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Homoclinic Orbits on Non-Compact Riemannian Manifolds for Second Order Hamiltonian Systems.

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ABSTRACT - We consider second order Hamiltonian systems on non-compact Riemannian manifolds. We prove the existence of a nontrivial homoclinic orbit under conditions related to the global superquadratic condition of Rabinowitz in [17].

0. Introduction.

In this paper, we study the existence of homoclinic orbits on a complete connected noncompact Riemannian manifold M of class C^3. For a given function $V(t, x) \in C^2(\boldsymbol{R} \times M, \boldsymbol{R})$, we consider the second order Hamiltonian system:

$$D_t \dot{x}(t) + \operatorname{grad}_x V(t, x(t)) = 0 \quad \text{in } \boldsymbol{R} \tag{0.1}$$

where $\dot{x}(t)$ denotes the derivative of $x(t)$ with respect to t, $D_t \dot{x}(t)$ the covariant derivative of $\dot{x}(t)$ and $\operatorname{grad}_x V(t, x)$ the gradient of $V(t, x)$ with respect to the variable x.

Let $x_0 \in M$ be a point such that

$$V(t, x_0) = 0, \qquad \operatorname{grad}_x V(t, x_0) = 0 \qquad \text{for all } t \in \boldsymbol{R}.$$

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We say a solution $x(t)$ of (0.1) is a homoclinic orbit emanating from x_0 if and only if

$$x(t) \to x_0\,, \quad \dot{x}(t) \to 0 \quad \text{as } t \to \pm\infty\,. \tag{0.2}$$

Our aim is to derive conditions which ensure the existence of a homoclinic orbit emanating from x_0.

This kind of problems has recently has recently been extensively studied via variational methods. See [1, 2, 5, 7, 11, 16, 17, 18, 22, 24] for homoclinic orbits or heteroclinic orbits in $\boldsymbol{R}^N$ and [3, 4, 8, 9, 12] for homoclinic orbits on Riemannian manifolds. See also [6, 10, 19-21, 23] for first order Hamiltonian systems.

To our knowledge, so far the existence of homoclinic orbits on Riemannian manifold has been studied under two types of conditions.

In [8, 9, 12], $V(t, x)$ is periodic in t and satisfies

$$V(t, x) \leq 0\,, \quad \text{for all } t, x\,,$$
$$V(t, x) = 0\,, \quad \text{if and only if } x = x_0\,,$$

while in [3], $V(t, x)$ verifies the above assumptions but it is time independent.

In [4] (cf. [1, 18]), the potential is time independent and $x_0 \in M$ is a local maximum of $V(x)$ such that $V(x_0) = 0$. The existence of a homoclinic orbit emanating from x_0 is proved under the assumptions that the set $\Omega = \{x \in M;\ V(x) < 0\} \cup \{x_0\}$ is open and bounded and grad $V(x) \neq 0$ for all $x \in \partial\Omega$.

In this paper, we consider the existence of a homoclinic orbit in the situation where $V(t, x)$ depends on t and changes sign on M. As far as we know, such a situation is studied only in the case $M = \boldsymbol{R}^N$ and the following global superquadratic condition is assumed ([7, 17])

(GSQ) $V(t, x)$ is a periodic function in t of the form

$$V(t, x) = -\frac{1}{2}(L(t)x, x) + W(t, x) \qquad \text{for all } t, x\,,$$

where $L(t)$ is a positive definite symmetric matrix depending on t continuously and $W(t, x)$ satisfies, for some $\mu > 2$,

$$W(t, 0) \equiv 0\,,$$
$$0 < \mu W(t, x) \leq (W_x(t, x), x) \quad \text{for all } t, x \neq 0\,.$$

Our main purpose is to shown the existence of a homoclinic orbit on a Riemannian manifold under a condition which is a generalization of (GSQ). We also show the existence of a homoclinic orbit in $\boldsymbol{R}^N$ under weaker condition than (GSQ). See Remark 0.2 and Example 0.4 below.

To state our result, we need some notations: let $\langle \cdot, \cdot \rangle_x$ be the Riemannian structure of M. For $W(x) \in C^2(M, \boldsymbol{R})$, $\operatorname{grad} W(x)$ and $H^W(x)[v, v]$ will denote the Riemannian gradient and the Riemannian Hessian of W, i.e.,

$$\langle \operatorname{grad} W(x), v \rangle_x = \frac{d}{ds}\bigg|_{s=0} W(\gamma(s)),$$

$$H^W(x)[v, v] = \frac{d^2}{ds^2}\bigg|_{s=0} W(\gamma(s)) \qquad \text{for } v \in T_x M,$$

where $\gamma(s)$ is the geodesic such that $\gamma(0) = x$, $\dot{\gamma}(0) = v$. In case $W(t, x) \in C^2(\boldsymbol{R} \times M, \boldsymbol{R})$ also depends on $t \in \boldsymbol{R}$, we denote by $\operatorname{grad}_x W(t, x)$ and $H_x^W(t, x)[v, v]$ the Riemannian gradient and the Riemannian Hessian of W respect to x.

We assume

(V0) $V \in C^2(\boldsymbol{R} \times M, \boldsymbol{R})$ is 1-periodic in t,

(V1) $V(t, x_0) = 0$, $\operatorname{grad}_x V(t, x_0)$ for all $t \in \boldsymbol{R}$ and

$$H_x^V(t, x_0)[v, v] < 0 \qquad \text{for all } v \in T_{x_0} M \setminus \{0\},$$

(V2) the set $\Omega(t) = \{x \in M;\, V(t, x) \leqslant 0\}$ is compact for all t and $\operatorname{grad}_x V(t, x) \neq 0$ for all $x \in \partial\Omega(t)$ and $t \in \boldsymbol{R}$,

(V3) $\liminf_{d(x, x_0) \to \infty} \inf_{t \in \boldsymbol{R}} V(t, x) > 0$.

(V4) $V(t, x)$ is of the form:

$$V(t, x) = -\psi(t, x) + W(t, x),$$

where $\psi(t, x)$, $W(t, x) \in C^2(\boldsymbol{R} \times M, \boldsymbol{R})$ are 1-periodic in t. Moreover there is a function $\varphi(x) \in C^2(M, \boldsymbol{R})$ such that

(φ1) $\operatorname{grad} \varphi(x_0) = 0$,

(φ2) there exists $\mu > 0$, $c_0 \in (0, 1/2)$, $c_0' > 0$ such that for all $x \in M$ and $v \in T_x M$,

$$\frac{1}{\mu} H^\varphi(x)[v, v] \leqslant \left(\frac{1}{2} - c_0\right) \langle v, v \rangle_x,$$

$$|H^\varphi(x)[v, v]| \leqslant c_0' \langle v, v \rangle_x,$$

(φ1) $\psi(t, x) \geqslant 0$ for all $t \in \boldsymbol{R}$ and $x \in M$,

(φ2) there exists $c_0'' \in (0, \mu)$ such that

$$\langle \operatorname{grad}_x \psi(t, x), \operatorname{grad} \varphi(x) \rangle_x \leqslant (\mu - c_0'')\psi(t, x)$$

for all $t \in \boldsymbol{R}$ and $x \in M$,

(w) $\langle \operatorname{grad}_x W(t, x), \operatorname{grad} \varphi(x) \rangle_x \geqslant \max\{\mu W(t, x), 0\}$ for all $t \in \boldsymbol{R}$ and $x \in M$.

Now we state our main result.

THEOREM 0.1. *Let M be as above and assume* (V0)-(V4). *Then* (0.1)-(0.2) *has at least one non- trivial homoclinic orbit emanating from x_0.*

The following observations clarify the meaning of conditions (V0)-(V4).

REMARK 0.2. Let $M = \boldsymbol{R}^N$ with a standard Euclidean metric and assume $V(t, x)$ satisfies condition (GSQ). Setting $\varphi(x) = 1/2\,|x|^2$ and $\psi(t, x) = 1/2\,(L(t)x, x)$, we see conditions (V0)-(V4) are satisfied. We obtain conditions (V0)-(V4) as a first trial to generalize condition (GSQ) of Rabinowitz [17]. We hope that they may be improved.

REMARK 0.3. (i) In the condition (V4), we can take $\varphi(t, x) \equiv 0$. In this case, $V(t, x) = W(t, x)$ and if we take a $\varphi(x) \in C^2(M, \boldsymbol{R})$ such that

$$\operatorname{supp} \operatorname{grad} \varphi(x) \subset \bigcup_{t \in \boldsymbol{R}} \{x \in M;\ \operatorname{dist}(x, M \backslash \Omega(t)) < \delta\} \equiv A$$

for some $\delta > 0$, the condition (V4)-(w) is clearly satisfied for $x \in M \backslash A$ and it can be regarded as a condition on the set A, that is, a condition on the behavior of $V(t, x)$ in a neighborhood of $\Omega(t)^c$. See Example 0.4 below.

(ii) By the condition (V4)-(w), $W(t, x)$ cannot take a positive maximum. Thus (V4)-(w) is satisfied only on non-compact manifolds.

EXAMPLE 0.4. Let $M = \boldsymbol{R}^N$ with a standard Euclidean metric. Let $\bar{\varphi}(r) \in C^2([0, \infty), \boldsymbol{R})$ be a function such that for some $\delta \in (0, 1/2)$

$$\bar{\varphi}(r) = \begin{cases} \frac{1}{2} r^2, & \text{for } r \geqslant 1 - \delta/2, \\ \text{constant}, & \text{for } r \leqslant 1 - \delta, \end{cases}$$

$$\bar{\varphi}'(r) \geqslant 0.$$

Suppose $V(t, x) \in C^2(\boldsymbol{R} \times \boldsymbol{R}^N, \boldsymbol{R})$ is 1-periodic in t and satisfies

$$V(t, x) = \begin{cases} \frac{1}{\mu}(|x|^\mu - 1), & \text{for } |x| \geqslant 1 - \delta, \\ -\frac{1}{2}|x|^2, & \text{for } |x| \leqslant \delta, \\ \text{negative}, & \text{for } |x| \in (\delta, 1 - \delta). \end{cases}$$

Then $V(t, x)$ satisfies (V0)-(V4) for $\varphi(x) = \bar{\varphi}(|x|)$ and $\psi(t, x) \equiv 0$ provided that

$$\mu > 2 \max_{x \in \boldsymbol{R}^n} |\varphi''(x)|.$$

We remark in the set $\{\delta < |x| < 1 - \delta\}$ conditions (V0)-(V4) are satisfied if $V(t, x)$ is negative and 1-periodic. In general, (GSQ) is not satisfied in this case.

REMARK 0.5. In the above example, putting a «handle» in the set $\{\rho < |x| < 1 - \rho\}$, we see that there exists a couple (M, V) satisfying conditions (V0)-(V4) with the Riemannian manifold M not being diffeomorphic to $\boldsymbol{R}^N$.

In the following sections, we prove Theorem 0.1. First we consider the problem on bounded intervals $[-n, n]$ of $\boldsymbol{R}$ and second we take a limit as $n \to \infty$.

1. Preliminaries.

Let $(M, \langle\cdot, \cdot\rangle)$ be a complete, connected finite dimensional Riemannian manifold of class C^3. By the well-known Nash embedding theorem ([13]), M can be embedded in $\boldsymbol{R}^N$ (with a standard Euclidean metric) for sufficiently large N. Thus we may assume M is a submanifold of $\boldsymbol{R}^N$ whose Riemannian structure is induced from the standard Euclidean metric on $\boldsymbol{R}^N$. We may also assume $x_0 \in M$ is corresponding to $0 \in \boldsymbol{R}^N$ by the embedding.

In what follows, we denote by $|\cdot|$ the Euclidean norm, by $\langle\cdot,\cdot\rangle$ the Euclidean s-calar product, by «$-$» the difference in $\boldsymbol{R}^N$, by $d(\cdot,\cdot)$ the distance on M indiced by the Riemannian structure, and we also write

$$B(x, r) = \{y \in M;\ d(x, y) < r\} \quad \text{for all } x \in M \text{ and } r > 0.$$

For a technical reason, we consider the Hamiltonian system (0.1) on bounded intervals first. For $n \in \boldsymbol{N}$, we define

$$W_0^{1,2}(n) = W_0^{1,2}([-n, n], \boldsymbol{R}^N) =$$

$$= \{x(t): [-n, n] \to \boldsymbol{R}^N;\ x(t), \dot{x}(t) \in L^2([-n, n], \boldsymbol{R}^N),\ x(\pm n) = 0\},$$

$$\Omega_n^1 = \Omega^1([-n, n], M, x_0) =$$

$$= \{x(t) \in W_0^{1,2}(n);\ x(t) \in M \quad \text{for all } t \in [-n, n]\}.$$

It is well-known (e.g. Palais [15]) that Ω_n^1 is a Hilbert manifold of class C^2 and its tangent space at $x \in \Omega_n^1$ is given by

$$T_x\Omega_n^1 = \{v \in W_0^{1,2}(n);\ v(t) \in T_{x(t)}M \quad \text{for all } t \in \boldsymbol{R} \text{ and } v(\pm n) = 0\}.$$

We define a scalar product and a norm on $T_x\Omega_n^1$ by

$$\langle v, v\rangle_n = \int_{-n}^{n} \langle D_t v, D_t v\rangle\, dt, \qquad \|v\|_n = \langle v, v\rangle_n^{1/2},$$

where D_t is the covariant derivative along the curve $x(t) \in \Omega_n^1$. We also define the distance $d_{\Omega_n^1}(x, y)$ between $x(t) \in \Omega_n^1$ and $y(t) \in \Omega_n^1$ by

$$d_{\Omega_n^1}(x, y) = \inf\left\{\int_0^1 \left\|\frac{\partial h}{\partial s}(s)\right\|_n ds;\ h \in C^1([0, 1], \Omega_n^1),\ h(0) = x,\ h(1) = y\right\}.$$

We have the following relation between $d_{\Omega_n^1}$ and d.

LEMMA 1.1. *For any* $x(t), y(t) \in \Omega_n^1$,

$$d(x(t), y(t)) \leq \min\{|t - n|^{1/2}, |t + n|^{1/2}\}\, d_{\Omega_n^1}(x, y)$$

for all $t \in [-n, n]$.

PROOF. For any $h \in C^1([0, 1], \Omega_n^1)$ with $h(0) = x$, $h(1) = y$,

$$d(x(t), y(t)) \leq \int_0^1 \left| \frac{\partial h}{\partial s}(s)(t) \right| ds \quad \textit{for all } t.$$

Since $h \in C^1([0, 1], \Omega_n^1)$, we have $(\partial h/\partial s)(s)(t) \in C([0, 1], W_0^{1,2}(n))$ with $(\partial h/\partial s)(s)(t) \in T_{h(s)(t)}M$ for all t, s and $\partial h/\partial s(s)(\pm n) = 0$. Now from

$$\left| \frac{d}{d\tau} \left| \frac{\partial h}{\partial s}(s)(\tau) \right| \right| = \\ = \frac{|\langle D_\tau(\partial h/\partial s)(s)(\tau), \partial h/\partial s\,(s)(\tau)\rangle|}{|\partial h/\partial s\,(s)(\tau)|} \leq \left| D_\tau \frac{\partial h}{\partial s}(s)(\tau) \right|,$$

we have

$$\left| \frac{\partial h}{\partial s}(s)(t) \right| \leq \int_{-n}^{t} \left| D_\tau \frac{\partial h}{\partial s}(s)(\tau) \right| d\tau \leq \\ \leq |t+n|^{1/2} \left(\int_{-n}^{n} \left| D_\tau \frac{\partial h}{\partial s}(s)(\tau) \right|^2 d\tau \right)^{1/2} = |t+n|^{1/2} \left\| \frac{\partial h}{\partial s}(s) \right\|_n.$$

Therefore

$$d(x(t), y(t)) \leq |t+n|^{1/2} \int_0^1 \left\| \frac{\partial h}{\partial s}(s) \right\|_n ds.$$

Since $h(s)$ is arbitrary, we get

$$d(x(t), y(t)) \leq |t+n|^{1/2}\, d_{\Omega_n^1}(x, y).$$

Similarly,

$$d(x(t), y(t)) \leq |t-n|^{1/2}\, d_{\Omega_n^1}(x, y).$$

Thus we get the conclusion. ■

We consider on Ω_n^1 the functional

$$I_n(x) = \int_{-n}^{n} \left[\frac{1}{2} \, |\dot{x}(t)|^2 - V(t, x(t)) \right] dt \colon \Omega_n^1 \to \boldsymbol{R} .$$

We can establish the following lemma in a standard way.

LEMMA 1.2. *$I_n(x) \in C^2(\Omega_n^1, \boldsymbol{R})$ and $x(t) \in \Omega_n^1$ is a critical point of $I_n(x)$ if and only if $x(t)$ solves*

$$D_t \dot{x} + \mathrm{grad}_x V(t, x(t)) = 0 , \qquad in \ (-n, n), \tag{1.1}$$

$$x(-n) = x(n) = x_0 (= 0) . \tag{1.2}$$

Here D_t is the covariant derivative along the curve $x(t)$. ∎

We also remark that

$$I_n'(x)[v] = \int_{-n}^{n} [\langle \dot{x}(t), D_t v(t) \rangle - \langle \mathrm{grad}_x V(t, x), v \rangle] \, dt$$

for all $x(t) \in \Omega_n^1$ and $v(t) \in T_x \Omega_n^1$.

REMARK 1.3. For $x \in M$, let $P(x)(\cdot)$ be the orthogonal projection from $\boldsymbol{R}^N$ onto $T_x M$ and let $Q(x) v = v - P(x) v$. Then we can write

$$D_t v(t) = P(x(t)) \, \dot{v}(t) = \dot{v}(t) - Q(x(t)) \, \dot{v}(t) \qquad \text{for all } v(t) \in T_x \Omega_n^1 .$$

Since $Q(x(t)) v(t) = 0$ for $v(t) \in T_x M$, we have

$$Q(x(t)) \, \dot{v}(t) = - dQ(x(t))[\dot{x}(t)] \, v(t) .$$

Thus

$$D_t v(t) = \dot{v}(t) + dQ(x(t))[\dot{x}(t)] \, v(t) . \tag{1.3}$$

2. The mountain pass structure of $I_n \in C^1(\Omega_n^1, \boldsymbol{R})$.

In this section, we prove that $I_n \in C^1(\Omega_n^1, \boldsymbol{R})$ satisfies the assumptions of the Mountain Pass Theorem.

LEMMA 2.1. *There exist $\delta_0, \rho_0 > 0$ independent of $n \in N$ such that if $x(t) \in \Omega_n^1$ satisfies*

$$\int_{-n}^{n} [\,|\dot{x}(t)|^2 + |x(t)|^2]\,dt = \delta_0\,, \tag{2.1}$$

then

$$I_n(x) \geqslant \rho_0 > 0\,.$$

Moreover

$$I_n(x) \geqslant 0$$

for $x(t) \in \Omega_n^1$ satisfying

$$\int_{-n}^{n} [\,|\dot{x}(t)|^2 + |x(t)|^2]\,dt \leqslant \delta_0\,.$$

PROOF. By the assumption (V1), we can choose $\delta > 0$ and $a > 0$ such that

$$-V(t, x) \geqslant a|x|^2 \qquad \text{for all } x \in B(x_0, \delta) = B(0, \delta)\,.$$

Now taking $\delta_0 > 0$ sufficiently small so that (2.1) implies

$$x(t) \in B(0, \delta) \qquad \text{for all } t \in [-n, n]\,,$$

for such a $\delta_0 > 0$, clearly it follows

$$I_n(x) \geqslant \int_{-n}^{n} \left[\frac{1}{2}\,|\,\dot{x}\,|^2 + a|x|^2\right] dt \geqslant \min\left\{\frac{1}{2}, a\right\}\delta_0 \equiv \rho_0 > 0\,.$$

We can deduce the second assertion in a similar way. ■

Next we take a point $x_\infty \in M$ such that $V(t, x_\infty) > 0$ for all t and choose a curve $q_1(t) \in \Omega_1^1$ such that

$$q_1(0) = x_\infty\,.$$

(Note that the existence of x_∞ follows from (V3).)

For $n \in N$, we set

$$q_n(t) = \begin{cases} q_1(t-n+1), & \text{for } t \in [n-1, n], \\ q_1(t+n-1), & \text{for } t \in [-n, -n+1], \\ x_\infty, & \text{for } t \in [-n+1, n-1]. \end{cases}$$

Then we can see easily

$$I_n(q_n) = I_1(q_1) - 2(n-1)\int_0^1 V(t, x_\infty)\,dt \quad \to -\infty \quad \text{as } n \to \infty .$$

Thus there exists an $n_0 \in N$ such that

(2.2) (i) $\int_{-n_0}^{n_0} [\,|\dot{q}_{n_0}|^2 + |q_{n_0}|^2]\,dt > \delta_0$, where $\delta_0 > 0$ is defined in Lemma 2.1.

(2.3) (ii) $I_{n_0}(q_{n_0}) < 0$.

We fix such an $n_0 \in N$ and write $\overline{q}_0(t) = q_{n_0}(t)$. Setting

$$\overline{q}_0(t) = x_0(=0) \quad \text{in } [-n, n]\backslash[-n_0, n_0],$$

we regard

$$\overline{q}_0(t) \in \Omega_n^1 \quad \text{for } n \geqslant n_0 .$$

We define for $n \geqslant n_0$

$$\Gamma_n = \{\gamma \in C([0,1], \Omega_n^1);\ \gamma(0)(t) = x_0,\ \gamma(1)(t) = \overline{q}_0(t) \quad \text{for all } t\},$$

$$b_n = \inf_{\gamma \in \Gamma_n} \max_{s \in [0,1]} I_n(\gamma(s)) .$$

By Lemma 2.1, and formulas (2.2), (2.3), we can see the following

LEMMA 2.2.

$$b_{n_1} \geqslant b_{n_2} \geqslant \rho_0 \quad \textit{for } n_2 \geqslant n_1 \geqslant n_0 ,$$

where $\rho_0 > 0$ *is given in Lemma* 2.1. *In particular the limit*

$$b_\infty = \lim_{n \to \infty} b_n \in [\rho_0, \infty)$$

exists.

PROOF. Since we can regard $\gamma_{n_1} \in \Gamma_{n_1} \subset \Gamma_{n_2}$ for $n_2 \geqslant n_1 \geqslant n_0$, the conclusion of Lemma 2.2 follows from the definition of b_n. ■

We can also verify the Palais-Smale compactness condition and we can see b_n is a critical value of $I_n(x)$, that is, there exists a non-trivial solution $x_n(t)$ of (1.1)-(1.2). One may expect after suitable shifts in time $\bar{x}_n(t) = x_n(t - t_n)$ ($t_n \in \{-n+1, \dots, 0, \dots, n-1\}$) converges to a homoclinic orbit. However, there is a possibility

$$n - t_n \to s_0^+ \neq \infty \qquad \text{or} \qquad -n - t_n \to s_0^- \neq -\infty$$

and $\bar{x}_n(t)$ may converge to a non-trivial solution of (0.1) in $(-\infty, s_0^+)$ with

$$\begin{cases} x(-\infty) = x_0\,, & \dot{x}(-\infty) = 0\,, \\ x(s_0) = x_0\,, & \end{cases}$$

or

$$\begin{cases} x(\infty) = x_0\,, & \dot{x}(\infty) = 0\,, \\ x(s_0) = x_0\,. & \end{cases}$$

To overcome this problem, we will get a homoclinic orbit as a limit of a special sequence of approximate solutions of (1.1)-(1.2) in the following section. For this we need.

LEMMA 2.3. *There exist constants* $\delta_1, C_0 > 0$ *independent of* $n \in N$ *such that if* $x(t) \in \Omega_n^1$ *satisfies for* $\varepsilon \in (0, 1]$

(2.4) (i) $x([-n, n]) \subset B(x_0, \delta_1)$,

(2.5) (ii) $\|I_n'(x)\| \leq \varepsilon$, *that is,* $|\langle I_n'(x), v\rangle| \leq \varepsilon\|v\|_n$ *for all* $v \in T_x\Omega_n^1$.

Then

$$I_n(x) \leq C_0\varepsilon^2\,. \tag{2.6}$$

PROOF. By the assumption (V1), there exist $\delta_2 > 0$, $C_1, C_2 > 0$ such that

$$\langle -\mathrm{grad}_x V(t, x), P(x)x\rangle \geq C_1|x|^2\,, \tag{2.7}$$

$$-V(t, x) \leq C_2|x|^2 \tag{2.8}$$

for all t and $x \in B(0, \delta_2)$.

For $x(t) \in \Omega_n^1$, we define

$$v(t) = P(x(t))x(t) \in T_x\Omega_n^1\,.$$

Then by (1.3)

$$D_t v(t) = \dot{v}(t) + dQ(x(t))[\dot{x}(t)]\, v(t) = P(x(t))\, \dot{x}(t) +$$

$$+ dP(x(t))[\dot{x}(t)]\, x(t) + dQ(x(t))[\dot{x}(t)]\, P(x(t))\, x(t) = \dot{x}(t) + f(x(t))\, \dot{x}(t),$$

where $f(x)\colon M \to \boldsymbol{R}^{N^2}$ is an $N \times N$ matrix-valued function satisfying

$$\lim_{x \to 0} |f(x)| = 0 .$$

Therefore there exist $\delta_3 > 0$ and $C_1', C_2' > 0$ such that

$$C_1' \,|\, \dot{x} \,|^2 \leqslant \langle \dot{x}, D_t(P(x(t))\, x(t)) \rangle, \tag{2.9}$$

$$|D_t(P(x(t))\, x(t))|^2 \leqslant C_2' \,|\, \dot{x}(t) \,|^2 \tag{2.10}$$

for all $x(t) \in \Omega_n^1$ with $x([-n, n]) \in B(0, \delta_3)$.

Now we set $\delta_1 = \min\{\delta_2, \delta_3\} > 0$ and we assume $x(t) \in \Omega_n^1$ satisfies (2.4) and (2.5). By (2.5), we have

$$\langle I_n'(x), P(x)\, x \rangle \leqslant \varepsilon \| P(x)\, x \|_n ,$$

that is,

$$\int_{-n}^{n} [\langle \dot{x}, D_t(P(x)\, x) \rangle - \langle \operatorname{grad}_x V(t, x), P(x)\, x \rangle]\, dt \leqslant$$

$$\leqslant \varepsilon \left(\int_{-n}^{n} |D_t(P(x)\, x)|^2\, dt \right)^{1/2} .$$

Thus using (2.7), (2.9), (2.10), we get

$$\int_{-n}^{n} |\, \dot{x} \,|^2\, dt \leqslant C_3\, \varepsilon^2 , \qquad \int_{-n}^{n} |x|^2\, dt \leqslant C_4\, \varepsilon^2 ,$$

where $C_3, C_4 > 0$ are independent of n and ε. Using these bounds, we get by (2.8)

$$I_n(x) = \int_{-n}^{n} \left[\frac{1}{2} \,|\, \dot{x} \,|^2 - V(t, x) \right] dt \leqslant \left(\frac{C_3}{2} + C_2 C_4 \right) \varepsilon^2 \equiv C_0\, \varepsilon^2 . \qquad \blacksquare$$

3. Approximate solutions and their estimates.

For $k \in \boldsymbol{N}$, by Lemma 2.2, we can choose $n_k \geqslant n_0$ and $\gamma_k \in \Gamma_{n_k}$ such that

1) $b_\infty \leqslant b_{n_k} \leqslant b_\infty + 1/2k$,
2) $\max_{s \in [0,1]} I_{n_k}(\gamma_k) \leqslant b_{n_k} + 1/2k$.

For $\varepsilon_0 \equiv \delta_1/2 > 0$, where $\delta_1 > 0$ is given in Lemma 2.3, we set

$$l_k = [\varepsilon_0^2 k],$$

where $[a]$ denotes the integer part of $a \in \boldsymbol{R}$. We regard $\gamma_k \in \Gamma_{n_k} \subset \Gamma_{n_k + l_k}$ and by Lemma 2.2 and 1), 2), we can see that

$$\max_{s \in [0,1]} I_{n_k + l_k}(\gamma_k(s)) \leqslant b_{n_k + l_k} + \frac{1}{k}.$$

Now we apply the following proposition, which is a consequence of the usual deformation argument.

PROPOSITION 3.1. *Suppose $J(x) \in C^1(\Omega_n^1, \boldsymbol{R})$ satisfies*

$$J(x_0) = 0, \qquad J(x_1) \leqslant 0$$

for some $x_0, x_1 \in \Omega_n^1$ and set

$$\Gamma = \{\gamma \in C([0,1], \Omega_n^1);\ \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

$$b = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} J(\gamma(s)).$$

Assume $b > 0$ and there exists a $\gamma \in \Gamma$ such that for some $\varepsilon > 0$

$$\max_{s \in [0,1]} J(\gamma(s)) \leqslant b + \varepsilon.$$

Then there exists $x(t) \in \Omega_n^1$ such that

(i) $J(x) \in [b - \varepsilon, b + \varepsilon]$,
(ii) $\|J'(x)\| \leqslant \sqrt{\varepsilon}$,
(iii) $\min_{s \in [0,1]} d_{\Omega_n^1}(\gamma(s), x)) \leqslant \sqrt{\varepsilon}$. ∎

We apply the above proposition to the case $n = n_k + l_k$. $J = I_{n_k + l_k}$, $\gamma = \gamma_k$, $\varepsilon = 1/k$ and obtain the existence of $x_k \in \Omega_{n_k + l_k}^1$ such that

(3.1) (i) $I_{n_k + l_k}(x_k) \in \left[b_{n_k + l_k} - \frac{1}{k},\ b_{n_k + l_k} + \frac{1}{k}\right]$, (3.2) (ii) $\|I_{n_k+l_k}+(x_k)\| \leq \frac{1}{\sqrt{k}}$, i.e.,

$$\left|\int_{-n_k-l_k}^{n_k+l_k} [\langle \dot{x}_k, D_t v\rangle - \langle \mathrm{grad}_x V(t, x_k), v\rangle]\, dt\right| \leq \frac{1}{\sqrt{k}} \|v\|_{n_k+l_k}$$

for all $v \in T_{x_k}\Omega^1_{n_k+l_k}$,

(3.3) (iii) $\min_{s\in[0,1]} d_{\Omega^1_{n_k+l_k}}(\gamma_k(s), x_k) \leq \frac{1}{\sqrt{k}}$.

We derive some properties of $x_k(t)$.

LEMMA 3.2.

$d(x_k(t), x_0) \leq \varepsilon_0$ *for all* $t \in [-n_k - l_k, n_k + l_k]\backslash[-n_k, n_k]$.

PROOF. Since $\gamma_s(t)(t) = 0$ for all $s \in [0, 1]$ and $t \in [-n_k - l_k, n_k + l_k]\backslash[-n_k, n_k]$, we can see from (3.3) and Lemma 1.1 that

$$d(x_k(t), x_0) \leq \sqrt{l_k}\, d_{\Omega^1_{n_k+l_k}}(x_k, \gamma_k(s))$$

for all $s \in [0, 1]$ and $t \in [-n_k - l_k, n_k + l_k]\backslash[-n_k, n_k]$. Thus

$$d(x_k(t), x_0) \leq \varepsilon_0 . \quad \blacksquare$$

PROPOSITION 3.3. *There exist constants* $C_1, C_2 > 0$ *independent of* $k \in N$ *such that*

$$\int_{-n_k-l_k}^{n_k+l_k} |\dot{x}_k|^2\, dt \leq C_1 , \qquad \int_{-n_k-l_k}^{n_k+l_k} |V(t, x_k)|\, dt \leq C_2 .$$

PROOF. Here we use the assumption (V4). We write

$$W_+(t, x) = \max\{W(t, x), 0\}, \qquad W_-(t, x) = \max\{-W(t, x), 0\}.$$

Then $W(t, x) = W_+(t, x) - W_-(t, x)$. By (V4)-(w) we have

$$\int_{n_k-l_k}^{n_k+l_k} \mu W_+(t, x_k)\, dt \leq \int_{-n_k-l_k}^{n_k+l_k} \langle \mathrm{grad}_x W(t, x_k)\, \mathrm{grad}\, \varphi(x_k)\rangle\, dt . \tag{3.4}$$

Setting $v = \operatorname{grad}\varphi(x_k) \in T_{x_k}\Omega^1_{n_k+l_k}$ in (3.2), we see

$$(3.5) \qquad \int_{-n_k-l_k}^{n_k+l_k} \langle \operatorname{grad}_x W(t, x_k), \operatorname{grad}\varphi(x_k)\rangle\, dt \leqslant$$

$$\leqslant \int_{-n_k-l_k}^{n_k+l_k} \langle \dot{x}_k, D_t \operatorname{grad}\varphi(x_k)\rangle\, dt + \int_{-n_k-l_k}^{n_k+l_k} \langle \operatorname{grad}_x \psi(t, x_k), \operatorname{grad}\varphi(x_k)\rangle\, dt +$$

$$+ \frac{1}{\sqrt{k}} \|\operatorname{grad}\varphi(x_k)\|_{n_k+l_k} =$$

$$= \int_{-n_k-l_k}^{n_k+l_k} H^{\varphi}(x_k)[\dot{x}_k, \dot{x}_k]\, dt + \int_{-n_k-l_k}^{n_k+l_k} \langle \operatorname{grad}_x \psi(t, x_k), \operatorname{grad}\varphi(x_k)\rangle\, dt +$$

$$+ \frac{1}{\sqrt{k}} \|\operatorname{grad}\varphi(x_k)\|_{n_k+l_k}\,.$$

Since $H^{\varphi}(x)[v, w] = \langle D_v \operatorname{grad}\varphi(x), w\rangle$ for all $v, w \in T_x M$ (e.g. Lemma 49 of [14, p.86]),

$$\|\operatorname{grad}\varphi(x_k)\|^2_{n_k+l_k} = \int_{-n_k-l_k}^{n_k+l_k} \langle D_t \operatorname{grad}\varphi(x_k), D_t \operatorname{grad}\varphi(x_k)\rangle\, dt =$$

$$\int_{-n_k-l_k}^{n_k+l_k} H^{\varphi}(x_k)[\dot{x}_k, D_t \operatorname{grad}\varphi(x_k)]\, dt\,.$$

Thus from (V4)-(φ2)

$$\|\operatorname{grad}\varphi(x_k)\|^2_{n_k+l_k} \leqslant c_0' \int_{-n_k-l_k}^{n_k+l_k} |\dot{x}_k|\; |D_t \operatorname{grad}\varphi(x_k)|\, dt \leqslant$$

$$\leqslant c_0' \left(\int_{-n_k-l_k}^{n_k+l_k} |\dot{x}_k|^2\, dt \right)^{1/2} \|\operatorname{grad}\varphi(x_k)\|_{n_k+l_k}\,,$$

that is,

$$\|\text{grad}\,\varphi(x_k)\|_{n_k+l_k} \leqslant \left(\int_{-n_k-l_k}^{n_k+l_k} |\dot{x}_k|^2\,dt\right)^{1/2}. \tag{3.6}$$

Therefore, combining (3.4)-(3.6) and (V4)-(ψ2), we have

$$\int_{-n_k-l_k}^{n_k+l_k} \mu W_+(t, x_k)\,dt \leqslant \tag{3.7}$$

$$\leqslant \int_{-n_k-l_k}^{n_k+l_k} H^\varphi(x_k)[\dot{x}_k, \dot{x}_k]\,dt + (\mu - c_0'') \int_{-n_k-l_k}^{n_k+l_k} \psi(t, x_k)\,dt +$$

$$+ \frac{1}{\sqrt{k}}\, c_0' \left(\int_{-n_k-l_k}^{n_k+l_k} |\dot{x}_k|^2\,dt\right)^{1/2}.$$

By (3.1) and (3.7), we have for large k

$$b_\infty + 1 \geqslant I_{n_k+l_k}(x_k) = \int_{-n_k-l_k}^{n_k+l_k} \left[\frac{1}{2}|\dot{x}_k|^2 + \psi(t, x_k) - W(t, x_k)\right] dt \geqslant$$

$$\geqslant \int_{-n_k-l_k}^{n_k+l_k} \left(\frac{1}{2}|\dot{x}_k|^2 - \frac{1}{\mu} H^\varphi(x_k(t))[\dot{x}_k, \dot{x}_k]\right) dt + \frac{c_0''}{\mu} \int_{-n_k-l_k}^{n_k+l_k} \psi(t, x_k)\,dt +$$

$$+ \int_{-n_k-l_k}^{n_k+l_k} W_-(t, x_k)\,dt - \frac{1}{\mu\sqrt{k}}\, c_0' \left(\int_{-n_k-l_k}^{n_k+l_k} |\dot{x}_k|^2\,dt\right)^{1/2}.$$

Thus we by (V4)-(φ2)

$$b_\infty + 1 \geqslant c_0 \int_{-n_k-l_k}^{n_k+l_k} |\dot{x}_k|^2\,dt - \frac{1}{\mu\sqrt{k}}\, c_0' \left(\int_{-n_k-l_k}^{n_k+l_k} dt + |\dot{x}_k|^2\,dt\right)^{1/2} +$$

$$+ \frac{c_0''}{\mu} \int_{-n_k-l_k}^{n_k+l_k} \psi(t, x_k)\,dt + \int_{-n_k-l_k}^{n_k+l_k} W_-(t, x_k)\,dt.$$

Thus we get from (V4)-(ψ1)

$$\int_{-n_k-l_k}^{n_k+l_k} |\dot{x}_k|^2\,dt\,,\qquad \int_{-n_k-l_k}^{n_k+l_k} \psi(t, x_k)\,dt\,,\qquad \int_{-n_k-l_k}^{n_k+l_k} W_-(t, x_k)\,dt \leqslant C_1\,,$$

where $C_1 > 0$ is independent of k. Thus we also get from (3.7)

$$\int_{-n_k-l_k}^{n_k+l_k} W_+(t, x_k)\,dt \leqslant C_1'\,,$$

for some $C_1' > 0$ independent of k. Therefore

$$\int_{-n_k-l_k}^{n_k+l_k} |V(t, x_k)|\,dt \leqslant C_2\,. \qquad \blacksquare$$

LEMMA 3.4. *For large k,*

$$x_k([-n_k - l_k, n_k + l_k]) \not\subset B(x_0, 2\varepsilon_0)\,.$$

PROOF. Recall that $\delta_1 = 2\varepsilon_0$. If $x_k([-n_k - l_k, n_k + l_k]) \subset B(x_0, 2\varepsilon_0) = B(x_0, \delta_1)$, we get from Lemma 2.3 and (3.2) that

$$I_{n_k+l_k}(x_k) \leqslant \frac{C_0}{k}\,.$$

But by Lemma 2.2 this contradicts (3.1) for large k. $\blacksquare$

In the next section, we take a limit as $t \to \infty$ to get a homoclinic solution.

4. Limit processes as $k \to \infty$.

By Lemmas 3.2 and 3.4, for large k we can find a $s_k \in \{-n_k, \ldots, 0, \ldots, n_k\}$ such that

$$x_k([s_k, s_k + 1]) \cap \partial B(0, 2\varepsilon_0) \neq \emptyset\,. \tag{4.1}$$

We define $y_k(t)\colon \boldsymbol{R} \to M$ by

$$y_k(t) = \begin{cases} x_k(t + s_k)\,, & \text{for } t \in [-n_k - l_k - s_k, n_k + l_k - s_k]\,,\\ x_0\,, & \text{for } t \notin [-n_k - l_k - s_k, n_k + l_k - s_k]\,. \end{cases}$$

We remark

$$-n_k - l_k - s_k \leqslant -l_k \to -\infty , \tag{4.2}$$

$$n_k + l_k - s_k \geqslant l_k \to \infty . \tag{4.3}$$

By Proposition 3.3, we have

$$\int_{\boldsymbol{R}} |\dot{y}_k|^2 \, dt \leqslant C_1 , \tag{4.4}$$

$$\int_{\boldsymbol{R}} |V(t, y_k(t))| \, dt \leqslant C_2 , \tag{4.5}$$

where $C_1, C_2 > 0$ are independent of $k \in \boldsymbol{N}$.

By (4.1) and (4.4), $y_k(t)$ is bounded in $W^{1,2}_{\text{loc}}(\boldsymbol{R}, \boldsymbol{R}^N)$ and we can extract a subsequence—still denote by k—such that

(4.6) $\quad y_k(t) \to y(t)$

$$\text{weakly in } W^{1,2}_{\text{loc}}(\boldsymbol{R}, \boldsymbol{R}^N) \quad \text{and strongly in } C_{\text{loc}}(\boldsymbol{R}, M).$$

From (4.4) and (4.5), we deduce

$$\int_{\boldsymbol{R}} |\dot{y}|^2 \, dt \leqslant \liminf_{k \to \infty} \int_{\boldsymbol{R}} |\dot{y}_k|^2 \, dt \leqslant C_1 , \tag{4.7}$$

$$\int_{\boldsymbol{R}} |V((t, y(t))| \, dt \leqslant \liminf_{k \to \infty} \int_{\boldsymbol{R}} |V((t, y_k(t))| \, dt \leqslant C_2 . \tag{4.8}$$

Moreover

LEMMA 4.1.

$$y_k(t) \to y(t) \quad \textit{strongly in } W^{1,2}_{\text{loc}}(\boldsymbol{R}, \boldsymbol{R}^N).$$

PROOF. It suffices to prove

$$\int_{\boldsymbol{R}} |\dot{y}_k(t) - \dot{y}(t)|^2 \, \phi(t) \, dt \to 0 \tag{4.9}$$

for any $\phi(t) \in C_0^\infty(\boldsymbol{R}, \boldsymbol{R})$.

For a given $\phi(t) \in C_0^\infty(\boldsymbol{R}, \boldsymbol{R})$, we define

$$v_k(t) = P(y_k(t))(y_k(t) - y(t)) \, \phi(t) \in T_{y_k(t)} M$$

and

$$\bar{v}_k(t) = v_k(t - s_k).$$

Since $\operatorname{supp}\phi(t) \subset (-l_k, l_k)$ for large k, $\operatorname{supp}\bar{v}_k(t) \subset (-n_k - l_k, n_k + l_k)$ and $\bar{v}_k \in T_{x_k}\Omega^1_{n_k + l_k}$ for large k. Now using the fact that

$$D_t v_k = P(y_k(t)) \frac{d}{dt} (P(y_k(t))(y_k(t) - y(t))\phi(t))$$

and $y_k(t)$ is bounded in $W^{1,2}_{\mathrm{loc}}(\boldsymbol{R}, \boldsymbol{R}^N)$, we have

$$\|D_t \bar{v}_k\|_{n_k + l_k} = \int_{\boldsymbol{R}} |D_t v_k(t)|^2 \, dt \quad \text{is bounded as } k \to \infty .$$

Therefore by (3.2)

$$(4.10) \qquad \int_{\boldsymbol{R}} [\langle \dot{y}_k, D_t v_k \rangle - \langle \operatorname{grad}_x V(t, y_k), v_k \rangle] \, dt =$$

$$= \int_{-n_k - l_k}^{n_k + l_k} [\langle \dot{x}_k, D_t \bar{v}_k \rangle - \langle \operatorname{grad}_x V(t, x_k), \bar{v}_k \rangle] \, dt 0 \qquad \text{as } k \to \infty .$$

By (4.6), we also have

$$\int_{\boldsymbol{R}} \langle \operatorname{grad}_x V(t, y_k), v_k \rangle \, dt =$$

$$= \int_{\boldsymbol{R}} \langle \operatorname{grad}_x V(t, y_k), P(y_k), P(y_k)(y_k(t) - y(t)) \rangle \phi(t) \, dt \to 0 \qquad \text{as } k \to \infty .$$

Thus it follows from (4.10)

$$(4.11) \qquad \int_{\boldsymbol{R}} \langle \dot{y}_k, D_t v_k \rangle \, dt \to 0 \qquad \text{as } k \to \infty .$$

On the other hand,

$$D_t v_k = D_t((1 - Q(y_k))(y_k - y)\phi) = P(y_k) \frac{d}{dt} ((1 - Q(y_k))(y_k - y)\phi) =$$

$$= P(y_k)(\dot{y}_k - \dot{y} - Q(y_k)(\dot{y}_k - \dot{y}) - dQ(y_k)[\dot{y}_k](y_k - y))\phi + \dot{\phi} P(y_k)(y_k - y) =$$

$$= (\dot{y}_k - \dot{y} + (1 - P(y_k))\dot{y} - P(y_k) \, dQ(y_k)[\dot{y}_k](y_k - y))\phi + \dot{\phi} P(y_k)(y_k - y).$$

Here we used the fact $\dot{y}_k \in T_{y_k}M$. From (4.6), (4.11) we can deduce

$$\int_R \langle \dot{y}_k, \dot{y}_k - \dot{y}\rangle \phi(t)\, dt \to 0 \quad \text{as } k \to \infty .$$

Thus we get (4.9). ■

Using the above Lemma, we can show the following

PROPOSITION 4.2. *$y(t)$ satisfies*

$$D_t\dot{y} + \mathrm{grad}_x V(t, y(t)) = 0 \quad \textit{in } \boldsymbol{R}.$$

PROOF. It suffices to prove

$$\int_R [\langle \dot{y}, D_t v\rangle - \langle \mathrm{grad}_x V(t, y), v\rangle]\, dt = 0$$

for all $v \in C_0^\infty(\boldsymbol{R}, \boldsymbol{R}^N)$ with $v(t) \in T_{y(t)}M$. We set

$$v_k(t) = P(y_k(t))\, v(t), \qquad \bar{v}_k(t) = v_k(t - s_k).$$

By (4.2)-(4.3), $\bar{v}_k(t) \in T_{x_k}\Omega^1_{n_k + l_k}$ for large k. We can see $\int_R |D_t v_k|^2$ is bounded as $k \to \infty$ and as in the proof of Lemma 4.1 we have

$$\text{(4.12)} \quad \int_R [\langle \dot{y}_k, D_t v_k\rangle - \langle \mathrm{grad}_x V(t, y_k), v_k\rangle]\, dt \to 0 \quad \text{as } k \to \infty .$$

Using Lemma 4.1, we have

$$\text{(4.13)} \quad \int_R \langle \dot{y}_k, D_t v_k\rangle\, dt = \int_R \langle \dot{y}_k, \dot{v}_k\rangle\, dt \to \int_R \langle \dot{y}, \dot{v}\rangle = \int_R \langle \dot{y}, D_t v\rangle\, dt,$$

$$\text{(4.14)} \quad \int_R \langle \mathrm{grad}_x V(t, y_k), v_k\rangle\, dt \to \int_R \langle \mathrm{grad}_x V(t, y), v\rangle\, dt,$$

as $k \to \infty$. Combining (4.12)-(4.14), we get the desired result. ■

END OF THE PROOF OF THEOREM 0.1. By (4.1), (4.6), we observe

$$y([0, 1]) \cap \partial B(x_0, 2\varepsilon_0) \neq \emptyset .$$

In particular, $y(t) \not\equiv x_0$.

To complete the proof of Theorem 0.1, we need to show

$$y(t) \to x_0 = 0 , \qquad \dot{y}(t) \to 0 ,$$

as $t \to \pm \infty$. We deal only with the case «+». (The case «−» can be treated in a similar way).

First, we remark $y(t)$ is bounded on $\boldsymbol{R}$. Indeed, by (4.7), $y(t)$ is uniformly continuous on $\boldsymbol{R}$ and using (V3) and (4.8), our claim follows.

From the equation (0.1), we have for any $h > 0$

$$\max_{s \in [0, h]} | \dot{y}(t + s) - \dot{y}(t)| \leqslant \int_t^{t+h} |D_\tau \dot{y}(\tau)| \, d\tau \leqslant$$

$$\leqslant \int_t^{t+h} |\mathrm{grad}_x \, V(\tau, y(\tau))| \, d\tau \leqslant mh ,$$

where $m = \max_{\tau \in \boldsymbol{R}} |\mathrm{grad}_x \, V(\tau, y(\tau))|$.

Thus, if there is a sequence $t_j \to \infty$ such that $|\dot{y}(t_j)| \not\to 0$. Setting

$$\delta = \min \{\limsup_{j \to \infty} | \dot{y}(t_j)|, 1\} > 0 , \qquad h = \frac{\delta}{2m} ,$$

we see

$$| \dot{y}(t_j + s)| \geqslant \frac{\delta}{3} > 0 \qquad \text{for all } s \in [0, h]$$

for large $j \in \boldsymbol{N}$. Therefore we have

$$\int_{\boldsymbol{R}} | \dot{y}(t)|^2 \, dt \geqslant \sum_{j=1}^{\infty} \int_{t_j}^{t_j + h} | \dot{y}(t)|^2 \, dt = \infty .$$

This contradicts (4.7). Thus

$$\dot{y}(t) \to 0 \quad \text{as } t \to \infty .$$

Next we set

$$Z = \{(t, x) \in [0, 1] \times \partial;\ \text{there exists some } (t_j)_{j=0}^{\infty} \text{ such that } t_j \to \infty,$$
$$\min_{i \in \mathbf{Z}} |t - t_j - i| \to 0,\ y(t_j) \to x \text{ as } j \to \infty\}.$$

Using the uniform continuity of $y(t)$ on $\mathbf{R}$ again, we can see that

$$Z \subset \{(t, x);\ V(t, x) = 0\}. \tag{4.15}$$

Let us show $Z \subset [0, 1] \times \{x_0\}$. For any $(s, x) \in Z$ there exists $(t_j)_{j=1}^{\infty}$ such that

$$t_j \to \infty, \quad \min_{i \in \mathbf{Z}} |s - t_j - i| \to 0, \quad y(t_j) \to x \quad \text{as } j \to \infty.$$

Then by the continuous dependence of solutions on data, we have

$$y(t + t_j) \to z(t) \quad \text{in } C^2_{\text{loc}}(\mathbf{R}, \mathbf{R}^N),$$

where $z(t)$ solves

$$D_t \dot{z} + \text{grad}_x V(t, z) = 0, \quad z(s) = x, \quad \dot{z}(s) = 0.$$

If $x \neq x_0$, using (4.15) and (V2), we can see $z(t) \not\equiv x$ and we have

$$\int_{\mathbf{R}} |\dot{y}|^2\, dt \geq \sum_{j=1}^{\infty} \int_{t_j}^{t_j + \delta} |\dot{y}(t)|^2\, dt = \infty,$$

since

$$\int_{t_j}^{t_j + \delta} |\dot{y}(t)|^2\, dt \to \int_{s}^{s + \delta} |\dot{z}|^2\, dt > 0 \quad \text{as } j \to \infty.$$

This contradicts (4.7). Therefore we get $y(t) \to x_0$ as $t \to \infty$. ■

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REFERENCES

[1] A. AMBROSETTI - M. L. BERTOTTI, *Homoclinics for second order conservative systems*, in *Partial Differential Equations and Related Subjects* (ed. M. MIRANDA), Pitman Research Note in Math. Ser. (1992).

[2] A. AMBROSETTI - V. COTI ZELATI, *Multiple homoclinic orbits for a class of conservative systems*, Rend. Sem. Mat. Univ. Padova, **89** (1993), pp. 177-194. See also *Multiplicité des orbites homoclines pour des Systèmes conservatifs*, C. R. Acad. Sci. Paris, **314** (1992), pp. 601-604.

[3] V. BENCI - F. GIANNONI, *Homoclinic orbits on compact manifolds*, J. Math. Anal. Appl., **157** (1991), pp. 568-576.

[4] M. L. BERTOTTI, *Homoclinics for Lagrangian systems on Riemannian manifolds*, Dyn. Sys. Appl., **1** (1992), pp. 341-368.

[5] P. CALDIROLI, *Existence and multiplicity of homoclinic orbits for singular potentials on unbounded domains*, Proc. Roy. Soc. Edinburgh (to appear).

[6] V. COTI ZELATI - I. EKELAND - E. SÉRÉ, *A variational approach to homoclinic orbits in Hamiltonian systems*, Math. Ann., **288** (1990), pp. 133-160.

[7] V. COTI ZELATI - P. H. RABINOEITZ, *Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials*, J. Amer. Math. Soc., 4 (1991), pp. 693-727.

[8] F. GIANNONI, *On the existence of homoclinic orbits on Riemannian manifolds*, Ergodic Theo. Dyn. Sys., **14** (1994), pp. 103-127.

[9] F. GIANNONI - P. H. RABINOWITZ, *On the multiplicity of homoclinic orbits on Riemannian manifolds for a class of second order Hamiltonian system*, Nonlinear Diff. Eq. Appl., **1** (1994), pp. 1-46.

[10] H. HOFER - K. WYSOCKI, *First order elliptic systems and the existence of homoclinic orbits in Hamiltonian system*, Math. Ann., **288** (1990), pp. 483-503.

[11] L. JEANJEAN, *Existence of connecting orbits in a potential well*, Dyn. Sys. Appl. (to appear).

[12] V. KOZLOV, *Calculus of variations in the large and classical mechanics*, Russ. Math. Surv., **40** (1985), pp. 37-71.

[13] J. NASH, *The embedding problem for Riemannian manifolds*, Ann. Math., **63** (1956), pp. 20-63.

[14] B. O'NEILL, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press. (1983).

[15] R. S. PALAIS, *Morse theory on Hilbert manifolds*, Topology, **2** (1963), pp. 299-340.

[16] P. H. RABINOWITZ, *Periodic and heteroclinic orbits for a periodic Hamiltonian system*, Ann. Inst. H. Poincaré: Analyse Non Linéaire, **6** (1989), pp. 331-346.

[17] P. H. RABINOWITZ, *Homoclinic orbits for a class of Hamiltonian systems*, Proc. Roy. Soc. Edinburg, **114** (1990), pp. 33-38.

[18] P. H. RABINOWITZ - K. TANAKA, *Some results on connecting orbits for a class of Hamiltonian system*, Math. Zeit., **206** (1991), pp. 473-499.

[19] E. SÉRÉ, *Existence of infinitely many homoclinic orbits in Hamiltonian systems*, Math. Zeit., **209** (1992), pp. 27-42.
[20] E. SÉRÉ, *Looking for the Bernoulli shift*, Ann. Inst. H. Poincaré: Analyse Non Linéaire (to appear).
[21] E. SÉRÉ, *Homoclinic orbits in compact hypersurface in* $\boldsymbol{R}^{2N}$ *of restricted contact type*, preprint.
[22] K. TANAKA, *Homoclinic orbits for a singular second order Hamiltonian system*, Ann. Inst. H. Poincaré: Analyse Non Linéaire, 7 (1990), pp. 427-438.
[23] K. TANAKA, *Homoclinic orbits in a first order superquadratic Hamiltonian system: Convergence of subharmonic orbits*, J. Diff. Eq., **94** (1991), pp. 315-339.
[24] K. TANAKA, *A note on the existence of multiple homoclinic orbits for a perturbed radial potential*, Nonlinear Diff. Eq. Appl. (to appear).

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