Rendiconti del Seminario Matematico della Università di Padova

FELIX LEINEN

Absolute irreducibility for finitary linear groups

Rendiconti del Seminario Matematico della Università di Padova, tome 92 (1994), p. 59-61

<http://www.numdam.org/item?id=RSMUP_1994__92__59_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 92 (1994)

Absolute Irreducibility for Finitary Linear Groups.

FELIX LEINEN (*)

Let V be a vector space over the field K. We denote by $FGL(V)$ the group of all *finitary linear transformations* of V, i.e., the group of all K-isomorphisms g of V satisfying $\dim_K [V, g] < \infty$. Subgroups of $FGL(V)$ are said to be *finitary linear*. During the last few years, these groups have been an area of intense and fruitful research, investigated by V. V. Belyaev, J. I. Hall, U. Meierfrankenfeld, R. E. Phillips, O. Puglisi and B. A. F. Wehrfritz. Our aim here is it to carry over the machinery of absolute irreducibility from linear groups to finitary linear groups.

For every field extension L/K, there is a natural action of the general linear group $GL(V)$ on the L-space $V^L = L \otimes_K V$, which embeds $GL(V)$ into $GL(V^L)$. A subgroup G of $GL(V)$ is said to be *absolutely irreducible* if, for every field extension L/K, the vector space V^L is an irreducible LG-module. We will extend [2, Theorem 1.19] literally to the following result about finitary linear groups.

THEOREM. *Let V be a vector space over the field K, and let G be an irreducible subgroup of $FGL(V)$. Then there exists a finite field extension L/K such that V is also a vector space over L, and such that V is absolutely irreducible as LG-module. Here, the degree $[L:K]$ of the field extension is finite, and it divides the K-dimension of every finite-dimensional L-subspace of V. Examples of L-subspaces are $C_V(F)$ and $[V, F]$ for every finitely generated subgroup F of G.*

Here, the finiteness of the degree $[L:K]$ may be a small surprise. In order to prove the Theorem, we will use the following Lemma, which is a counterpart to [2, Corollary 1.18].

(*) Fachbereich Mathematik, Universität Mainz, D-55099 Mainz, Germany. E-mail: leinen@mzdmza.zdv.uni-mainz.de.

LEMMA. *Let V be a vector space over the field K. Then, for every subgroup G of $FGL(V)$, the following are equivalent.*

(1) *G is absolutely irreducible.*

(2) *$V^{\overline{K}}$ is an irreducible $\overline{K}G$-module.*

(3) *V is an irreducible KG-module, and $C_{\mathrm{End}_K(V)}(KG) = K \cdot \mathrm{id}_V$.*

Here, $\overline{K}$ denotes the algebraic closure of the field K.

PROOF. For the proof of (2)$\Rightarrow$(3), let $\overline{E} = \mathrm{End}_{\overline{K}}(\overline{V})$ where $\overline{V} = V^{\overline{K}}$. Let $\phi \in C_{\overline{E}}(\overline{K}G)$ and $1 \neq g \in G$. Since $C_{\overline{V}}(g)$ is a proper ϕ-invariant subspace of finite codimension in $\overline{V}$, and since the field $\overline{K}$ is algebraically closed, the $\overline{K}$-endomorphism induced by ϕ on $\overline{V}/C_{\overline{V}}(g)$ has an eigenvalue $\lambda \in \overline{K}$. It follows, that the image of $\phi - (\lambda \cdot \mathrm{id}_{\overline{V}})$ has non-zero codimension in $\overline{V}$, whence $\phi - (\lambda \cdot \mathrm{id}_{\overline{V}})$ is not surjective. But $C_{\overline{E}}(\overline{K}G)$ is a division ring by Schur's Lemma [2, 1.1]. Thus $\phi = \lambda \cdot \mathrm{id}_{\overline{V}}$ and this shows that $C_{\overline{E}}(\overline{K}G) = \overline{K} \cdot \mathrm{id}_{\overline{V}}$. We can now continue as in the proof of [2, Corollary 1.18] in order to obtain (3) from (2).

For the proof of (3)$\Rightarrow$(1), we first apply the Density Theorem [1, p. 41, Theorem 2.1.2]: Under the hypothesis (3), the group algebra KG acts densely on V, i.e., whenever $v_1, \ldots, v_n$ are linearly independent vectors in V, and whenever $w_1, \ldots, w_n$ are arbitrary vectors in V, there exists some $x \in KG$ such that $w_i = v_i x$ for $1 \leqslant i \leqslant n$. Let L/K be a field extension of K. Suppose, that U is a non-trivial LG-submodule of V^L and fix any non-trivial $u \in U$. Then $u = \sum_{i=1}^{n} l_i \otimes v_i$ for suitable $l_i \in L - \{0\}$ and a linearly independent subset $\{v_1, \ldots, v_n\}$ of V. Let $w \in V$ be arbitrary. Then by the Density Theorem, there exists $x \in KG$ such that $v_1 x = w$ and $v_i x = 0$ for $2 \leqslant i \leqslant n$. It follows, that $l_1 \otimes w \in U$. This shows, that $U = V^L$. So V^L must be an irreducible LG-module. ■

PROOF OF THE THEOREM. By Schur's Lemma, $\Delta = C_{\mathrm{End}_K(V)}(KG)$ is a division ring. Let L be a maximal subfield of Δ. Then, using the above Lemma, we can follow the proof of [2, Theorem 1.19] in order to show, that V is a vector space over the field L, and that V is absolutely irreducible as LG-module. Clearly, $\dim_K U = \dim_L U \cdot [L : K]$ for every finite-dimensional L-subspace U of V. Moreover, for every finitely generated subgroup F of G, the K-subspaces $C_V(F)$ and $[V, F]$ are L-invariant. Finally the finiteness of $\dim_K [V, F]$ enforces the finiteness of $[L : K]$, and this completes the proof. ■

REFERENCES

[1] I. N. HERSTEIN, *Noncommutative Rings*, J. Wiley & Sons (1971).
[2] B. A. F. WEHRFRITZ, *Infinite Linear Groups*, Springer-Verlag, Berlin - Heidelberg - New York (1973).

Manoscritto pervenuto in redazione il 13 gennaio 1993.