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SALVATORE LEONARDI

The best constant in weighted Poincaré and Friedrichs inequalities

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The Best Constant in Weighted Poincaré and Friedrichs Inequalities.

SALVATORE LEONARDI (*)

SUMMARY - Our aim is to find suitable sufficient conditions under which the best constant in weighted Poincaré and Friedrichs inequalities are achieved.

0. Introduction.

In this paper we deal with the weighted Poincaré and Friedrichs inequalities. Our aim is to find suitable sufficient conditions under which the best constant in these inequalities are achieved. We work in weighted Sobolev spaces. The main tool in our proof is the variational characterization of these best constants combined with compact imbeddings between weighted spaces. We work with general weight functions which may have degeneracies or singularities both at the boundary points or interior points of the domain. Let us point out that the integral inequalities of the type mentioned have been studied by many authors. Let us mention e.g. the works of Edmunds, Evans [3], Edmunds, Opic [4], Evans, Harris [5], Hurri [7], Kufner [8], Opic [11], Opic Kufner [12] and others. For a bounded domain we use natural imbedding theorems between weighted spaces which were used in order to prove the existence results for degenerate elliptic problems (see e.g. Leonardi [9], Drabek, Nicolosi [2], Guglielmino, Nicolosi [6]).

For unbounded domains we use sufficient conditions for a compact imbedding used for more special weights in the book Opic, Kufner [12] and in the paper Edmunds, Opic [4].

(*) Dipartimento di Matematica, Città Universitaria, Viale A. Doria 6, 95125 Catania, Italy.

1. Notation.

Let Ω be a domain in $\mathbb{R}^N$. Denote by $W(\Omega)$ the set of all weight functions in Ω, i.e. the set of all measurable functions on Ω which are positive and finite almost everywhere in Ω. Let $1 \leqslant p < +\infty$. For $w \in W(\Omega)$ denote by $L^p(\Omega; w)$ the weighted Lebesgue space of all real-valued functions u on Ω with the norm

$$\|u\|_{p,\Omega,w} = \left[\int_\Omega |u(x)|^p w(x)\,dx \right]^{1/p}. \tag{1.1}$$

Then $L^p(\Omega, w)$ with the norm (1.1) is a uniformly convex Banach space. For $w, v \in W(\Omega)$ denote by $W^{1,p}(\Omega; w, v)$ the weighted Sobolev space which consists of all real valued measurable functions u on Ω for which the distributional derivatives $\partial u/\partial x_i$ $(i = 1, 2, \ldots, N)$ exist on Ω and for which the norm

$$\|u\|_{1,p,\Omega,w,v} = \left[\int_\Omega |u|^p w(x)\,dx + \int_\Omega |\nabla u|^p v(x)\,dx \right]^{1/p} \tag{1.2}$$

is finite. It is known (see Leonardi [9]) that $W^{1,p}(\Omega; w, v)$ is a uniformly convex Banach space, provided

$$p > 1,\ w^{-1},\ v^{-1} \in L^{1/(p-1)}_{\mathrm{loc}}(\Omega) \tag{1.3}$$

and, moreover

$$C_0^\infty(\Omega) \subset W^{1,p}(\Omega; w, v)$$

if and only if

$$w, v \in L^1_{\mathrm{loc}}(\Omega). \tag{1.4}$$

Hence, under the assumption (1.4) we can define the space $W_0^{1,p}(\Omega; w, v)$ as the closure of the set $C_0^\infty(\Omega)$ with respect to the norm (1.2). The space $W_0^{1,p}(\Omega; w, v)$ with the norm (1.2) is also a Banach space.

We shall write $X \hookrightarrow Y$ or $X \hookrightarrow \hookrightarrow Y$ if the imbedding of a Banach space X into Y is continuous or compact, respectively.

2. Natural imbeddings.

Let us assume $p > 1$ and $g^* \in \mathbb{R}$, $g^* \geqslant 1/(p-1)$. Applying the

Hölder inequality, we obtain that the imbedding

$$L^p(\Omega, w) \hookrightarrow L^{p_1}(\Omega), \qquad p_1 = \frac{pg^*}{g^*+1}$$

holds, provided

$$w^{-1} \in L^{g^*}(\Omega). \tag{2.1}$$

As a consequence we obtain the imbedding

$$W^{1,p}(\Omega; w, v) \hookrightarrow W^{1,p_1}(\Omega) \tag{2.2}$$

provided

$$w^{-1}, v^{-1} \in L^{g^*}(\Omega). \tag{2.3}$$

Let us assume that Ω is a bounded domain. The following assertion is a direct consequence of the Sobolev imbedding theorem.

LEMMA 2.1. (i) *Let* $N(g^*+1) < pg^*$ *and* Ω *has locally Lipschitz boundary* (see e.g. Adams [1]).
Then

$$W^{1,p_1}(\Omega) \hookrightarrow\hookrightarrow C^0(\overline{\Omega}).$$

(ii) *Let* $N(g^*+1) = pg^*$ *and* Ω *has a cone property* (see e.g. Miranda [10]).
Then

$$W^{1,p_1}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$$

with arbitrary $1 \leqslant r < +\infty$.

(iii) *Let* $N(g^*+1) > pg^*$ *and* Ω *has a cane property.*
Then

$$W^{1,p_1}(\Omega) \hookrightarrow L^q(\Omega)$$

with $1 \leqslant q = pg^*N/[N(g^*+1) - pg^*]$ *and*

$$W^{1,p_1}(\Omega) \hookrightarrow\hookrightarrow L^{\overline{q}}(\Omega)$$

with arbitrary $1 \leqslant \overline{q} < q$.

LEMMA 2.2. *Let* Ω *be a bounded domain with locally Lipschitz boundary and let us assume* (1.3), (1.4), (2.3),

$$g^* \in \left[\frac{N}{p}, +\infty\right[\cap \left[\frac{1}{p-1}, +\infty\right[. \tag{2.4}$$

Moreover, in the case $N(g^* + 1) > pg^*$, *assume that*

$$w \in L^{f^*}(\Omega) \tag{2.5}$$

with

$$f^* = \frac{Ng^*}{pg^* - N} \left(= \frac{q}{q-p} \right).$$

Then the imbedding

$$W^{1,p}(\Omega; w, v) \hookrightarrow L^p(\Omega, w) \tag{2.6}$$

holds.

If we assume

$$w \in L^{\bar{f}^*}(\Omega) \tag{2.7}$$

with $\bar{f}^* > f^*$, *instead of (2.5), the imbedding (2.6) is compact:*

$$W^{1,p}(\Omega; w, v) \hookrightarrow\hookrightarrow L^p(\Omega, w). \tag{2.6'}$$

It follows from the Hölder inequality and (2.5) that

$$L^q(\Omega) \hookrightarrow L^p(\Omega, w). \tag{2.8}$$

The imbedding (2.6) now follows from (2.8) and Lemma 2.1. The assumption (2.7) implies

$$L^{\bar{q}}(\Omega) \hookrightarrow L^p(\Omega, w) \tag{2.9}$$

with some $\bar{q}$ satisfying $p < \bar{q} < q$. The compact imbedding (2.6′) now follows from (2.9) and Lemma 2.1.

Remark 2.1. Under the assumptions of Lemmas 2.1 and 2.2 we have

$$W^{1,p}(\Omega; w, v) \hookrightarrow W^{1,p_1}(\Omega) \hookrightarrow L^q(\Omega) \hookrightarrow L^p(\Omega, w) \tag{2.10}$$

and

$$W^{1,p}(\Omega; w, v) \hookrightarrow W^{1,p_1}(\Omega) \hookrightarrow\hookrightarrow L^{\bar{q}}(\Omega) \hookrightarrow L^p(\Omega, w). \tag{2.10'}$$

Applying Lemma 2.2 we obtain the following assertion.

Proposition 2.3. *Let* Ω *be a bounded domain with a locally Lipschitz boundary and assume* (1.3), (1.4), (2.3), (2.5) *and* (2.7).

Then there exists a constant $F_1 > 0$ *such that the weighted*

Friedrichs inequality

$$\int_\Omega |u(x)|^p\, w(x)\, dx \leqslant F_1 \int_\Omega |\nabla u(x)|^p\, v(x)\, dx \tag{2.11}$$

holds for any $u \in W_0^{1,p}(\Omega, w, v)$.

Let $u \in C_0^\infty(\Omega)$. Then it follows from (2.10) that there are constants $c_1, c_2 > 0$ independent of u such that

$$\left[\int_\Omega |u(x)|^p\, w(x)\, dx\right]^{1/p} \leqslant c_1 \left[\int_\Omega |u(x)|^q\, dx\right]^{1/q} \leqslant \tag{2.12}$$

$$\leqslant c_2 \left[\int_\Omega |u(x)|^{p_1}\, dx + \int_\Omega |\nabla u(x)|^{p_1}\, dx\right]^{1/p_1}.$$

It follows from the Friedrichs inequality in the nonweighted space $W_0^{1,p_1}(\Omega)$ that there is a constant $c_3 > 0$ independent of u such that

$$\left[\int_\Omega |u(x)|^{p_1}\, dx + \int_\Omega |\nabla u(x)|^{p_1}\, dx\right]^{1/p_1} \leqslant c_3 \left[\int_\Omega |\nabla u(x)|^{p_1}\, dx\right]^{1/p_1}. \tag{2.13}$$

The assumption (2.3) and Hölder inequality yield

$$\left[\int_\Omega |\nabla u(x)|^{p_1}\, dx\right]^{1/p_1} \leqslant c_4 \left[\int_\Omega |\nabla u(x)|^p\, v(x)\, dx\right]^{1/p} \tag{2.14}$$

with $c_4 > 0$ independent of u. Now, the assertion (2.11) follows from (2.12)-(2.14) and from the fact that $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, w, v)$. This completes the proof.

In what follows we will study the question when the best (i.e. the least) constant $\overline{F}_1 > 0$ in (2.11) is achieved, and also when the best (the least) constant $\overline{P}_1$ in the weighted Poincaré inequality

$$\int_\Omega |u(x)|^p\, w(x)\, dx \leqslant \tag{2.15}$$

$$\leqslant P_1 \left[\left|\int_\Omega u(x)\, w(x)\, dx\right|^p + \int_\Omega |\nabla u(x)|^p\, v(x)\, dx\right],$$

for any $u \in W^{1,p}(\Omega; w, v)$, is achieved.

Let us assume that $w \in L^1(\Omega)$ and let $(u_n) \subset W^{1,p}(\Omega; w, v)$ be the sequence such that $u_n \to u_1$ in $L^p(\Omega, w)$. Then we have, applying the Hölder inequality,

$$\left| \int_\Omega u_n(x) w(x)\, dx \right| \leq \int_\Omega |u_n(x) - u_1(x)| w(x)\, dx + \left| \int_\Omega u_1(x) w(x)\, dx \right| \leq$$

$$\leq \left[\int_\Omega |u_n(x) - u_1(x)|^p w(x)\, dx \right]^{1/p} \left[\int_\Omega w\, dx \right]^{1-1/p} + \left| \int_\Omega u_1(x) w(x)\, dx \right|$$

i.e.

$$\limsup_{n\to\infty} \left| \int_\Omega u_n(x) w(x)\, dx \right| \leq \left| \int_\Omega u_1(x) w(x)\, dx \right|. \tag{2.16}$$

Similarly, we obtain

$$\left| \int_\Omega u_1(x) w(x)\, dx \right| \leq \int_\Omega |u_1(x) - u_n(x)| w(x)\, dx + \left| \int_\Omega u_n(x) w(x)\, dx \right| \leq$$

$$\leq \left[\int_\Omega |u_1(x) - u_n(x)|^p w(x)\, dx \right]^{1/p} \left[\int_\Omega w\, dx \right]^{1-1/p} + \left| \int_\Omega u_n(x) w(x)\, dx \right|,$$

i.e.

$$\liminf_{n\to\infty} \inf \left| \int_\Omega u_n(x) w(x)\, dx \right| \geq \left| \int_\Omega u_1(x) w(x)\, dx \right|. \tag{2.17}$$

It follows from (2.16), (2.17) that

$$\lim_{n\to\infty} \left| \int_\Omega u_n(x) w(x)\, dx \right|^p = \left| \int_\Omega u_1(x) w(x)\, dx \right|. \tag{2.18}$$

THEOREM 2.4. *Let Ω be a bounded domain with locally Lipschitz boundary and assume $v \in L^1_{loc}(\Omega)$, $w^{-1}, v^{-1} \in L^{g^*}(\Omega)$, $w \in L^1(\Omega) \cap L^{h^*}(\Omega)$ $h^* > Ng^*/(pg^* - N)$, $g^* \in]N/p, \infty[\cup [1/(p-1), \infty[$. Then there* *exists* $\overline{u} \in W^{1,p}(\Omega; w, v)$, $\overline{u} \neq 0$, *such that*

$$\int_\Omega |\overline{u}(x)|^p w(x)\,dx = \tag{2.19}$$

$$= \overline{P}_1 \left[\left| \int_\Omega \overline{u}(x)\, w(x)\,dx \right|^p + \int_\Omega |\nabla \overline{u}(x)|^p v(x)\,dx \right].$$

Let us denote

$$I = \inf \left\{ \left| \int_\Omega u(x)\, w(x)\,dx \right|^p + \int_\Omega |\nabla u(x)|^p v(x)\,dx; \right.$$

$$\left. u \in W^{1,p}(\Omega; w, v),\ \|u\|_{p,\Omega,w} = 1 \right\}.$$

Obviously $I \geqslant 0$. Let $(u_n) \subset W^{1,p}(\Omega; w, v)$ be a minimizing sequence for I, more precisely let $\|u_n\|_{p,\Omega,w} = 1$ and

$$\left| \int_\Omega u_n(x)\, w(x)\,dx \right|^p + \int_\Omega |\nabla u_n(x)|^p v(x)\,dx = I + \delta_n \tag{2.20}$$

where $\delta_n \to 0$ for $n \to \infty$. It follows from (2.20) that $\|u_n\|_{1,p,\Omega,w,v} \leqslant$ $\leqslant$ const. and hence the reflexivity of $W^{1,p}(\Omega; w, v)$ yields that (u_n) converges weakly in $W^{1,p}(\Omega; w, v)$ to some element $\overline{u} \in W^{1,p}(\Omega; w, v)$ at least for some subsequence. The compact imbedding (2.6) implies the strong convergence

$$u_n \to \overline{u} \quad \text{in } L^p(\Omega, w). \tag{2.21}$$

Now, the definition of I, (2.18), (2.21) and the weak convergence in $W^{1,p}(\Omega; w, v)$ yield

$$I + 1 - \left| \int_\Omega \overline{u}(x)\, w(x)\,dx \right|^p \leqslant \int_\Omega |\nabla \overline{u}(x)|^p v(x)\,dx + \|\overline{u}\|_{p,\Omega,w} = \tag{2.22}$$

$$= \|\overline{u}\|_{1,p,\Omega,w,v} \leqslant \liminf_{n\to\infty} \|u_n\|_{1,p,\Omega,w,v} =$$

$$= \liminf_{n\to\infty} \int_\Omega |\nabla u_n(x)|^p v(x)\,dx + 1 =$$

$$= \liminf_{n\to\infty}\left[\left|\int u_n(x)\,w(x)\,dx\right|^p + \int_\Omega |\nabla u_n(x)|^p v(x)\,dx\right] +$$

$$+1 - \lim_{n\to\infty}\left|\int_\Omega \overline{u}(x)\,w(x)\,dx\right|^p = I + 1 - \left|\int_\Omega \overline{u}(x)\,w(x)\,dx\right|^p .$$

It follows from (2.22) and from the fact that $\|u\|_{p,\Omega,w} = 1$ the inequality

$$\left|\int_\Omega \overline{u}(x)\,w(x)\,dx\right|^p + \int_\Omega |\nabla\overline{u}(x)|^p v(x)\,dx = I > 0 .$$

Setting $\overline{P}_1 = 1/I$ the equality (2.19) follows, which completes the proof of the theorem.

Similarly for the weighted Friedrichs inequality we obtain the following assertion.

THEOREM 2.5. *Assume the same as in Theorem* 2.4. *Then there exists* $\overline{u} \in W_0^{1,p}(\Omega; w, v)$, $\overline{u} \neq 0$, *such that*

$$\int_\Omega |\overline{u}(x)|^p w(x)\,dx = \overline{F}_1 \int_\Omega |\nabla\overline{u}(x)|^p v(x)\,dx . \tag{2.23}$$

Set

$$\overline{I} = \inf\left\{\int_\Omega |\nabla u(x)|^p v(x)\,dx;\quad u \in W_0^{1,p}(\Omega; w, v),\ \|u\|_{p,\Omega,w} = 1\right\}.$$

Using the same method as in the proof of the preceding theorem we get $\overline{u} \in W_0^{1,p}(\Omega; w, v)$ satisfying (2.23) with $\overline{F}_1 = 1/\overline{I}$.

REMARK 2.6. It follows from the proof of Theorem 2.4 that u_n converges weakly to $\overline{u}$ in $W^{1,p}(\Omega; w, v)$ and $\|u_n\|_{1,p,\Omega,w,v} \to \|\overline{u}\|_{1,p,\Omega,w,v}$. The uniform convexity of $W^{1,p}(\Omega; w, v)$ then implies the strong convergence $u_n \to \overline{u}$ of in $W^{1,p}(\Omega; w, v)$. Hence $\overline{u}$ is obtained as the strong limit of some minimizing sequence. Similarly for $\overline{u}$ from Theorem 2.5.

EXAMPLE 2.7. Consider the square $\Omega \subseteq \mathbb{R}^2$, $\Omega =]-1, 1[\times]-1, 1[$

and the weight functions defined by

$$w(x_1, x_2) = \begin{cases} 1 & \text{for } x_1 \leqslant 0\,, \\ x_2^{\lambda}(1-x_1)^{\gamma} & \text{for } x_1 > 0\,, \quad x_2 > 0\,, \\ |x_2|^{\mu}(1-x_1)^{\gamma} & \text{for } x_1 > 0\,, \quad x_2 < 0\,, \end{cases}$$

$$v(x_1, x_2) = \begin{cases} 1 & \text{for } x_1 \leqslant 0\,, \\ x_2^{\bar\lambda}(1-x_1)^{\bar\gamma} & \text{for } x_1 > 0\,, \quad x_2 > 0\,, \\ |x_2|^{\bar\mu}(1-x_1)^{\bar\gamma} & \text{for } x_1 > 0\,, \quad x_2 < 0\,. \end{cases}$$

Let us note that both weight functions may have degeneracies or singularities not only on the boundary of Ω (more precisely on $\Gamma_1 = \{(x_1, x_2);\ x_1 = 1,\ x_2 \in]-1, 1[\}$) but also in the interior of Ω (more precisely on $\Gamma_2 = \{(x_1, x_2);\ x_2 = 0,\ x_1 \in]-0, 1[\}$).

An easy calculation yields that

$$\lambda, \mu, \bar\lambda, \bar\mu > -1 \tag{2.24}$$

implies $w, v \in L^1_{\mathrm{loc}}(\Omega)$, and

$$\lambda, \mu, \gamma, \bar\lambda, \bar\mu, \bar\gamma < \min(p/2, p-1) \tag{2.25}$$

implies $w, v \in L^{g^*}(\Omega)$ with

$$g^* > \max(1/p, 1/(p-1))\,. \tag{2.26}$$

Moreover, if

$$\lambda, \mu, \gamma > \min(0, p/2, 1) \tag{2.27}$$

then

$$\lambda, \mu, \gamma > (2 - pg^*)/2g^*$$

and hence there exists $\bar f^* > (pg^* - 1)/2g^*$ such that $w \in L^{\bar f^*}(\Omega)$.

It follows from (2.24), (2.25), (2.27) that the assumptions of Proposition 2.3 are satisfied provided

$$\begin{cases} \min(0, p/2 - 1) < \lambda, \mu, \gamma < \min(p/2, p-1)\,, \\ -1 < \bar\lambda, \bar\mu < \min(p/2, p-1)\,, \\ \bar\gamma < \min(p/2, p-1)\,. \end{cases} \tag{2.28}$$

Since $(pg^* - 2)/2g^* > 1$ for g^* satisfying (2.26) the conditions (2.28) guarantee also the validity of the assumptions of Theorems (2.4) and (2.5).

3. The case of arbitrary domain.

In the previous section we dealt with the bounded domain $\Omega \subset \mathbb{R}^N$ and the main tool in order to prove our results were natural imbeddings between weighted and nonweigthed Sobolev spaces.

In this section we will deal with possibly unbounded domain in $\mathbb{R}^N$ and we will prove the existence of the best constant in weighted Poincaré (or Friedrichs) inequality without making use of imbedding theorems from Section 2.

We use the compact imbedding $W^{1,p}(\Omega; w, v) \hookrightarrow\hookrightarrow L^p(\Omega; w)$ as in Opic, Kufner [12]) but rather more general weight functions can be considered in our paper.

Let us denote by $\widetilde{\Omega}_{w,v}$ the set of all $x \in \Omega$ which there exists a sequence $(x_n) \subset \Omega$ such that $x_n \to x$ and at least one of the following cases occurs

(i) $w(x_n) \to 0$,

(ii) $v(x_n) \to 0$,

(iii) $w(x_n) \to \infty$,

(iv) $v(x_n) \to \infty$.

By the definition of $\widetilde{\Omega}_{w,v}$ the weight functions w, v are bounded from below and from above by positive constants on each compact set $\mathcal{C} \subset \Omega \backslash \widetilde{\Omega}_{w,v}$.

We denote by $\widetilde{W}(\Omega) \subset W(\Omega)$ the set of all weight functions w and v with the following property

$$(3.1) \qquad \Omega \backslash \widetilde{\Omega}_{w,v} = \bigcup_{k=1}^{\infty} \Omega_k$$

where Ω_k is a bounded domain whose boundary is locally Lipschitz and

$$\Omega_k \subset \overline{\Omega}_k \subset \Omega_{k+1} \subset \Omega \backslash \widetilde{\Omega}_{w,v}$$

for each $k \in N$.

REMARK 3.1. Note that meas $\widetilde{\Omega}_{w,v} = 0$ in many practical applications. This situation occurs e.g. when $\widetilde{\Omega}_{w,v}$ consists of isolated segments or points (see Example 3.1).

In what follows we shall restrict ourselves to such weight functions w and v for which

$$\operatorname{meas} \widetilde{\Omega}_{w,v} = 0 .$$

Put $\Omega^k = \Omega \backslash \Omega_k$ and define

$$A_k = \sup_{\|u\|_{1,p,\Omega,w,v} \leqslant 1} \|u\|_{p,\Omega^k,w} . \tag{3.2}$$

It follows from (3.2) that

$$0 \leqslant A_{k+1} \leqslant A_k \leqslant 1$$

for any $k \in N$ and hence the limit

$$A = \lim_{k \to \infty} A_k$$

exists and it is $A \in [0, 1]$.

LEMMA 3.1. *Let us assume that $A = 0$. Then the following compact imbedding holds*:

$$W^{1,p}(\Omega; w, v) \hookrightarrow\hookrightarrow L^p(\Omega; w). \tag{3.3}$$

Let $\varepsilon_1 > 0$ be arbitrary. Then $A = 0$ jelds the existence of such $k_{\varepsilon_1} \in N$ that for any $k \geqslant k_{\varepsilon_1}$

$$\mathit{sup}_{\|u\|_{1,p,\Omega,w,v} \leqslant 1} \|u\|_{p,\Omega^k,w} \leqslant \varepsilon_1^{1/p} ,$$

i.e.

$$\|u\|_{p,\Omega^k,w} \leqslant \varepsilon_1^{1/p} \|u\|_{1,p,\Omega,w,v} .$$

Hence

$$\|u\|^p_{p,\Omega,w} \leqslant \|u\|^p_{p,\Omega^k,w} + \|u\|^p_{p,\Omega_k,w} \leqslant \varepsilon_1 \|u\|^p_{1,p,\Omega,w,v} + \|u\|^p_{p,\Omega_k,w} . \tag{3.4}$$

Now, let (u_k) be a bounded sequence in $W^{1,p}(\Omega; w, v)$, i.e.

$$\|u_k\|_{1,p,\Omega,w,v} \leqslant c$$

for every $k \in N$. For a given $\varepsilon > 0$ choose $\varepsilon_1 \in \left]0, \dfrac{\varepsilon^p}{2c^p + 1}\right[$. There exists $k_0 \in N$ such that (3.4) holds for any $k = k_0$ and for every $u \in W^{1,p}(\Omega; w, v)$. The imbedding

$$W^{1,p}(\Omega_k; w, v) \hookrightarrow\hookrightarrow L^p(\Omega_k; w)$$

implies the existence of a subsequence $(u_{k_j}) \subset (u_k)$ which is a Cauchy one in $L^p(\Omega_{k_0}; w)$.

Consequently there exists $i_0 \in N$ such that

$$\|u_{h_i} - u_{h_j}\|^p_{p, \Omega_{k_0}, w, v} < \varepsilon_1 \qquad \text{for } i, j \geq i_0 ,$$

which together with (3.4) (where we put $k = k_0$ and $u = u_{h_i} - u_{h_j}$) yields

$$\|u_{k_i} - u_{k_j}\|^p_{p, \Omega, w} \leq \varepsilon_1 \|u_{h_i} - u_{h_j}\|^p_{1, p, \Omega, w, v} + \|u_{h_i} - u_{h_j}\|^p_{p, \Omega_{k_0}, w} \leq$$

$$\leq \varepsilon_1 2c^h + \varepsilon_1 \leq \varepsilon^h .$$

Thus (u_{h_j}) is a Cauchy sequence in $L^p(\Omega; w)$.

THEOREM 3.3. *Let* $w, v \in \widetilde{W}(\Omega)$, $w \in L^1(\Omega)$, $v \in L^1_{\text{loc}}(\Omega)$, $1/w$, $1/v \in L^{1/p-1}(\Omega)$ *and* (3.3) *holds. Then the assertion of Theorem* 2.4. *remains true.*

The proof of this assertion follows the lines of the proof of Theorem 2.4. However, instead of the compact imbedding (2.10′) we use the assertion of Lemma 3.1.

Analogously we have the following

THEOREM 3.4. *Assume the same as in Theorem* 3.3. *Then the assertion of Theorem* 2.5 *remains true.*

EXAMPLE 3.1. Let Ω be the complement of unit ball centred at the origin in $\mathbb{R}^N$, i.e.

$$\Omega = \mathbb{R}^n \backslash \{x \in \mathbb{R}^N;\ |x| \leq 1\} .$$

Consider the weight functions defined by

$$w(x) = |x|^\alpha, \quad v(x) = |x|^{\alpha+p}, \quad x \in \Omega$$

where α is appropriate real number to be specified later. It follows from the Example 20.6 in Opic, Kufner [12] that the compact imbedding

$$W^{1,p}(\Omega;\ |x|^\alpha,\ |x|^{\alpha+p}) \hookrightarrow\hookrightarrow L^p(\Omega;\ |x|^\alpha) \tag{3.5}$$

holds if and only if

$$\frac{\alpha}{q} - \frac{\alpha+p}{p} + \frac{N}{q} - \frac{N}{p} + 1 < 0 \tag{3.6}$$

and

$$N\left(\frac{1}{q} - \frac{1}{p}\right) + 1 > 0. \tag{3.7}$$

Let us consider, now, $q > p$ and that (3.7) holds. Then (3.6) is fulfilled for any α satisfying

$$\alpha > -N. \tag{3.8}$$

If $\alpha < -1$ then $|x|^\alpha \in L^1(\Omega)$ and the Hölder inequality yields

$$L^q(\Omega; |x|^\alpha) \hookrightarrow L^p(\Omega; |x|^\alpha). \tag{3.9}$$

Hence it follows from (3.5), (3.8) and (3.9) that the compact imbedding

$$W^{1,p}(\Omega; |x|^\alpha, |x|^{\alpha+p}) \hookrightarrow\hookrightarrow L^p(\Omega; |x|^\alpha)$$

holds provided

$$\alpha \in]-N, -1[. \tag{3.10}$$

Thus (3.10) is the only assumption which guarantees that hypotheses of Theorems 3.3 and 3.4 are verified.

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