RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

JOHN C. LENNOX
JAMES WIEGOLD

A result about cosets

Rendiconti del Seminario Matematico della Università di Padova, tome 93 (1995), p. 185-186
<http://www.numdam.org/item?id=RSMUP_1995__93__185_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 93 (1995)

A Result About Cosets.

JOHN C. LENNOX - JAMES WIEGOLD (*) (*)

Marking some final year honours exercises on right coset representations has led us to the following problem:

When is it the case that every proper non-trivial subgroup H of a finite group G has a coset Hx consisting of elements of one and the same order $a(x, H)$?

We call finite groups with this property CSO-groups. It is not surprising that CSO-groups are rare. However, they are not hugely uncommon either.

THEOREM. A soluble group G is a CSO-group if and only if G is a p-group and $G \backslash \Phi(G)$ consists of elements of the same order. Therefore, for every soluble CSO-group, there exists a number α depending only on G such that every proper non-trivial subgroup has a coset Hx consisting of elements of order α.

Solubility is an essential ingredient in our proof. Indeed we would make the following

CONJECTURE. Every CSO-group is soluble.

Quite possibly, one would need the classification theorem for simple groups to verify this! It is easy to see that the alternating groups of degree more than 4 are not CSO-groups.

Turning now to the proof of the theorem, let G be a soluble CSO-

(*) Indirizzo degli AA.: School of Mathematics, University of Wales, College of Cardiff, Cardiff CF2 4YH, U.K.

group and M a maximal normal subgroup. Then M is of prime index p, say, and there is a coset Mx consisting of elements of the same order. Since x has order p mod M, x must have p-power order p^t, say, so that Mx consists of elements of that order.

We claim that $G \backslash M$ consists of elements of order p^t. To see this, consider any coset Mx^i with $1 \leq i < p$: every element of $G \backslash M$ is in such a coset. Let j be a positive integer such that $ji \equiv 1 \bmod p$. For every element mx^i of Mx^i, we have $(mx^i) = m'x$ for some $m' \in M$. But $m'x$ has order p^t; since $(j, p) = 1$, so does mx^i.

Thus $G \backslash M$ consists of elements of order p^t. A simple count shows that every maximal normal subgroup N must have the same index p as M and that the elements of $G \backslash N$ have order p^t. Therefore G/G' is a p-group.

We claim that G is a p-group. If not, we can choose a non-trivial Hall p'-subgroup Q of G inside G' and a Sylow p-subgroup P permuting with Q, so that $G = PQ$. By the CSO-property, P has a coset Py consisting of elements of p'-order. Thus $Py \subseteq G'$; since $y \in G'$, we have $P \subseteq G'$ and $G = G'$, a contradiction. Thus G is a p-group after all, and by the first part of the proof, $G \backslash \Phi(G)$ consists of elements of the same order $p^t = \alpha$.

Conversely, let G be a p-group such that $G \backslash \Phi(G)$ consists of elements of the same order α. Let H be a proper non-trivial subgroup and M a maximal subgroup containing H. For $x \in G \backslash M$ we have $Hx \subseteq G \backslash M$, so that Hx consists of elements of order α, as required. This completes the proof.

Obvious examples of CSO-groups are groups of prime exponent. Less obvious are the second nilpotent products of cyclic p-groups of the same odd order. A full classification is probably out of the question.

Manoscritto pervenuto in redazione il 28 settembre 1993.