0 \to \mathcal{O}(iL_0) \to \mathcal{O}((i+1)L_0) \to \mathcal{O}_C((i+1)L_0) \to 0\,. \tag{8.1}$$

We have $h^1(\mathcal{O}_S) \leqslant h^1(L_0)$ by (8.1) with $i=0$. By Clifford's theorem, we have $h^0(2L_0|_C) \leqslant 4$. If $h^0(2L_0|_C)=4$, then it follows from Lemma 8.3 that $|K|_C|$ cannot induce a birational map of C. This contradicts that S is canonical. Hence $h^0(2L_0|_C)=3$. Then (8.1) with $i=1$ shows $h^1(L_0)=0$ since $h^1(2L_0)=h^1(L)=0$. It follows that S is a regular surface. Q.E.D.

LEMMA 8.5. *Let S be a surface of type* (IV) *with $n=2$. Then the canonical model of S is a weighted complete intersection in Theorem* 8.1, (1). *Hence the canonical model is not cut out by hyperquadrics. More precisely, the quadric hull of the canonical image is a cone over the Veronese surface* (*defined by $v=0$*).

PROOF. See [11], or one can show it is similar as in the next lemma.

LEMMA 8.6. *Let S be a surface of type* (IV) *with $n=3$. Then the canonical model of S is a weighted complete intersection in Theorem* 8.1, (2). *In particular, the canonical image is cut by hyperquadrics.*

PROOF. Let x_i, $0 \leqslant i \leqslant 3$, be a basis for $H^0(L)$. Since V is a quadric surface in $\boldsymbol{P}^3$, we have a quadric relation $A^2=0$ in the x_i. Therefore, the products x_ix_j span a 9-dimensional subspace of $H^0(2L)=H^0(K)$. Since $p_g=11$, we have two elements ξ, η in $H^0(K)$ which together with x_ix_j form a basis for $H^0(K)$. In $H^0(2K) \simeq \boldsymbol{C}^{44}$, we have the following elements:

quartics in the x_i, (quadrics in the $x_i)\xi$, (quadrics in the $x_i)\eta$,
ξ^2, η^2, $\xi\eta$.

Modulo $A_2=0$, these present 46 sections. Hence we have two non-trivial relations of the form

$$q(\xi, \eta) + B_2\xi + B_2'\eta + B_4 = 0\,,$$

where $q(\xi,\eta)$ is a quadratic form in ξ, η and the B_i are homogeneous forms of degree i in the x_j. We have shown that Φ_L can be lifted to a holomorphic map of S into $\boldsymbol{P}(1,1,1,1,2,2)$ by putting $u=\xi$, $v=\eta$. Since Φ_L induces a holomorphic map of degree 4 onto the quadric surface, the defining equation of the image can be normalized as in Theorem 8.1, (2). Q.E.D.

LEMMA 8.7. *Let S be a surface of type* (IV) *with $n = 5$, and assume that V is the Veronese surface. Then the canonical model of S is a weighted complete intersection in Theorem* 8.1, (3). *In particular, the canonical image is cut out by hyperquadrics.*

PROOF. Let L_0 be as in the proof of Lemma 8.4. As we saw there, we have $h^1(L_0) = 0$. By the Riemann-Roch theorem, we have $\chi(L_0) = 14$ and we get $h^0(3L_0) = 11$. Let x_0, x_1, x_2 denote a basis for $H^0(L_0)$. Then the products $x_i x_j x_k$ generate a subspace of dimension 10 in $H^0(3L_0)$. Hence, we have a new element $\xi \in H^0(3L_0)$. We have $h^0(4L_0) = h^0(K) = 19$. In $H^0(4L_0)$, we have 18 elements:

$$(\text{quartics in the } x_i), \quad x_i \xi$$

which are clearly linearly independent. Therefore, we have a new element $\psi \in H^0(4L_0)$. We have $h^0(6L_0) = 44$ by the Riemann-Roch theorem and Ramanujam's vanishing theorem. In $H^0(6L_0)$, we have the following elements:

$$\text{sextics in the } x_i, \qquad (\text{cubics in the } x_i)\xi,$$
$$(\text{quadrics in the } x_i)\psi, \quad \xi^2.$$

It follows that there is a non-trivial relation among them. In this relation, the coefficient of ξ^2 cannot be zero. Hence, we can assume that it is of the form

$$\xi^2 + A_2\psi + A_6 = 0.$$

We look at $H^0(8L_0) = H^0(2K)$ which is of dimension 84. In $H^0(8L_0)$, we have the following elements modulo the above relation:

$$\text{octics in the } x_i, \qquad (\text{quintics in the } x_i)\xi,$$
$$(\text{quartics in the } x_i)\psi, \quad x_i\xi\psi, \psi^2.$$

These represent 85 sections. Hence, we have a non-trivial relation of the form

$$\psi^2 + B_5\xi + B_8 = 0.$$

We have shown that $S \to \boldsymbol{P}^2$ can be lifted to a holomorphic map into $\boldsymbol{P}(1, 1, 1, 3, 4)$ by putting $u = \xi$ and $v = \psi$. The image is as in Theorem 8.1, (3). It is easy to see that it coincides with the canonical model. Q.E.D.

9. Surfaces of type (IV): general case.

We assume that $n \geqslant 4$ and that V is a rational normal scroll of dimension 2.

The following can be shown as in [8, Lemma 1.5]:

LEMMA 9.1. *Assume that V is a cone over a rational curve of degree $n-1$ in $\boldsymbol{P}^{n-1}$, $n \geqslant 4$. Then the semi-canonical map can be lifted to a holomorphic map $\tilde{f}\colon S \to \Sigma_{n-1}$ of degree 4 satisfying $L = \tilde{f}^*[\Delta_0 + (n-1)\Gamma]$.*

Hence we have a holomorphic map $f\colon S \to \Sigma_d$ of degree 4 satisfying $L = f^*[\Delta_0 + ((n-1+d)/2)\Gamma]$ where $n-1-d$ is a nonnegative even integer. We put $D = f^*\Gamma$. Since $LD = 4$, we see that $|D|$ is a pencil of curves of genus 5. We let $\lambda\colon S \to \boldsymbol{P}^1$ denote the fibration induced by $|D|$. Then $\lambda_* \mathcal{O}(K)$ is of rank 5. Since $K = f^*[2\Delta_0 + (n-1+d)\Gamma]$, we have a subsheaf $p_*\mathcal{O}(2\Delta_0 + (n-1+d)\Gamma)$ of $\lambda_*\mathcal{O}(K)$, where $p\colon \Sigma_d \to \boldsymbol{P}^1$ is the projection map. The quotient bundle is of the form $\mathcal{O}(n_1) \oplus \mathcal{O}(n_2)$, $n_1 \leqslant n_2$, and we have an exact sequence

$$(9.1) \quad 0 \to p_*\mathcal{O}(2\Delta_0 + (n-1+d)\Gamma) \to \lambda_*\mathcal{O}(K) \to \mathcal{O}(n_1) \oplus \mathcal{O}(n_2) \to 0\,.$$

Note that we have

$$p_*\mathcal{O}(2\Delta_0 + (n-1+d)\Gamma) \simeq \mathcal{O}(n-1-d) \oplus \mathcal{O}(n-1) \oplus \mathcal{O}(n-1+d)\,.$$

LEMMA 9.2. *If V is a rational normal scroll of dimension 2, then S is a regular surface.*

PROOF. Assume that S is an irregular surface. By [15, Theorem 1], the slope of λ is not less than 4, that is, $K^2 \geqslant 4\chi(\mathcal{O}_S) - 16$. Since the equality holds here, it follows from [15, Theorem 3] that $q = 1$ and $\lambda_*\omega_{S/\boldsymbol{P}^1}$ is a direct sum of $\mathcal{O}$ and a semi-stable sheaf of rank 4. Hence $\lambda_*\mathcal{O}(K) \simeq \mathcal{O}(n-1)^{\oplus 4} \oplus \mathcal{O}(-2)$. Since (9.1) shows that $\mathcal{O}(n-1+d)$ is a subsheaf of $\lambda_*\mathcal{O}(K)$, we get $d = 0$. Since $h^0(\lambda_*\mathcal{O}(K)) = 4n$ and $h^1(\lambda_*\mathcal{O}(K)) = 1$, we have $n_1 = -2$ and $n_2 = n-1$ in (9.1). This implies that there is an element $\xi \in H^0(K - (n-1)D)$ which is not contained in $f^*H^0(\Sigma_0, \mathcal{O}(2\Delta_0))$. Let $D_0 \in |f^*\Delta_0|$ be a general member. Then $h^0(2D_0)$ must be strictly greater than $3 = h^0(2\Delta_0)$. On the other hand, it follows from the cohomology long exact sequence for

$$0 \to \mathcal{O}(D_0) \to \mathcal{O}(2D_0) \to \mathcal{O}_{D_0} \to 0$$

that $h^0(2D_0) \leqslant h^0(D_0) + h^0(\mathcal{O}_{D_0}) = 3$. Hence S is not an irregular surface. Q.E.D.

LEMMA 9.3. *Let S and D be as above. Then any general member of $|D|$ is a weighted complete intersection of type* (4,4) *in the weighted projective space* $\boldsymbol{P}(1, 1, 2, 2)$ *defined by*

$$u^2 = A_2 v + A_4\,, \qquad v^2 = B_2 u + B_4\,,$$

where (z_0, z_1, u, v) is a system of coordinates with $\deg z_0 = \deg z_1 = 1$, $\deg u = \deg v = 2$, *and the A_i, B_i are homogeneous forms of degree i in* (z_0, z_1).

PROOF. Let L_D denote the restriction of L to a general member D of $|D|$. Then L_D induces a holomorphic map $f_D\colon D \to \boldsymbol{P}^1$ of degree 4. Let ζ_0, ζ_1 be a basis for $H^0(L_D)$. Then ζ_0^2, $\zeta_0\zeta_1$ and ζ_1^2 generated a subspace of dimension 3 in $H^0(2L_D)$. Since $h^0(2L_D) = h^0(K_D) = 5$, there exist two new elements ξ, η in $H^0(2L_D)$. We have $h^0(4L_D) = 12$. In $H^0(4L_D)$, we have the following 14 elements:

$$\text{quartics in the } \zeta_i\,, \qquad (\text{quadrics in the } \zeta_i)\xi\,,$$
$$(\text{quadrics in the } \zeta_i)\eta\,, \quad \xi^2, \eta^2, \xi\eta\,.$$

Among them, the first 11 elements are clearly linearly independent. Hence, we have two non-trivial relations of the form

$$c_{11}\xi^2 + c_{12}\xi\eta + c_{13}\eta^2 = A_2\xi + A_2'\eta + A_4\,,$$
$$c_{21}\xi^2 + c_{22}\xi\eta + c_{23}\eta^2 = B_2\xi + B_2'\eta + B_4\,,$$

where the c_{ij} are constants and the A_k, B_k are homogeneous forms of degree k in (ζ_0, ζ_1). Since the matrix (c_{ij}) is of rank 2, by a suitable linear change among ξ and η, the relations can be normalized as

(i) $\xi^2 = A_2\xi + A_2'\eta + A_4\,,\ c_{22}\xi\eta + \eta^2 = B_2\xi + B_2'\eta + B_4$, or

(ii) $\xi^2 = A_2\eta + A_2'\xi + A_4\,,\ \xi\eta = B_2\xi + B_2'\eta + B_4$.

We can lift f_D to a holomorphic map into $\boldsymbol{P}(1,1,2,2)$ by putting $u = \xi$, $v = \eta$. Since D is nonhyperelliptic, the image is isomorphic to D. Since f_D is of degree 4, the case (ii) is inadequate. Thus we have (i). Then, replacing ξ by $\xi + A_2'/2$, we can assume that $A_2' = 0$. By a similar change of η, we can assume that $B_0 = B_2' = 0$. Hence D is isomorphic to the weighted complete intersection as in the statement. Q.E.D.

We relative the above lemma in order to get a birational model of S. Since $p_g = 4n - 1$ and $q = 0$, we have $-1 \leqslant n_1 \leqslant n_2$ and $n_1 + n_2 = n - 3$ in (9.1). Putting $k := n_1 + 2$, we get $n_2 = n - 1 - k$ and $1 \leqslant k \leqslant (n + 1)/2$. By the construction, we have two independent elements $\xi \in H^0(K - n - 1 - k)D)$ and $\eta \in H^0(K - (k-2)D)$ which are not contained in $f^* H^0(2\Delta_0 + (d+k)\Gamma)$ and $f^* H^0(2\Delta_0 + (n+1+d-k)\Gamma)$, respectively. Let ζ_0 and ζ_1 be sections of $[\Delta_0]$ and $[\Delta_0 + d\Gamma]$, respectively, such that they form a system of homogeneous coordinates on fibres of $p: \Sigma_d \to \boldsymbol{P}^1$. Then ζ_0^2, $\zeta_0\zeta_1$, ζ_1^2, ξ, η induce a basis for $H^0(K_D)$.

LEMMA 9.4. *Let m be an integer with $m \geqslant d$. Then $h^1(2K-(n-1-m)D)=0$ and $h^0(2K-(n-1-m)D) = 8n+12m-4$, unless $(d, m) = (0, 0)$.*

PROOF. By the Riemann-Roch theorem, we have

$$\chi(2K-(n-1-m)D) = 8n-4+12m\,.$$

Note that $K-(n-1-)D = f^*[2\Delta_0 + (d+m)\Gamma]$. Hence $h^2(2K-(n-1-m)D)=0$. Since $(d, m) \neq (0, 0)$, $|K-(n-1-m)D|$ contains a connected member. Hence $H^1(2K-(n-1-m)D)=0$. Q.E.D.

Now, we are ready to prove Theorem 8.1, (4). For simplicity, we shall use the following notation: For any nonnegative integer i, we let α_i, β_i and γ_i represent elements in $H^0(4\Delta_0 + (2n+2d-i)\Gamma)$, $H^0(2\Delta_0 + (2n+d-k-i)\Gamma)$ and $H^0(2\Delta_0 + (n-1+d+k-i)\Gamma)$, respectively.

Since $d \leqslant n-1$, it follows from Lemma 9.4 that the restriction map $H^0(2K+2D) \to H^0(2K_D)$ is surjective and that $H^0(2K+2D)$ is of dimension $20n+8$. In $H^0(2K+2D)$, we have elements of the forms

$$\alpha_0\,, \qquad \beta_0\xi\,, \qquad \gamma_0\eta$$

which generate a subspace of dimension $19n+8$, and

$$\begin{aligned} &x_0^i x_1^{2n-2k-i}\xi^2 && (0 \leqslant i \leqslant 2n-2k)\,,\\ &x_0^i x_1^{n-1-i}\xi\eta && (0 \leqslant i \leqslant n-1)\,,\\ &x_0^i x_1^{2k-2-i}\eta^2 && (0 \leqslant i \leqslant 2k-2)\,, \end{aligned}$$

where (x_0, x_1) is a basis for $H^0(D)$. Since there are $3n$ elements in the second group, we have at least $2n$ relations of the form

$$a_{2n-2k}\xi^2 + b_{n-1}\xi\eta + c_{2k-2}\eta^2 = \alpha_0 + \beta_0\xi + \gamma_0\eta\,, \tag{9.2}$$

where a_i, b_i and c_i are homogeneous forms of degree i in (x_0, x_1). When restricted to a general fiber D, they give only two independent relations described in the proof of Lemma 9.3. Let μ and ν be nonnegative integers with $\mu \geqslant \nu$ such that we can find relations

$$a_{2n-2k-\mu}\xi^2 + b_{n-1-\mu}\xi\eta + c_{2k-2-\mu}\eta^2 = \alpha_\mu + \beta_\mu\xi + \gamma_\mu\eta \tag{9.3}$$

in $H^0(2K - (\mu - 2)D)$ and

$$(9.4) \qquad a'_{2n-2k-\nu}\xi^2 + b'_{n-1-\nu}\xi\eta + c'_{2k-2-\nu}\eta^2 = \alpha'_\nu + \beta'_\nu\xi + \gamma'_\nu\eta$$

in $H^0(2K - (\nu - 2)D)$ which induce the independent relations in $H^0(2K_D)$. (In (9.4), we put «dash» in order to distinguish the notation when $\mu = \nu$.) We can choose (μ, ν) so that it is maximum in lexicographic order among those with such a property.

LEMMA 9.5. $\mu = 2n - 2k$ and $\nu = 2k - 2$.

PROOF. If $\mu > 2n - 2k$, then the left hand side of (9.3) is zero, which is absurd. If $\nu > 2k - 2$, then, in the left hand sides of (9.3) and (9.4), the coefficients of η^2 are both zero. Then, restricting to D, we would get relations of the form (ii) in the proof of Lemma 9.3, which is impossible. Hence we have $\mu \leqslant 2n - 2k$ and $\nu \leqslant 2k - 2$. On the other hand, since all the $2n$ relations of the form (9.2) must be derived from (9.3) and (9.4), we get $(\mu + 1) + (\nu + 1) \geqslant 2n$. Hence $4m = 2n - 2k$ and $\nu = 2k - 2$. Q.E.D.

When $2k < n + 1$, in (9.3), the coefficients of $\xi\eta$ and η^2 are both zero. Hence we can assume $a_0 = 1$ and eliminate ξ^2 from (9.4) by using (9.3). In this way, we get two independent relations

$$(9.5) \qquad \begin{cases} \xi^2 = \alpha_{2m-2k} + \beta_{2n-2k}\xi + \gamma_{2n-2k}\eta\,, \\ \eta^2 + b_{n+1-2k}\xi\eta = \alpha'_{2k-2} + \beta'_{2k-2}\xi + \gamma_{2k-2}\eta\,. \end{cases}$$

Even when $2k = n + 1$, (9.3) and (9.4) can be reduced to (9.5) by a suitable linear change among ξ and η as in the proof of Lemma 9.3.

LEMMA 9.6. *The condition* $k \geqslant \max(d, 2)$ *holds. In particular, the exact sequence* (9.1) *splits.*

PROOF. We put $G = f^*\varDelta_0$.

We first show that $k \geqslant d$. Since $k \geqslant 1$, we assume that $d \geqslant 2$. We assume that $k < d$ and show that this leads us to a contradiction. Since $\xi \in H^0(2G + (d + k)D)$, we have

$$(9.6) \qquad h^0(2G + (d + k)D) > h^0(2\varDelta_0 + (d + k)\varGamma) = d + 2k + 2\,.$$

We have

$$(2\Delta_0 + (d+k)\Gamma)\Delta_0 = k - d < 0\,,$$

$$(2\Delta_0 + (d+3k-n-1)\Gamma)\Delta_0 = (2k-n-1) + (k-d) < 0\,.$$

Hence $\alpha_{2n-2k}, \beta_{2n-2k}$ and γ_{2n-2k} vanish identically on Δ_0 It follows from (9.5) that ξ vanishes identically on G. When $2k = n+1$, we similarly see that ξ and η both vanish identically on G. Hence, we have $|2G + (d+k)D| = G + |G + (d+k)D|$.

Assume that $2(d+k) \leqslant n-1+d$. Since $L = G + ((n-1+d)/2)D$ and since $h^0(L) = h^0(\Delta_0 + ((n-1+d)/2)\Gamma) = n+1$, we get $h^0(2G + (d+k)D) = h^0(\Delta_0 + (d+k)\Gamma) = d + 2k + 2$, which contradicts (9.6). Assume that $2(d+k) > n-1+d$. We put $m = d+k-(n-1+d)/2$ and choose m general members $D_1, \dots, D_m$ of $|D|$. Since $H^1(L) = 0$ by Proposition 1.1, we get $h^0(G + (d+k)D) = n+1+2m = d+2k+2$ from the cohomology long exact sequence for

$$0 \to \mathcal{O}(L) \to \mathcal{O}(G + (d+k)D) \to \bigoplus_{i=1}^{m} \mathcal{O}_{D_i}(L) \to 0\,.$$

Hence we get $h^0(2G + (d+k)D) = d + 2k + 2$ which again contradicts (9.6).

We next show that $k = 1$ is inadequate. Assume that $(d, k) = (0, 1)$. We can assume that G is a nonsingular curve which is non-hyperelliptic. We consider the cohomology long exact sequence for

$$0 \to \mathcal{O}(G + D) \to \mathcal{O}(2G + D) \to \mathcal{O}_G(D) \to 0\,.$$

Since $DG = 4$, we get $h^0(\mathcal{O}_G(D)) \leqslant 2$ by Clifford's theorem. Since $G + D = L - ((n-3)/2)$, we have $h^0(G + D) = h^0(\Delta_0 + \Gamma) = 4$ and $h^2(G + D) = h^0(L + ((n-3)/2)D) = 2n-2$. It follows from the Riemann-Roch theorem that $H^1(G + D) = 0$. Hence, $h^0(2G + D) = h^0(G + D) + h^0(\mathcal{O}_G(D)) \leqslant 6$. On the other hand, we have $h^0(2\Delta_0 + \Gamma) = 6$. Hence $h^0(2G + D) = h^0(2\Delta_0 + \Gamma)$ and ξ cannot be in $H^0(2G + D)$.

Assume that $(d, k) = (1, 1)$. We choose a general member $C \in |G + D|$ and consider the cohomology long exact sequence for

$$0 \to \mathcal{O}(G + D) \to \mathcal{O}(2G + 2D) \to \mathcal{O}_G(2D) \to 0\,.$$

We can assume that C is a nonsingular curve of genus $2n+3$ which is nonhyperelliptic. Then, we have $h^0(2D|_C) \leqslant 4$ by Clifford's theorem. Similarly as in the previous case, we can show that $h^0(G + D) = 3$ and $h^1(G + D) = 0$. It follows $h^0(2G + 2D) = h^0(G + D) + h^0(2D|_C) \leqslant 7$. On the other hand, we have $h^0(2\Delta_0 + 2\Gamma) = 6$. Hence, if ξ is in $H^0(2G + 2D)$, we must have $h^0(2G + 2D) = 7$ and $h^0(2D|_C) = 4$. Since the canonical bundle of C is induced by $(n+1)D$ and since $K|_C = nD|_C$, we have $h^0(K|_C) = 2n$ by the Riemann-Roch theorem. We denote the restrictions of the x_i and ξ to C by the same symbols. Then x_0^2, x_0x_1, x_1^2 and ξ form a basis for $H^0(2D|_C)$. Restricting the first equation of (9.5) to C, we have

$$\xi^2 + \phi_2\xi + \phi_4 = 0\,, \tag{9.7}$$

where the ϕ_i are homogeneous forms of degree i in (x_0, x_1). This is a relation in $H^0(4D|_C)$. Modulo (9.7), we have $2n$ elements $x_0^i x_1^{n-i}$, $x_0^i x_1^{n-2-i}\xi$ in $H^0(K|_C)$. Since these are clearly linearly independent, (9.7) implies that the image of C under the canonical map of S is an elliptic curve. This contradicts that S is a canonical surface. Q.E.D.

Let $W_{d,k}$ and (u, v) be the same as in Theorem 8.1, (4). If we put $u = \xi$ and $v = \eta$, we can lift $f\colon S \to \Sigma_d$ to a map $h\colon S \to W_{d,k}$. By (9.5), we see that $S^* = h(S)$ is defined in $W_{d,k}$ by the equation in Theorem 8.1, (4). We remark that there are no further relations among ξ, η. Hence S^* is the canical model (possibly partially resolved its rational double points). Conversely, it follows from (9.5) and Lemma 9.6 that the surface S^* defined by the equation in Theorem 8.1, (4) is nonsingular provided that the coefficients are sufficiently general. Furthermore, its dualizing sheaf ω_{S^*} is induced by $[2\Delta_0 + (n-1+d)\Gamma]$. To see this, we consider a compactification

$$\varpi\colon \widehat{W}_{d,k} =$$

$$= \boldsymbol{P}(\mathcal{O} \oplus \mathcal{O}(-2\Delta_0 - (d+k)\Gamma) \oplus \mathcal{O}(-2\Delta_0 - (n+1+d-k)\Gamma)) \to \Sigma_d$$

of $W_{d,k}$. Let T_0 denote a tautological divisor. Then S^* is a complete intersection of two hypersurfaces linearly equivalent to $2T_0 + 2\varpi^*(2\Delta_0 + (d+k)\Gamma)$ and $2T_0 + 2\varpi^*(2\Delta_0 + (n+1+d-k)\Gamma)$, respectively. Since the canonical bundle of $\widehat{W}_{d,k}$ is induced by $-3T_0 - \varpi^*(6\Delta_0 + (n+3+3d)\Gamma)$, we see that the dualizing sheaf of S^* is induced by $T_0 + \varpi^*(2\Delta_0 + (n-1+d)\Gamma)$. Since $[T_0]$ is trivial on S^* we see that ω_{S^*} comes from $[2\Delta_0 + (n-1+d)\Gamma]$ as desired. By a spectral sequence argument using the Koszul resolution

$$0 \to \mathcal{O}(-4T_0 - 2\varpi^*(4\Delta_0 + (n+1+2d)\Gamma)) \to$$

$$\to \mathcal{O}(-2T_0 - 2\varpi^*(2\Delta_0 + (d+k)\Gamma)) \oplus$$

$$\oplus \mathcal{O}(-2T_0 - 2\varpi^*(2\Delta_0 + (n+1+d-k)\Gamma)) \to \mathcal{O}_{\widehat{W}_{d,k}} \to \mathcal{O}_{S^*} \to 0\,,$$

we can show that $H^0(T_0 + \varpi^*(2\Delta_0 + (n-1+d)\Gamma))$ is restricted onto $H^0(\omega_{S^*})$ isomorphically. Note that $|T_0 + \varpi^*(2\Delta_0 + (n+1+d-k)\Gamma)|$ induces a birational map of $\widehat{W}_{d,k}$ onto the image. Since we have $k \geqslant 2$ by Lemma 9.6, S^* is surely a canonical surface. Hence S^* is an even canonical surface with $K^2 = 4p_g - 12$, $q = 0$ as a standard calculation shows.

It should be clear from (9.5) that the canonical image of S is cut out by hyperquadrics. In sum, Theorem 8.1, (4) has been shown.

10. Final remarks.

The results in the preceding sections imply the following:

THEOREM 10.1. *Let S be an even canonical surface with $K^2 = 4p_g - 12$, $q = 0$. Assume that the canonical image X is cut out by hyperquadrics. Then $|K|$ is free from base points and X is projectively normal.*

THEOREM 10.2. *Let S be an irregular even canonical surface with $K^2 < 4\chi(\mathcal{O}_S) - 12$. Then S a pencil of trigonal curves and the quadric hull of the canonical image X containing X is of dimension* 3. *In particular, Reid's conjecture* [14, p. 541] *is true for even surfaces with* $q = 1$.

PROOF. Since S is an even surface, we can assume that $K^2 = 4\chi - 16$ by [Part I, Theorem 8.4]. Then, as we saw, S is of type (III) or (V). Hence S has a pencil of trigonal curves. The rest follows from [Part I, Theorem 8.3]. Q.E.D.

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