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Wiener Tauberian Theorems for Ultradistributions.

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SUMMARY - The purpose of this paper is the extension of Wiener Tauberian theorems for distributions ([6]) on ultradistribution spaces. Because of that, we give the versions of Beurling's and Wiener's theorems for bounded ultradistributions. The corollary of our main theorem is the following one. Let f be an ultradistribution such that f/c is a bounded ultradistribution, where c is a smooth function which behaves as $L(e^x) e^{\alpha x}$, $x \to \infty$, L is a slowly varying function at ∞ and $\alpha \in \mathbf{R}$. If for an ultradifferentiable function ϕ with the property $\mathcal{F}[\phi](\xi - i\alpha) \neq 0$, $\xi \in \mathbf{R}$,

$$\lim_{x \to \infty} \frac{(f * \phi)(x)}{L(e^x) e^{\alpha x}} = a \int \phi(t) e^{-\alpha t} \, dt, \qquad a \in \mathbf{R},$$

then for every ultradifferentiable function ψ

$$\frac{(f * \psi)(x)}{L(e^x) e^{\alpha x}} \to a \int \psi(t) e^{-\alpha t} dt, \qquad x \to \infty.$$

1. Notation and notions.

With N and $\mathbf{R}$ are denoted the sets of natural and real numbers; $N_0 = N \cup \{0\}$. If f is a function on $\mathbf{R}$, then $\overset{\vee}{f}$ denotes the function defined by $\overset{\vee}{f}(x) = f(-x)$, $x \in \mathbf{R}$. $\mathcal{C}^\infty$ denotes the space of smooth functions on $\mathbf{R}$ and $\mathcal{L}^1$ is the space of Lebesgue integrable funstions (classes) on $\mathbf{R}$ with the usual norm $\| \ \|_{\mathcal{L}^1}$. For an f from $\mathcal{L}^1$ the Fourier transform is denoted by $\mathcal{F}f$ or $\hat{f}$. $\mathcal{L}^1_{\text{loc}}$ and $\mathcal{L}^\infty$ are defined in a usual way.

We shall always denote by L a slowly varying function ([1]). Recall,

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L is measurable, positive and

$$\frac{L(xh)}{L(h)} \to 1, \qquad h \to \infty, \quad x > 0.$$

For every $\delta > 0$ there is $C_\delta > 0$ such that

$$(1) \quad \frac{1}{C_\delta} \min\left\{\left(\frac{x}{y}\right)^\delta, \left(\frac{y}{x}\right)^\delta\right\} \leq \frac{L(x)}{L(y)} \leq C_\delta \max\left\{\left(\frac{x}{y}\right)^\delta, \left(\frac{y}{x}\right)^\delta\right\},$$

$$x > 0, \quad y > 0.$$

For the notation and properties of the spaces $\mathcal{D}^{(M_\alpha)}$, $\mathcal{D}^{\{M_\alpha\}}$ and their strong duals $\mathcal{D}'^{(M_\alpha)}$ (the space of Beurling ultradistributions) and $\mathcal{D}'^{\{M_\alpha\}}$ (the space of Roumieau ultradistributions) we refer to [3]. We shall assume that the sequence M_α satisfies conditions (M.1), (M.2) and (M.3)' ([3]).

As in [5] we put

$$\mathcal{D}_{\mathcal{L}^1}^{(M_\alpha)} = \text{proj}_{h \to \infty} \mathcal{D}_{\mathcal{L}^1, h}^{M_\alpha}, \qquad \mathcal{D}_{\mathcal{L}^1}^{\{M_\alpha\}} = \text{ind}_{h \to 0} \mathcal{D}_{\mathcal{L}^1, h}^{M_\alpha},$$

where $\mathcal{D}_{\mathcal{L}^1, h}^{M_\alpha}$, $h > 0$, is a Banach space of smooth functions φ on $\mathbf{R}$ with the finite norm

$$\|\varphi\|_{\mathcal{L}^1, h} = \sup\left\{\frac{h^\alpha}{M_\alpha} \|\phi^{(\alpha)}\|_{\mathcal{L}^1}; \alpha \in \mathbf{N}_0\right\}.$$

The common notation for (M_α) and $\{M_\alpha\}$ will be $*$. The space $\mathcal{D}^*$ is dense in $\mathcal{D}_{\mathcal{L}^1}^*$ and the inclusion mapping is continuous. The strong dual of $\mathcal{D}_{\mathcal{L}^1}^*$ is denoted by $\mathcal{B}'^*$.

The spaces of tempered ultradistributions are defined as the strong duals of the following testing function spaces ([4])

$$S^{(M_\alpha)} = \text{proj}_{h \to \infty} S_h^{M_\alpha}, \qquad S^{\{M_\alpha\}} = \text{ind}_{h \to 0} S_h^{M_\alpha},$$

where $S_h^{M_\alpha}$, $h > 0$, is the Banach space of smooth functions φ on $\mathbf{R}$ with the finite norm:

$$\gamma_h(\varphi) = \sup\left\{\frac{h^{\alpha + \beta}}{M_\alpha M_\beta} \|(1 + x^2)^{\alpha/2} \varphi^{(\beta)}\|_{\mathcal{L}^\infty}, \alpha, \beta \in \mathbf{N}_0\right\}.$$

The Fourier transformation is an isomorphism of S^* onto S^* and $\mathcal{D}^*$ is dense in S^*. Clearly, S^* is dense in $\mathcal{D}_{\mathcal{L}^1}^*$ and the inclusion mapping is continuous.

Recall from [2] that a sequence of continuous and bounded functions

f_n on $\boldsymbol{R}$ converges narrowly to a continuous bounded function f_0 if and only if f_n converges to f_0 uniformly on bounded sets in $\boldsymbol{R}$ and

$$\|f_n\|_{\mathscr{L}^\infty} \to \|f_0\|_{\mathscr{L}^\infty}, \qquad n \to \infty.$$

We shall always assume that $\alpha, \beta \in \boldsymbol{R}$ and $\alpha > \beta$. Put

(2)
$$c_0(x) = \begin{cases} L(e^x) e^{\alpha x}, & x \geqslant 0, \\ e^{\beta x}, & x < 0. \end{cases}$$

The following regularization of this function will be used.

(3)
$$c(x) = (c_0 * \omega)(x), \qquad x \in \boldsymbol{R},$$

where $\omega \in \mathscr{D}^*$, $\operatorname{supp} \omega \subset [-1, 1]$, $\omega \geqslant 0$ and $\int_{\boldsymbol{R}} \omega(t)dt = 1$.

We shall denote by η a function from $\mathscr{E}^*$ with the properties

(4)
$$\eta(x) = 1, \qquad x > x_0 > 0, \qquad \eta(x) = 0, \qquad x < -x_0.$$

2. Assertions.

THEOREM 1. *Let $f \in \mathscr{B}'^*$ and $K \in \mathscr{D}^*_{\mathscr{L}^1}$ such that $\mathscr{F}[K](\xi) \neq 0$, $\xi \in \boldsymbol{R}$. If*

$$\lim_{x \to \infty} (f * K)(x) = a \int_{\boldsymbol{R}} K(t)\,dt, \qquad a \in \boldsymbol{R},$$

then for every $\psi \in \mathscr{D}^*_{\mathscr{L}^1}$,

$$\lim_{x \to \infty} (f * \psi)(x) = a \int_{\boldsymbol{R}} \psi(t)\,dt.$$

THEOREM 2. *Let $f \in \mathscr{D}'^*$ and $K \in C^\infty$. Assume:*

(i) $f/c \in \mathscr{B}'^*$.

(ii) *There exists $\delta > 0$ such that*

$$\eta \overset{\vee}{K} e^{(\alpha + \delta)\cdot}, (1 - \eta) \overset{\vee}{K} e^{\beta \cdot} \in \mathscr{D}^*_{\mathscr{L}^1}.$$

(iii) $\mathscr{F}[K](\xi - i\alpha) \neq 0$, $\xi \in \boldsymbol{R}$.

(iv) $\lim_{x \to \infty} (f * K)(x)/(L(e^x) e^{\alpha x}) = a \int_{\boldsymbol{R}} K(t) e^{-\alpha t}\,dt, \qquad a \in \boldsymbol{R}.$

Then for every $\psi \in C^\infty$ for which

(*) $\qquad\qquad\qquad \eta \overset{\vee}{\psi} e^{(\alpha + \delta)\cdot} \, , \; (1 - \eta) \overset{\vee}{\psi} e^{\beta\cdot} \in \mathcal{O}^*_{\mathcal{L}^1} \, ,$

there holds

$$\lim_{x \to \infty} \frac{(f * \psi)(x)}{L(e^x) e^{\alpha x}} = a \int \psi(t) e^{-\alpha t} \, dt \, .$$

REMARK. It is an open problem whether the assumption that the set $\{(f(\cdot + h))/c(h); \; h \in \mathbf{R}\}$ is bounded in $\mathcal{O}'^*$ implies that $f/c \in \mathcal{B}'^*$. Note that for distributions the corresponding assertion holds (see [7]).

COROLLARY 1. *Let $f \in \mathcal{O}'^*$ such that $f/c \in \mathcal{B}'^*$ and let $\phi \in \mathcal{O}^*$ such that $\mathcal{F}[\phi](\xi - i\alpha) \neq 0, \; \xi \in \mathbf{R}$. If*

$$\lim_{x \to \infty} \frac{(f * \phi)(x)}{L(e^x) e^{\alpha x}} = a \int \phi(t) e^{-\alpha t} \, dt \, , \qquad a \in \mathbf{R} \, ,$$

then for every $\psi \in \mathcal{O}^*$

$$\lim_{x \to \infty} \frac{(f * \psi)(x)}{L(e^x) e^{\alpha x}} = a \int \psi(t) e^{-\alpha t} \, dt \, .$$

3. Proofs.

PROOF OF THEOREM 1. First we need the following version of Beurling's theorem ([2]) for bounded ultradistributions.

«Let $f \in \mathcal{B}'^*$. A point ξ_0 belongs to $\text{supp}\,\widehat{f}$ if and only if there is a sequence of functions $\{\varphi_n\}$ from $\mathcal{S}^*$ such that

$$f_n(x) = (f * \varphi_n)(x), \qquad x \in \mathbf{R}, \qquad n \in \mathbf{N},$$

convergences narrowly to $f_0(x) = e^{ix\xi_0}, \; x \in \mathbf{R}, \; n \to \infty$».

The proof of this assertion is the same as for bounded distributions since all the properties of Schwartz's test functions which were used in [2, pp. 230-231], have been proved in [3] and [5] for ultradifferentiable functions. The same holds for the next assertion, based on the previous one, which is analogous to the Theorem on p. 232 in [2].

«Let $f \in \mathcal{B}'^*$ and $K \in \mathcal{O}_{\mathcal{L}^1}^*$. If $K * f \equiv 0$ on $\mathbf{R}$, then $\widehat{K}(\xi) = 0$ for $\xi \in \operatorname{supp}\widehat{f}$».

First, we shal prove that the set $\mathcal{M}$ which consists of finite linear combinations of translations of $K \in \mathcal{O}_{\mathcal{L}^1}^*$ is dense in $\mathcal{O}_{\mathcal{L}^1}^*$. By the property of dual pairing, $\mathcal{M}$ is dense in $\mathcal{O}_{\mathcal{L}^1}^*$ if and only if for every $S \in \mathcal{B}'^*$, $S * \check{K} = 0 \Leftrightarrow S = 0$. For, if $\mathcal{M}$ is not dense, there exists an $S_0 \in \mathcal{B}^*$, $S_0 \neq 0$ such that $S_0 * \check{K} = 0$. Thus $\mathcal{F}[\check{K}](\xi) = 0$, $\xi \in \operatorname{supp} \mathcal{F}[S_0]$. Since we assume $\mathcal{F}[\check{K}](\xi) = \mathcal{F}[K](-\xi)$ is never zero, we conclude that $\mathcal{M}$ is dense in $\mathcal{O}_{\mathcal{L}^1}^*$.

From that and previous statement we obtain the proof of the quoted Wiener theorem.

Note ([5]), $f \in \mathcal{B}'^*$ if and only if it is of the form

$$f = \sum_{\alpha = 0}^{\infty} D^\alpha F_\alpha, \qquad F_\alpha \in \mathcal{L}^\infty, \qquad \alpha \in N_0,$$

where D is derivative in $\mathcal{B}'^*$ and F_α, $\alpha \in N_0$, are such that for some $h > 0$ (in the (M_α)-case), respectively, for every $h > 0$ (in the $\{M_\alpha\}$-case)

$$(5) \qquad \sum_{\alpha = 0}^{\infty} \frac{M_\alpha}{h^\alpha} \|F_\alpha\|_{\mathcal{L}^\infty} = K_h < \infty.$$

Let $\psi \in \mathcal{O}_{\mathcal{L}^1}^*$. Since $\mathcal{M}$ is dense in $\mathcal{O}_{\mathcal{L}^1}^*$, then: In the (M_p) case, for every $\varepsilon > 0$ and every $h > 0$, there is $H_h \in \mathcal{M}$ such that

$$(6) \qquad \|H_h - \psi\|_{\mathcal{L}^1, h} < \varepsilon.$$

In the $\{M_p\}$ case we have that for every $\varepsilon > 0$ there is $h > 0$ and H such that (6) holds.

In the $\{M_p\}$ case, the assumption of the theorem and Lebesgue's theorem give that for $x > x_0(\varepsilon)$, where $x_0(\varepsilon)$ is large enough,

$$\left| (f * \psi)(x) - a \int_R \psi(\xi)\, dt \right| \leq \left| ((\psi - H) * f)(x) - a \int_R (\psi(t) - H(t))\, dt \right| +$$

$$+ \left| (H * f)(x) - a \int_R H(t)\, dt \right| \leq$$

$$\leq \left| \sum_{i=0}^{\infty} \int_R (\psi - H)^{(\alpha)}(t)\, F_\alpha(x - t)\, dt \right| + a \int_R |\psi(t) - H(t)|\, dt +$$

$$+ \left| (H * f)(x) - a \int_R H(t)\, dt \right| \leq$$

$$\leq \sup_\alpha \frac{h^\alpha}{M_\alpha} \| (\psi - H)^{(\alpha)} \|_{\mathcal{L}^1} \sum_{\alpha=0}^\infty \frac{M_\alpha}{h^\alpha} \| F_\alpha \|_{\mathcal{L}^\infty} + a\varepsilon +$$

$$+ \left| (H * f)(x) - a \int H(t)\, dt \right| \leq \varepsilon K_h + a\varepsilon + \varepsilon .$$

The (M_p) - case can be proved similarly. The proof is completed.

PROOF OF THEOREM 2. We shall only prove the (M_α)-case since this proof can be simply transfered to the $\{M_\alpha\}$-case.

The proof is organized as follows. In Part I we shall prove estimations (7), (7') and (8) which will be used in Part II for the proof that $\mathcal{F}(Ke^{-\alpha \cdot})(\xi)$, $\xi \in R^n$, and the convolution $f * K$ exist. In Part III we will prove the assertion of Theorem 2.

Part I. Note, from the assumption that $(1 - \eta)\overset{\vee}{K} e^{\beta \cdot}$ and $\eta \overset{\vee}{K} e^{(\alpha + \delta)\cdot}$ belong to $\mathcal{D}_{\mathcal{L}^1}^{(M_\alpha)}$ and (M.2) it follows that for every $r > 0$

$$(7) \quad \sup \left\{ \frac{r^m}{M_m} \left[\| e^{\beta x}((1 - \eta(x)\overset{\vee}{K}(x))^{(m)} \|_{\mathcal{L}^1} + \right. \right.$$

$$\left. \left. + \| e^{(\alpha + \delta)x}(\eta(x)\overset{\vee}{K}(x))^{(m)} \|_{\mathcal{L}^1} \right], \ m \in N_0 \right\} < \infty .$$

Since $e^{(\alpha + \delta)x} \leq e^{\beta x}$, for $x < 0$, we also have that for every $r > 0$

$$(7') \quad \sup \left\{ \frac{r^m}{M_m} \| e^{(\alpha + \delta)x}((1 - \eta(x))\overset{\vee}{K}(x))^{(m)} \|_{\mathcal{L}^1(-\infty, 0)}, \ m \in N \right\} < \infty .$$

We need the following estimate:
For every $r > 0$ there is $C > 0$ such that

$$(8) \quad \sup_{k \in N_0} \left\{ \frac{r^k}{M_k} \left| \left(\frac{c(x+h)}{L(e^h) e^{\alpha h}} - e^{\alpha x} \right)^{(k)} \right| \right\} \leq \begin{cases} Ce^{\alpha x + \delta |x|}, & x + h > 0, \\ Ce^{\beta x}, & x + h < 0, \end{cases}$$

where we choose δ such that $0 < \delta < \alpha - \beta$.

Let $r > 0$, $k \in N_0$ and $x + h > 1$. By using (3) and (1) we have (with

suitable constants)

$$\frac{r^k}{M_k} \left| \left(\frac{c(x+h)}{L(e^h)e^{\alpha h}} - e^{\alpha x} \right)^{(k)} \right| \leqslant$$

$$\leqslant \frac{r^k}{M_k} \int_{-1}^{1} \frac{L(e^{x+h-t})}{L(e^h)} e^{\alpha(x-t)} \left| \omega^{(k)}(t) \right| dt + \frac{|r\alpha|^k}{M_k} e^{\alpha x} \leqslant$$

$$\leqslant C_1 \, e^{\alpha x + \delta|x|} \, \|\omega\|_{[-1,1],r} + C_2 \, e^{\alpha x} \leqslant C e^{\alpha x + \delta|x|} \, .$$

Similarly, for $x + h < -1$ we get that for given $r > 0$ there is $C > 0$ such that

$$\sup_{k \in N_0} \left\{ \frac{r^k}{M_k} \left| \left(\frac{c_0(x+h)}{L(e^h)e^{\alpha h}} - e^{\alpha x} \right)^{(k)} \right| \right\} < C e^{\beta x} \, .$$

Let $u = x + h \in [-1, 1]$, $r > 0$ and $k \in N_0$. From (1) we have that

$$\frac{1}{L(e^{u-x})} \leqslant C e^{\delta|x|} \, , \qquad x \in R \, ,$$

where $C > 0$ and $\delta > 0$. Thus,

$$\frac{r^k}{M_k} \left| \left(\frac{c(x+h)}{L(e^h)e^{\alpha h}} - e^{\alpha x} \right)^{(k)} \right| \leqslant$$

$$\leqslant \|\omega\|_{[-1,1],r} \sup_{t \in [-1,1]} \{ c_0(x+h-t) \} \frac{1}{L(e^{u-x})e^{\alpha(u-x)}} +$$

$$+ C_2 \, e^{\alpha x} \leqslant C_3 \, e^{\alpha x + \delta|x|} + C_2 \, e^{\alpha x} \, .$$

These inequalities and the assumption $\delta \in (0, \alpha - \beta)$ imply (8).

Part II. Let $\psi \in \mathscr{E}^{(M_\alpha)}$ be such that $\psi(x) = 1$ on $(-\infty, -1)$ and $\psi(x) = 0$ on $[0, \infty)$. We have

$$(9) \quad e^{\alpha x} \overset{\vee}{K}(x) = e^{\alpha x} \overset{\vee}{K}(x)(1 - \eta(x)) \psi(x) +$$

$$+ e^{\alpha x} \overset{\vee}{K}(x)(1 - \eta(x))(1 - \psi(x)) + e^{\alpha x} \overset{\vee}{K}(x) \eta(x), \qquad x \in R \, .$$

Since the multiplication in $\mathscr{O}_{\mathscr{L}^1}^{(M_\alpha)}$ is the inner operation, one can easily

prove that all the members on the right side of (9) are from $\mathcal{O}_{\mathcal{L}^1}^{(M_\alpha)}$ and so the same holds for $e^{\alpha \cdot} \overset{\vee}{K}$. This implies that $\mathcal{F}(Ke^{-\alpha \cdot})(\xi) = \mathcal{F}(\overset{\vee}{K})(\xi - i\alpha)$, $\xi \in \mathbf{R}^n$, exists.

Since

$$(f * K)(h) = \left\langle \frac{f(x + h)}{c(x + h)}, \ c(x + h) \overset{\vee}{K}(x) \right\rangle, \qquad h \in \mathbf{R},$$

the existence of the convolution $f * K$ will be proved if we prove that for every $h \in \mathbf{R}$,

$$c(. + h) \overset{\vee}{K} \in \mathcal{O}_{\mathcal{L}^1}^{(M_\alpha)},$$

because by (i) $f(. + h)/c(. + h) \in \mathcal{B}'^{(M_\alpha)}$.

For a fixed $h \subset \mathbf{R}$ and a ψ as in (9) we have

$$(10) \qquad c(x + h) \overset{\vee}{K}(x) = c(x + h) \overset{\vee}{K}(x)(1 - \eta(x)) \psi(x) +$$

$$+ c(x + h) \overset{\vee}{K}(x)(1 - \eta(x))(1 - \psi(x)) + c(x + h) \overset{\vee}{K}(x) \eta(x), \qquad x \in \mathbf{R}.$$

By using (8), (7), (7′) and that $e^{\alpha \cdot} \overset{\vee}{K} \in \mathcal{O}_{\mathcal{L}^1}^{(M_\alpha)}$ we shall prove that $c(. + h) \overset{\vee}{K} \in \mathcal{O}_{\mathcal{L}^1}^{(M_\alpha)}$. We shall only prove that $c(. + h) \overset{\vee}{K}(1 - \eta) \psi$ is from $\mathcal{O}_{\mathcal{L}^1}^{(M_\alpha)}$ for every $h \in \mathbf{R}$, because the proof that $c(. + h) \overset{\vee}{K} \eta$ belongs to $\mathcal{O}_{\mathcal{L}^1}^{(M_\alpha)}$ is similar and then one can easily see that

$$c(. + h) \overset{\vee}{K}(1 - \eta)(1 - \psi) \in \mathcal{O}^{(M_\alpha)}.$$

Since

$$c(x + h) \overset{\vee}{K}(x)(1 - \eta(x)) \psi(x) = \left[e^{\alpha h} L(e^h) \left(\frac{c(x + h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right) \overset{\vee}{K}(x) + \right.$$

$$\left. + e^{\alpha(x + h)} L(e^h) \overset{\vee}{K}(x) \right] (1 - \eta(x)) \psi(x), \qquad x \in \mathbf{R},$$

we have to prove that

$$(1 - \eta) \psi \left(\frac{c(. + h)}{e^{\alpha h} L(e^h)} - e^{\alpha \cdot} \right) \overset{\vee}{K} \in \mathcal{O}_{\mathcal{L}^1}^{(M_\alpha)}.$$

For every $r > 0$, $x \in \mathbf{R}$, $k \in N_0$, by using (M.2) and (8) we have

$$\frac{r^k}{M_k} \left\| \sum_{j=0}^{k} \binom{k}{j} (\overset{\vee}{K}(x)(1-\eta(x))\,\psi(x))^{(k-j)} \left(\frac{c(x+h)}{e^{\alpha h}\,L(e^h)} - e^{\alpha x} \right)^{(j)} \right\|_{\mathscr{L}^1} \leq$$

$$\leq \frac{AC}{2^k} \sum_{j=0}^{k} \binom{k}{j} \frac{(2rH)^{k-j}}{M_{k-j}} \left\{ \|(\overset{\vee}{K}(x)(1-\eta(x))\,\psi(x))^{(k-j)}\, e^{\beta x}\|_{\mathscr{L}^1} \right\} \leq$$

$$\leq C_1 \sup_{\alpha \in N_0} \left\{ \frac{(2rH)^\alpha}{M_\alpha} \|(\overset{\vee}{K}(x)(1-\eta(x))\,\psi(x))^{(\alpha)}\, e^{\beta x}\|_{\mathscr{L}^1} \right\},$$

where C and C_1 are suitable constants.

For the proof that the last supremum is bounded we have to use the following estimates

$$\frac{(4rH)^\alpha}{M_\alpha} \frac{1}{2^\alpha} \sum_{j=0}^{\alpha} \binom{\alpha}{j} \|\overset{\vee}{K}(x)(1-\eta(x))^{(j)}\, e^{\beta x}\, \psi^{(\alpha-j)}\|_{\mathscr{L}^1} \leq$$

$$\leq C_2 \sup_{j \in N_0} \left\{ \frac{(4rH^2)^j}{M_j} \|\overset{\vee}{K}(x)(1-\eta(x))^{(j)}\, e^{\beta x}\|_{\mathscr{L}^1} \right\} \cdot$$

$$\cdot \sup_{\substack{\alpha \in N_0 \\ j \leq \alpha}} \left\{ \frac{(4rH^2)^{k-j}}{M_{\alpha-j}} \|\psi^{(\alpha-j)}\|_{\mathscr{L}^\infty(-1,\,0)} \right\} \leq C_3 \,,$$

where C_2 and C_3 are suitable constants. Thus, we have proved that the convolution $f * K$ exists.

Part III. We are going to prove that the assumptions of the theorem imply

(11) $$\left(\frac{f}{c} * Ke^{-\alpha \cdot} \right)(h) = \left\langle \frac{f(x+h)}{c(x+h)}, \overset{\vee}{K}(x)\, e^{\alpha x} \right\rangle \to a \int_{\mathbf{R}} K(t)\, e^{-\alpha t}\, dt \,,$$

$$h \to \infty \,.$$

It is enough to prove that

$$\left| \left\langle \frac{f(x+h)}{c(x+h)}, \left(\frac{c(x+h)}{e^{\alpha h}\,L(e^h)} - e^{\alpha x} \right) \overset{\vee}{K}(x) \right\rangle \right| \to 0 \,, \qquad \text{as } h \to 0 \,.$$

Since $\mathcal{B}'^* \ni f/c = \sum\limits_{i=0}^{\infty} D^i F_i$ such that (5) holds, we have to prove that

$$S_h = \sum_{i=0}^{\infty} (-1)^i \int\limits_R F_i(x+h) \left(\left(\frac{c(x+h)}{L(e^h)\,e^{\alpha h}} - e^{\alpha x} \right) \overset{\vee}{K}(x) \right)^{(i)} dx \to 0 \,,$$

$$h \to \infty \,.$$

We have

$$S_h = \sum_{i=0}^{N} (-1)^i \int\limits_R F_i(x+h) \left(\left(\frac{c(x+h)}{L(e^h)\,e^{\alpha h}} - e^{\alpha x} \right) \overset{\vee}{K}(x) \right)^{(i)} dx +$$

$$+ \sum_{i=N+1}^{\infty} (-1)^i \int\limits_R F_i(x+h) \left(\left(\frac{c(x+h)}{L(e^h)\,e^{\alpha h}} - e^{\alpha x} \right) \overset{\vee}{K}(x) \right)^{(i)} dx = S_{h,N} + S_{h,\infty} \,.$$

Because the sum in $S_{h,N}$ is finite, the proof that $S_{h,N} \to 0$, $h \to \infty$, is the same as in the main assertion of [6]. By using (5) we obtain

$$S_{h,\infty} \leqslant \sum_{i=N+1}^{\infty} \frac{1}{2^i} \frac{M_i \|F_i\|_{\mathcal{L}^{\infty}}}{r^i} \frac{(2r)^i}{M_i} \left\| \left[\left(\frac{c(x+h)}{L(e^h)\,e^{\alpha h}} - e^{\alpha x} \right) \overset{\vee}{K}(x) \right]^{(i)} \right\|_{\mathcal{L}^1} \leqslant$$

$$\leqslant \frac{C}{2^{N+1}} \sum_{i=0}^{\infty} \frac{(2r)^i}{M_i} \left\| \left[\left(\frac{c(x+h)}{L(e^h)\,e^{\alpha h}} - e^{\alpha x} \right) \overset{\vee}{K}(x) \right]^{(i)} \right\|_{\mathcal{L}^1} \,.$$

So, if we prove that the last series is bounded with respect to h for $h \geqslant h_0$, the proof that $S_{h,\infty} \to 0$, $h \to \infty$, simply follows. Put

$$I_{m,h} = \left\| \left(\frac{c(\,.\,+h)}{e^{\alpha h} L(e^h)} - e^{\alpha\,.} \right) \overset{\vee}{K} \right)^{(m)} \right\|_{\mathcal{L}^1} \,, \qquad m \in N_0 \,, \qquad h \geqslant h_0 \,.$$

We are going to prove that for every $r > 0$, there is $C > 0$ such that

(12) $$\sup_{m \in N_0} \left\{ \frac{r^m}{M_m} I_{m,h} \right\} < C \,, \qquad h > h_0 \,.$$

This implies that the quoted series is bounded.

Let $x_0 > 0$ be as in (4). We have

$$I_{m,h} = \int\limits_{-\infty}^{-x_0} \left| \left[\left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right)(1 - \eta(x)) \overset{\vee}{K}(x) \right]^{(m)} \right| dx +$$

$$+ \int\limits_{-x_0}^{x_0} \left| \left[\left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right) \overset{\vee}{K}(x) \right]^{(m)} \right| dx +$$

$$+ \int\limits_{x_0}^{\infty} \left| \left[\left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right) \eta(x) \overset{\vee}{K}(x) \right]^{(m)} \right| dx = I_1 + I_2 + I_3 \,.$$

By the Leibniz formula, (8), (7), (7') and (1) there are constants C_1 and C which do not depend on m and p (but depend on r) such that for $\delta \in (0, \alpha - \beta)$

$$I_1 \leq \sum_{p=0}^{m} \binom{m}{p} \left[\int\limits_{-\infty}^{-h} \left| \left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right)^{(p)} e^{-\beta x} \right| \cdot \right.$$

$$\cdot \left| e^{\beta x} ((1 - \eta(x)) \overset{\vee}{K}(x))^{(m-p)} \right| dx +$$

$$+ \int\limits_{-h}^{-x_0} \left| \left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x} \right)^{(p)} e^{-(\alpha x - \delta x)} \right| \left| e^{\alpha x - \delta x} ((1 - \eta(x)) \overset{\vee}{K}(x))^{(m-p)} \right| dx \bigg] \leq$$

$$\leq \sum_{p=0}^{m} \binom{m}{p} C \left[\frac{M_p}{r^p} \frac{M_{m-p}}{r^{m-p}} \sup_{\substack{m,p \\ p \leq m}} \left\{ \frac{r^{m-p}}{M_{m-p}} \| e^{\beta x} (1 - \eta(x))(\overset{\vee}{K}(x))^{(m-p)} \|_{\mathcal{L}^1} \right\} + \right.$$

$$+ \frac{M_p}{r^p} \frac{M_{m-p}}{r^{m-p}} \sup_{\substack{m,p \\ p \leq m}} \left\{ \frac{r^{m-p}}{M_{m-p}} \| e^{\alpha x - \delta x} (1 - \eta(x)) \overset{\vee}{K}(x) \|_{\mathcal{L}^1_{(-\infty, 0)}} \right\} \bigg] \leq$$

$$\leq C_1 \sum_{p=0}^{m} \binom{m}{p} \frac{M_m}{r^m} = C_1 \frac{M_m}{(r/2)^m} \,.$$

This gives that $\sup\limits_{m \in N_0} \{ (r/2)^m / (M_m) I_1 \} < C_1$.

In a similar way one can prove the corresponding estimates for I_2 and I_3 and the proof of (12) is completed.

Thus, we have proved (11).

If $\psi \in \mathcal{E}^{(M_\alpha)}$ satisfies the assumption given in $(*)$ then $\psi e^{\alpha \cdot} \in \mathcal{O}_{\mathcal{L}^1}^{(M_\alpha)}$ and

we have

$$\left(\frac{f}{c} * \psi e^{-\alpha\cdot}\right)(h) \to a \int_R \psi(x) e^{-\alpha x} dx, \qquad h \to \infty.$$

As above we can prove that $(f * \psi)(h)$, $h \in \mathbf{R}$, exists.

For the proof that

$$\left\langle \frac{f(x+h)}{L(e^h) e^{\alpha h}}, \overset{\vee}{\psi}(x) \right\rangle \to a \int \psi(x) e^{-\alpha x} dx, \qquad h \to \infty,$$

we have to prove that

$$\left\langle \frac{f(x+h)}{c(x+h)}, \left(\frac{c(x+h)}{e^{\alpha h} L(e^h)} - e^{\alpha x}\right)\psi(x) \right\rangle \to 0, \qquad h \to \infty$$

but this is already done (with K instead of ψ) and the proof of the Theorem is completed.

The proof of Corollary 1 simply follows from the given Theorem 2 since the ϕ in the Corollary satisfies conditions assumed for K and functions from $\mathcal{D}^*$ satisfy condition (c) of Theorem 2.

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