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Centralizers of Semisimple Subgroups in Locally Finite Simple Groups.

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The classification of finite simple groups has led to considerable progress in the study of the locally finite simple groups or LFS-groups as we will call them. In [7], B. Hartley and the author studied the centralizing properties of elements in LFS-groups. LFS-groups are usually studied in two classes; infinite linear LFS-groups and infinite non-linear LFS-groups. Infinite linear LFS-groups are the Chevalley groups and their twisted analogues over infinite locally finite fields [1], [2], [6] and [12]. Here we are mainly interested in non-linear LFS-groups.

In [9] we have defined semisimple elements for LFS-groups and studied the centralizers of these elements. Here we extend the definition of a semisimple element given in [9] to semisimple subgroups.

DEFINITION. Let G be a countably infinite LFS-group and F be a finite subgroup of G. The group F is called a K-semisimple subgroup of G, if G has a Kegel sequence $K = (G_i, M_i)_{i \in N}$ such that $(|M_i|, |F|) = 1$, M_i are soluble for all i and if G_i/M_i is a linear group over a field of characteristic p_i, then $(p_i, |F|) = 1$.

This definition is a generalization of the K-semisimple element in [9]. In particular every element in a K-semisimple group is a K-semisimple element in the sense of [9]. B. Hartley and the author proved in [7], Theorem B that centralizers of K-semisimple elements in non-linear LFS-groups involve infinite non-linear LFS-groups.

In [5], the centralizers of subgroups are studied and the following questions are asked:

Is it the case that in a non-linear LFS-group the centralizer of every finite subgroup is infinite?

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Does the centralizer of every finite subgroup involve an infinite non-linear simple group?

A finite abelian group F in a finite simple group G of classical type or alternating is called a *nice group* if whenever G is of type B_l or D_l, then Sylow 2-subgroup of F is cyclic. If G is alternating group or of type A_l or C_l, then every abelian subgroup is a nice group. In particular every abelian group of odd order is a nice group.

A finite abelian group in a countably infinite locally finite simple group G is called a K-nice group if F is a nice group in almost all Kegel components of a Kegel sequence K of G. We prove here:

THEOREM 1. *If F is a K-nice abelian subgroup and K-semisimple in a non-linear LFS-group G, then $C_G(F)$ has a series of finite length in which the factors are either non-abelian simple or locally soluble moreover one of the factors is non-linear simple. In particular $C_G(F)$ is an infinite group.*

THEOREM 2. *Suppose that G is infinite non-linear and every finite set of elements of G lies in a finite simple group. Then*

(i). *There exist infinitely many finite abelian semisimple subgroups F of G and local systems L of G consisting of simple subgroups such that F is nice in every member of L.*

(ii) *There exists a function f from natural numbers to natural numbers independent of G such that $C = C_G(F)$ has a series of finite length in which at most $f(|F|)$ factors are simple non-abelian groups for any F as in* (i). *Furthermore C involves a non-linear simple group.*

Let us recall the definition of the group theoretical classes T_n and $T_{n,r}$ given in [7].

DEFINITION. $T_{n,r}$ consists of all groups (not necessarily locally finite) having a series of finite length in which at most n factors are non-abelian simple and the rest are soluble groups, the sum of whose derived lengths is at most r.

DEFINITION. T_n consist of all locally finite groups having a series of finite length in which there are at most n non-abelian simple factors and the rest are locally soluble.

The following Lemma is given in [7] Lemma 2.1.

LEMMA 1. (i) *The classes* T_n *and* $T_{n,r}$ *are closed under taking normal subgroups and quotients.*

(ii) *Let* $N \lhd M \lhd G$. *If* $G \in T_{n,r}$ *and* M/N *is soluble, then the derived length of* M/N *is at most* r.

(iii) *If* $M \lhd G$, $M \in T_m$ *and* $G/M \in T_n$, *then* $G \in T_{m+n}$.

LEMMA 2. *Let G be a group and A be a finite automorphism group of G. Let N be a normal A-invariant subgroup of G and* $C/N = C_{G/N}(A)$.

(i) *If* $N \leq Z(G)$, *then* $C_G(A) \lhd C$ *and* $C/C_G(A)$ *is isomorphic to a direct product of subgroups of N. In particular* $C/C_G(A)$ *is an abelian group.*

(ii) *If* $[N, G, G\ldots, G] = 1$ *with a finite number of terms of G and* $C \in T_n$ *(respectively* $T_{n,r}$*), then* $C_G(A) \in T_n$ *(respectively* $T_{n,r}$*).*

The commutator in (ii) is left normed, so that N lies in the hypercenter of G.

PROOF (i) Let $A = \langle a_1, \ldots, a_n \rangle$. For each $i = 1, 2, \ldots, n$ define a map

$$\phi_{a_i}\colon C \to N,$$

$$\phi_{a_i}(g) = [a_i, g],$$

ϕ_{a_i} is a homomorphism with $\operatorname{Ker}\phi_{a_i} = C_C(a_i)$. So $C/C_C(a_i)$ is isomorphic to a subgroup of N. Then we get

$$C/\cap C_C(a_i) \hookrightarrow C/C_C(a_1) \times C/C_C(a_2) \times \ldots \times C/C_C(a_n).$$

Since each of $C/C_C(a_i)$ is isomorphic to a subgroup of $Z(G)$, the group $C/C_G(A)$ is abelian.

(ii) Let $N_0 = N$, $N_1 = [N, G]$, $\ldots$, $N_{i+1} = [N_i, G]$. Then $N_k = 1$. We get each $N_i \lhd G$ and a series

$$1 = N_k \lhd N_{k-1} \lhd \ldots N_1 \lhd N_0 = N.$$

Let $C_{G/N_i}(A) = C_i/N_i$ and $C_k = C_G(A)$. Since $N_{k-1} \leq Z(G)$, by (i) we have $C_{k-1}/C_{C_{k-1}}(A)$ is abelian.

$C_{i+1} \lhd C_i$; to see this we define a map ϕ_{a_j} for each $a_j \in A$ as in the first case:

$$\phi_{a_j}\colon C_i \to N_i/N_{i+1},$$

$$g \to g^{-1}a_j^{-1}ga_jN_{i+1},$$

the intersection of the kernels of these maps is C_{i+1}; and C_i/C_{i+1} is abelian. Hence $C_G(A) \lhd \lhd C$ and by Lemma 1 we get $C_G(A) \in T_n$.

LEMMA 3. [7], 2.3) (i) *If $G \in T_{n,r}$, then G has a finite series of length at most $2n+1$, the factors of which comprise at most n non-abelian simple factors, at most $n+1$ soluble groups of derived length at most r and no others.*

(ii) $L\ T_n = T_n$.

Centralizers of elements in symmetric groups are well known.

LEMMA 4 ([7], 2.4). – *Let G be the symmetric group Sym(l) and x be an element of order n in G. Suppose that the cycle decomposition of x involves k_i cycles of length i $(1 \leq i \leq n\ i|n)$. Then*

$$C_G(x) \cong Dr_{i|n} L_i$$

where Dr denotes direct product , and L_i is a permutational wreath product $C_i \wr \mathrm{Sym}(k_i)$ of the cyclic group C_i of order i and the symmetric group $\mathrm{Sym}(k_i)$ acting naturaly on k_i points. If $k_i = 0$, then L_i is to be interpreted as 1.

LEMMA 5. *Let $F = \langle a_1, \ldots, a_n \rangle$ be an abelian subgroup of $G = \mathrm{Sym}(l)$ and $|F| = m$. Then $C_G(F) \in T_{g(m)}$ where g is a function of m independent of G.*

The proof of the Lemma 5 goes along the lines of the proof of the Lemma 4. We replace the argument on cycles of an element of equivalent length with the equivalent representations of F on the orbits of F. But the bound in the Lemma 4 is no longer valid; the number of non-abelian simple factors in Lemma 5 is less than or equal to the number of subgroups of F.

Similarly this Lemma holds for alternating group $\mathrm{Alt}(l)$.

If l is sufficiently large, then $C_G(F)$ involves alternating groups of arbitrary high orders.

LEMMA 6. *Suppose that G is infinite and every finite set of elements of G lies in a finite alternating subgroup. Let F be an abelian subgroup of order m in G. Then $C = C_G(F)$ has a series of finite length in which the factors consist of at most $g(m)$ simple non-abelian groups. Further C involves a non-linear simple group.*

PROOF. G has a local system consisting of alternating subgroups and each subgroup in the local system contains F. Now by Lemma 5 we have $C_{G_i}(F) \in T_{g(m)}$ where G_i is isomorphic to an alternating group and i

is taken from the index set I. $C_G(F)$ becomes locally $T_{g(m)}$. By Lemma 3 we get $C \in T_{g(m)}$ and we are done.

Now we will mention some of the facts about infinite LFS-groups. Some of the questions about infinite LFS-groups can be reduced to questions about countably infinite LFS-groups by using [8], Theorem 1.L.9 and Theorem 4.4. The question of whether the centralizer of a finite subgroup involves an infinite simple group or not is one of these types of questions. If in every countably infinite non-linear LFS-group the centralizer of every finite subgroup involves an infinite simple group, then in any infinite non-linear LFS-group centralizer of a finite subgroup also involves an infinite simple group. Therefore we confine ourselves to countable LFS-groups. For countable LFS-groups [8] Theorem 4.5 says that for every countably infinite LFS-group there exists a Kegel sequence $K = (G_i, N_i)$ where G_i's form a tower of finite subgroups of G satisfying $G = \bigcup_{i=1}^{\infty} G_i$, $N_i \lhd G_i$, such that G_i/N_i is a finite simple group and $G_i \cap N_{i+1} = 1$ for each i. By [8] , Theorem 4.6 if G is an infinite linear LFS-group one can always choose an infinite subsequence (G_j, N_j) such that $N_j = 1$ for all j.

By using classification of finite simple groups one can find that every LFS-group is either linear or G_i/N_i are all alternating or a fixed type of classical linear group over various fields with unbounded rank parameter. See [7] for more details about Kegel sequences.

THEOREM 3. *Let G be a connected reductive linear algebraic group and F be a finite subgroup of order m contained in a maximal torus T in G. Then $C_G(F) \in T_{f(m)k}$ where k is the number of simple components of the semisimple part of G when it is written as a product of simple linear algebraic groups and f is a function from natural numbers to natural numbers and is independent of G.*

By using the above theorem we prove:

THEOREM 4. *Let G be a connected simple linear algebraic group of classical type. Let F be a finite subgroup of order m contained in a maximal torus of G. If F is fixed pointwise by a Frobenius automorphism σ of G, then $(C_G(F))^\sigma \in T_{f(m)}$ where f is a function from natural numbers to natural numbers and is independent of G.*

PROOF OF THEOREM 3. Let G be a connected reductive linear algebraic group. Then by [10] (E 1.4) $G = Z^0 G'$ where Z^0 is the connected component of the centre of G and G' is the commutator subgroup. G' is a connected semisimple group, moreover $G' \cap Z^0$ is a finite normal subgroup of G. If

$$C/Z^0 = C_{G/Z^0}(F) \in T_{f(m)k}$$

then by Lemma 1 the group $C \in T_{f(m)k}$. But G/Z^0 is a semisimple group. Hence we may assume that G is semisimple. Then $G = G_1 G_2 \dots G_k$ where G_i are simple linear algebraic groups.

Let $Z = Z_1 \dots Z_k = Z(G)$ where $Z_i = Z(G_i)$. Then

$$G/Z = G_1 Z/Z \times \dots \times G_k Z/Z .$$

But $G_i Z/Z \cong G_i / G_i \cap Z$

Hence $\overline{G} = G/Z = \overline{G_1} \times \dots \times \overline{G_k}$. Then

$$C_{G/Z}(F) = C_{\overline{G_1}}(F_1) \times \dots \times C_{\overline{G_k}}(F_k)$$

where F_i's are the images of F under the projection of G onto G_i. Now if the number of non-abelian simple factors in $C_{G_i Z/Z}(F_i)$ is at most $f(m)$, then the number of non-abelian simple factors of $C_{G/Z}(F)$ is $f(m)k$. Then by Lemma 1 we have $C_G(F) \in T_{f(m)k}$. For exceptional types the connected components of the Dynkin diagram is already fixed so we may assume that the simple components of the semisimple part of G are of classical type.

Therefore it is enough to prove the following:

If G is a simple linear algebraic group of classical type, F a finite subgroup of G of order $\leq m$ and contained in a maximal torus T of G, then $C_G(F) \in T_{f(m)}$.

Let $F = \langle a_1, \dots, a_n \rangle$ where $|a_i| = m_i$ and $|F| = m = m_1 m_2 \dots m_n$. Then by [10] Theorem 4.1

$$C_G(F) = \langle T, X_\alpha, \ n_w \mid \alpha(a_i) = 1, \alpha \in \Phi, a_i^w = a_i, \ \ i = 1, 2, \dots, n \rangle,$$

$$C_G(F)^0 = \langle T, X_\alpha \mid \alpha(a_i) = 1, \alpha \in \Phi, i = 1, 2, \dots n \rangle$$

where X_α's are the root subgroups with respect to the torus T. The group $C_G(F)^0$ is a reductive group. Since every element in F is semisimple and $C_G(a_i)/C_G(a_i)^0$ is an abelian group by [10], Corollary 4.4, we get that $C_G(F)/C_G(F)^0$ is an abelian group. Now by Lemma 1, it is enough to show that $C_G(F)^0 \in T_{f(m)}$.

Since the maximal torus T and the character group of the root lattice are isomorphic as abelian groups, for every element $a_i \in T$, there exists a character χ_{a_i} of the root lattice corresponding to a_i.

Let

$$\Psi = \{\alpha \mid \alpha(a_i) = 1,\ i = 1, 2, \dots, n\}.$$

Ψ is a subroot system of Φ in the sense that Ψ is itself a root system and if the sum of any two roots in Ψ is a root in Φ, then their sum is again in Ψ. The subroot system may not be connected but it can be written as a union of connected root systems. But by [4], page 25 every root system determines the simple group up to isogeny and the groups corresponding to disjoint connected components centralize each other. Each connected component of Ψ corresponds to a subgroup K of $C_G(F)^0$ such that $K/Z(K)$ is simple.

Hence in order to find the number of non-abelian simple factors of $C_G(F)^0$ it is enough to find the number of connected components of Ψ.

Let L_E be the corresponding Lie algebra of the linear algebraic group G over an algebraically closed field E. Then χ_{a_i} acts on the Lie algebra as $\chi_{a_i}(h) = h$ for all h in the Cartan subalgebra of L_E and $\chi_{a_i}(e_r) = \chi_{a_i}(r)(e_r)$ for all $e_r \in L_r$.

Given a connected root system and non-trivial characters χ_{a_i} of order m_i, $i = 1, 2, \dots, n$, we need to show that the number of irreducible components of

$$\Psi = \{\alpha \in \Phi \mid \chi_{a_i}(\alpha) = 1 \text{ for all } i = 1, 2 \dots, n\}$$

is less than $f(m)$.

So the problem reduces actually to a root system problem.

In [9] we found that for each χ_{a_i} the number of connected components of Ψ is at most $m_i + 2$. Here by using similar methods as in [9] we show that the number of connected components of Ψ is at most $f(m) = m^n$, $n \leqslant m$.

We give the proof only for the type A_l because the other classical types can be handled easily by adapting the same technique.

Let s be the least common multiple of $(m_1, m_2, \dots, m_n)$. Since each χ_{a_i} is of order m_i, for each i, $\chi^s_{a_i}$ is identity on the root lattice. So for each $r \in \Psi$, $\chi_{a_i}(r)$ is s^{th} root of unity.

Now let Φ be the root system of type A_l. By [3] page 45

$$\Phi = \{e_i - e_j \mid i \neq j,\ i, j \in \{1, 2, \dots, l+1\}\}$$

where $e_1, e_2, \dots, e_{l+1}$ is an orthonormal basis of an Euclidean space of dimension $l + 1$. The following vectors form a fundamental system for A_l

$$e_1 - e_2,\quad e_2 - e_3,\ \dots,\ e_l - e_{l+1}.$$

$\chi_{a_i}(e_i - e_k)$ is an s^{th} root of unity. In order to make calculations easier we would like to extend χ_{a_i} for all i from root lattice to $\sum_{i=1}^{l+1} Ze_i$. As root lattice and $\sum_{i=1}^{l} Ze_i$ are abelian groups and χ_{a_i} is a homomorphism from the root lattice to the divisible abelian group K^* of the multiplicative group of the field K, χ_{a_i} can be extended from root lattice to $\sum_{i=1}^{l} Ze_i$. We can define $\chi_{a_i}(e_{l+1})$ for case A_l as we please. Let $\chi_{a_i}(e_{l+1}) = 1$. So $\chi_{a_i}(e_l - e_{l+1}) = \chi_{a_i}(e_l)\chi_{a_i}(e_{l+1})^{-1} = \lambda_l$. Hence $\chi_{a_i}(e_l) = \lambda_l$. Therefore $\chi_{a_i}(e_i)$ is an s^{th} root of unity for all $i = 1, 2, \dots l+1$. For each n-tuple $(\lambda_1, \dots, \lambda_n)$ of the s^{th} roots of 1 the sets

$$S(\lambda_1, \dots, \lambda_n) = \{j \ : \ \chi_{a_i}(e_j) = \lambda_i \ \text{ for all } \ i = 1, 2, \dots, n\}$$

form a partition of $\{1, 2, \dots, l+1\}$ into not more than s^n disjoint sets. Since the roots of A_l are of the form $e_i - e_j$, $i \neq j$, we have

$$\chi_a(e_i - e_j) = 1 \quad \text{ iff } \chi_a(e_i)\chi_a(e_j)^{-1} = 1 \quad \text{ iff } \chi_a(e_i) = \chi_a(e_j)$$

if and only if i and j belong to the same $S(\lambda_1, \dots, \lambda_n)$. Then the set

$$\{e_i - e_j \ : \ i \neq j \ i \ \textit{and} \ j \ \textit{belong to the same} \ S(\lambda_1, \dots, \lambda_n)\}$$

forms a subroot system of Φ.

The elements $e_j - e_k$ of Ψ having index in $S(\lambda_1, \dots, \lambda_n)$, with the property that $k, j \in S(\lambda_1, \dots, \lambda_n)$ and there exists no m between k and j in $S(\lambda_1, \dots, \lambda_n)$, form a basis Δ for Ψ.

Observe that any two roots in Δ having their indices from different $S(\lambda_1, \dots, \lambda_n)$ are orthogonal i.e. If $i, j \in S(\lambda_1, \dots, \lambda_n)$ and $n, m \in S(\mu_1, \dots, \mu_n)$ where $S(\lambda_1, \dots, \lambda_n) \neq S(\mu_1, \dots, \mu_n)$, then $e_i - e_j$ and $e_n - e_m$ are orthogonal. We also observe that the roots in Δ having their indices from a fixed $S(\lambda_1, \dots, \lambda_n)$ form a connected root system. If $S(\lambda_1, \dots, \lambda_n) = \{j_1, j_2 \dots, j_{t(i)}\}$ with $j_1 < j_2 \dots < j_{t(i)}$, then $\{e_s - e_t : s, t \in S(\lambda_1, \dots, \lambda_n)\}$ is a root system of type $A_{t(i)-1}$ as the root system arising from $S(\lambda_1, \dots, \lambda_n)$ contains only one type of root. Therefore each nonempty $S(\lambda_1, \dots, \lambda_n)$ with $|S(\lambda_1, \dots, \lambda_n)| \geqslant 2$ gives a connected component and the connected components arising from different $S(\lambda_1, \dots, \lambda_n)$ are orthogonal to each other. Hence the number of irreducible components of Ψ is less than or equal to the number of nonempty $S(\lambda_1, \dots, \lambda_n)$ with $|S(\lambda_1, \dots, \lambda_n)| \geqslant 2$ which is less than or equal to $s^n \leqslant m^n \leqslant m^m$.

Hence Ψ has less than or equal to m^n connected components and each of them is of type A_k for some k.

It is clear that if the rank parameter l is greater than m^n, then Ψ is non-empty. If moreover $l > t(m^n)$, then $C_G(F)$ involves simple groups of rank greater than t. Hence $C_G(F)$ involves subgroups isomorphic to alternating groups Alt (t).

For the other cases $C_G(F)^0$ is also reductive and we have $C_G(F)^0 = Z^0 C$ where C is a semisimple connected linear algebraic group and $C = C_1 C_2 \dots C_k$, $k \leq f(m)$ where each C_i is a simple linear algebraic group corresponding to the roots in the corresponding irreducible root system. Hence C has a series of finite length consisting of at most $f(m)$ non-abelian simple factors. Since the type of the root system determines the type of the linear algebraic group up to isogeny, we know the possible types as well.

PROOF OF THEOREM 4. Let F be the subgroup satisfying the assumptions of the theorem. Then by [10] Lemma 5.9 there exists a maximal σ-invariant torus T of G containing F. Then F becomes a semisimple subgroup in the linear algebraic group G. So we can use all the theory for the semisimple subgroups of linear algebraic groups. By Theorem 3, $C_G(F) \in T_{f(m)}$ and by previous observation $C_G(F)/C_G(F)^0$ is an abelian group. It is clear that $C_G(F)$ is σ-invariant. Hence $C_G(F)^0$ is σ-invariant moreover $C_G(F)/C_G(F)^0$ is σ-invariant.

$C_G(F)$ is closed and $C_G(F)^0$ is closed and connected, then by [4] page 33

$$((C_G(F))/(C_G(F)^0))^\sigma \cong (C_G(F))^\sigma/(C_G(F)^0)^\sigma$$

which is abelian. Since we are interested in the number of non-abelian simple factors it is enough to find the number of non-abelian simple factors of $(C_G(F)^0)^\sigma$. The group $C_G(F)^0$ is a reductive group. So $C_G(F)^0 = CZ^0$ where C is a connected semisimple subgroup of G. Since Z^0 and C are σ invariant and Z^0 is abelian, by Lemma 1 it is enough to find the number of non-abelian composition factors of C^σ. Let $C = C_1 C_2 \dots C_k$ where each C_i is a simple linear algebraic group $k \leq f(m)$. Let $Z = Z(C) = Z(C_1) \dots Z(C_k)$. Then $C/Z = \overline{C} = \overline{C}_1 \overline{C}_2 \dots \overline{C}_k$ and $\overline{C}_i = C_i Z/Z$. By Krull Schmidt Theorem $(C_i Z)^\sigma = C_j Z$, then by taking the derived group we see that $(C_i)^\sigma = ((C_i Z)')^\sigma = ((C_j Z)')^\sigma = C_j$. Therefore σ permutes the C_i's. Let O_i, $i = 1, 2, \dots, r$ be the orbits of σ on $\{C_1, C_2, \dots, C_k\}$ and let $K_i = \prod_{D \in O_i} D$. Hence C is the central product of $K_1 \dots K_r$. Let $\overline{K}$ be any one of the orbits of σ on $\overline{C}$ say for simplicity the one containing $\overline{C}_1$

$$\overline{K} = \overline{C}_1 \overline{C}_1^{\sigma} \overline{C}_1^{\sigma^2} \dots \overline{C}_1^{\sigma^{t(1)-1}}$$

and $(\overline{C}_1)^{\sigma^{t(1)}} = \overline{C}_1$. Then $\overline{K}$ is the direct product of groups $\overline{C}_1^{\sigma^i}$ and

$$\overline{K}^{\sigma} = \{c_0 c_1^{\sigma} \dots c_{t-1}^{\sigma^{t-1}} \mid (c_0 c_1^{\sigma} \dots c_{t-1}^{\sigma^{t-1}})^{\sigma} = c_0 c_1^{\sigma} \dots c_{t-1}^{\sigma^{t-1}}\}$$

where $c_i \in \overline{C}_1$. This implies that $c_i = c_0$ for all $i = 1, \dots, t-1$. Hence

$$\overline{K}^{\sigma} = \{c\ c^{\sigma} \dots c^{\sigma^{t-1}} \mid c \in \overline{C}_1,\ c^{\sigma^t} = c\} \cong C_{\overline{C}_1^{\sigma^t}}$$

Since σ is a Frobenius automorphism, σ^t is also a Frobenius automorphism. $\overline{C}_1$ is a simple group and the fixed points of a Frobenius automorphism of a simple linear algebraic group form a group of the same type possibly the twisted version of it. So $C_{\overline{C}_1}(\sigma^{t(1)}) \in T_1$. Since $\overline{C} = \overline{K}_1 \overline{K}_2 \dots \overline{K}_r$. We have $\overline{C}^{\sigma} \in T_r$ where $r \leqslant f(m)$. Hence $C^{\sigma} \in T_{f(m)}$ as required.

Corollary. *Let X be a finite simple group of classical type and F be a nice abelian subgroup of order m consisting of semisimple elements. Then $C_X(F) \in T_{f(m)}$.*

Proof. Let X be a simple group of classical type and F be an abelian subgroup as above. We may assume that $X = O^{p'}(G^{\sigma})$ where G is a linear algebraic group of adjoint type and σ is a Frobenius automorphism of G. Then F becomes an abelian subgroup of commuting semisimple elements of G. Now by [10], Theorem 5.8(c) and 5.11 there exist a maximal torus of G containing F. By [10], Corollary 5.9 there exists a σ-invariant maximal torus of G containing F. Then $C_X(F) = C_G(F) \cap O^{p'}(G^{\sigma}) = C_{G^{\sigma}}(F) \cap O^{p'}(G^{\sigma})$. Now by Theorem 3, Theorem 4 and Lemma 1 we have the result.

Proof of Theorem 2. If all finite subgroups are contained in an alternating group, then every finite subgroup is semisimple and by Lemma 6 we are done. Hence we may assume that, all finite subgroups are contained in linear groups of fixed classical type. Now if there exists a prime p such that every finite subset of G is contained in a finite simple group of fixed classical type over a field of characteristic p, then F can be chosen as any abelian subgroup of G such that p does not divide $|F|$; in case simple groups of type B_l or D_l we choose F abelian with cyclic Sylow 2-subgroup. If there exists no such a prime p, then each prime appears only finitely many times as a characteristic of the field. Now choose any abelian subgroup F of G, let $|F| = m$ and consider $\{p_i \mid p_i \mid m\}$.

Since each p_i appears only finitely many times we may discard finitely many of the G_i's and obtain a local system in such a way that $F \leqslant G_i$ for all i where i is in some index set I and F is semisimple in every member G_i of the new local system. If necessary we take a subgroup of F with cyclic Sylow 2-subgroup. Now by Corollary we have $C_i = C_{G_i}(F) \in T_{f(m)}$ for all i. Now using Lemma 3 $C \in T_{f(m)}$. Hence we are done.

PROOF OF THEOREM 1. Let F be a K-nice abelian subgroup and $K = (G_i, M_i)$ be the given Kegel sequence of G. If necessary by passing to a subsequence, we may assume that G_i/M_i are all alternating or all belong to a fixed classical family. Then either Theorem 4 applies and $C_{G_i/M_i}(F) \in T_{f(m)}$ or Lemma 4 applies and $C_{G_i/M_i}(F) \in T_{g(m)}$. Since $(|M_i|, |F|) = 1$ we get $C_{G_i/M_i}(F) = C_{G_i}(F)M_i/M_i \in T_{f(m)}$ in the first case and in $T_{g(m)}$ in the second case. By assumption M_i's are soluble hence by Lemma 1 either $C_{G_i}(F) \in T_{f(m)}$ for all i or $C_{G_i}(F) \in T_{g(m)}$ for all i. By Lemma 3 we get $C_G(F) \in T_{f(m)}$ or $T_{g(m)}$.

Since $C_{G_i}(F)$ involves alternating groups of unbounded orders and $C_G(F)$ has a series of finite length in which each factor is either non-abelian simple or locally soluble, one of the factors of the series of $C_G(F)$ must be non-linear.

We may extend Theorem 1 to somewhat more general LFS-groups. This will be the extension of [7] Theorem B' from a single semisimple element to an abelian subgroup consisting of semisimple elements.

THEOREM 5. *Let G be a non-linear* LFS-*group and F be an abelian subgroup of order m in G. Let π be the set of prime divisors of the order of F. Suppose there exists a Kegel sequence $K = (G_i, N_i)$ of G such that F is nice for all Kegel components G_i/N_i and*

(i) *$O_{\pi'}(N_i)$ is soluble.*

(ii) *$N_i/O_{\pi'}(N_i)$ is hypercentral in G_i.*

(iii) *G_i/N_i is either an alternating group or a classical group defined over a field of characteristic not in π.*

Then $C_G(F)$ belongs to $T_{f(m)}$ and involves a non-linear simple group.

We omit the proof as the technique is similar to the proof of Theorem 1.

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