RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Well posedness in C^∞ for a weakly hyperbolic second order equation

Rendiconti del Seminario Matematico della Università di Padova, tome 91 (1994), p. 65-83

<http://www.numdam.org/item?id=RSMUP_1994__91__65_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 91 (1994)

Well Posedness in C^∞ for a Weakly Hyperbolic Second Order Equation.

PIERO D'ANCONA (*)

§ 1. Introduction.

We consider here the Cauchy problem on $[0, T] \times \boldsymbol{R}_x$ $(T > 0)$

$$u_{tt} = (a(t, x) u_x)_x + b(t, x) u_x + d(t, x) u + f(t, x), \tag{1}$$

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x), \tag{2}$$

where the coefficients a, b, d, f are C^∞ functions on $[0, T] \times \boldsymbol{R}_x$. We will assume that Pb. (1), (2) is *weakly hyperbolic* and that the coefficient $a(t, x)$ is bounded, namely

$$\Lambda_0 \geqslant a(t, x) \geqslant 0, \tag{3}$$

and that the first order term satisfies a Levi-type condition of the form

$$b^2(t, x) \leqslant M(K) a(t, x), \qquad (t, x) \in K, \ \forall K \subset\subset [0, T] \times \boldsymbol{R}_x. \tag{4}$$

We will be interested in the *global solvability* of problem (1), (2) in C^∞.

It is well known that the above assumptions are not sufficient for the well posedness in C^∞ of Pb. (1), (2); see e.g. [CS], where it is constructed an equation of the form

$$u_{tt} = a(t) u_{xx}$$

with $a(t)$ non negative and infinitely differentiable, which is not locally solvable in C^∞.

In this counterexample the coefficient $a(t)$ oscillates an infinite number of times in a neighbourhood of the initial line. On the other

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hand, in the case when the coefficient $a(t, x) = a(t)$ depends only on time, if it oscillates only a finite number of times on $[0, T]$, it is possible to prove that the Cauchy problem (1), (2) is well posed in C^∞, with a finite loss of derivatives depending on the number of oscillations. Moreover, in the proof of Nishitani's result (see [Ni1]) about the local well posedness in C^∞ for (1), (2) with analytic coefficients, a crucial step is a careful analysis of the zeroes of $a(t, x)$.

These considerations lead to the following question: is it possible to obtain a *global* existence result for (1), (2) in C^∞, under some a priori knowledge of the oscillating behaviour of the coefficient $a(t, x)$?

We propose here a partial answer. Denote with G_R the rectangle $[0, T] \times [-R, R]$. Let $\phi_j(x)$, $1 \leqslant j \leqslant k-1$ (k depending of R) be absolutely continuous, Hölder continuous functions, such that the set $\{x\colon \phi_j'(x) = 0\}$ has a boundary with measure 0 (or more generally, coincides almost everywhere with a set whose boundary has measure 0). Let moreover $0 \leqslant \phi_1(x) \leqslant \ldots \leqslant \phi_{k-1}(x) \leqslant T$. Then, writing ϕ_0 for the line $t = 0$, ϕ_k for the line $t = T$, and denoting with G_R^j the set

$$G_R^j = \{(t, x)\colon \phi_{j-1}(x) < t < \phi_j(x)\}, \qquad j = 1, \ldots, k$$

we shall assume that:

Assumption (A).

1) $G_R = \cup G_R^j$;

2) $a(\phi_j(x), x)\,\phi_j'(x)^2 < 1$ on $[-R, R]$;

3) in each region G_R^j, one of the following inequalities holds, for some constant K (depending on j):

$$a_t \geqslant -Ka \qquad \text{or} \qquad a_t \leqslant Ka\,.$$

In § 2 we shall prove the following.

Theorem 1. *Under assumptions* (3), (4) *and* (A), *problem* (1), (2) *is globally well posed in* C^∞.

Moreover, in § 3 we will show that a real analytic, nonnegative function $a(t, x)$ satisfies assumption (A). As a consequence, we shall obtain the following theorem, which extends the local result in [Ni1]:

Theorem 2. *Assume* (3), (4) *hold, and that* $a(t, x)$ *is real analytic. Then problem* (1), (2) *is globally well posed in* C^∞.

Remark 1. Assumption (A) is of course modelled on the behaviour of real analytic functions, but is in fact more general than that. For instance, it is satisfied by any equation of the form

$$u_{tt} = [(t^p - \phi(x))^{2q} u_x]_x$$

for any $p, q \in \boldsymbol{N}$, $\phi \in C^\infty(\boldsymbol{R})$: choose

$$\phi_1(x) = (0 \vee \phi(x))^{1/p} \wedge T$$

(and of course $\phi_0 \equiv 0$, $\phi_2 \equiv T$).

REMARK 2. Condition A.2 is invariant under a regular change of variables. More precisely, suppose we are given the partial differential operator

$$\partial_t^2 - \partial_x(a(t, x)\,\partial_x)$$

and a curve $t = \phi(x)$ and perform a C^∞ change of variables $(t, x) \to$ $\to (s, y)$ leaving the form of the principal part unchanged:

$$\partial_s^2 - \partial_y(\tilde{a}(s, y)\,\partial_y) + \text{L.O.T.}.$$

Denote the transformed curve with $s = \tilde{\phi}(y)$. Then

$$a(t, x)\,\phi'(x)^2 = \tilde{a}(s(t, x), y(t, x))\,\tilde{\phi}'(y(t, x))^2\,.$$

Using an equivalent terminology, A.2 can be restated by saying that the function $t - \phi(x)$ is a *time function* with respect to the operator $\partial_t^2 - a(t, x)\,\partial_x^2$.

REMARK 3. As it is well known, condition (4) is not necessary for the well posedness in C^∞ of (1), (2). Necessary and sufficient conditions on lower order terms can be found in [Ni3], where the problem is investigated in detail.

Acknowledgments. We would like to thank Prof. S. Spagnolo for many useful discussions about the subject of this work.

§ 2. Proof of Theorem 1.

The idea of the proof is the following. We prove a priori estimates for the solution of (1), (2). These estimates are of two different types. In the regions where $a_t(t, x) \geqslant -Ka(t, x)$, we use a method due to O. Oleinik (see [O]). She made a similar assumption about $a(t, x)$, but the form of the region was very simple (a strip). Here the region is bounded by two absolutely continuous curves, and this causes some additional difficulties. We prove that any derivative of the solution in the region can be estimated by some norm of the solution restricted to the lower boundary of the region. In the complementary regions we get a similar result, but the technique used is the usual energy estimate of Petrowski (modified, of course, to take into account the particular form of the region). Thus we can estimate, in a finite number of steps, the solution on the whole strip $[0, T] \times \boldsymbol{R}_x$ by its norms at the time $t = 0$.

Suppose firstly that $u(t, x)$ is a solution of (1), (2) with compact support (in $[0, T] \times \boldsymbol{R}_x$) say with support contained in $[0, T] \times [-R, R]$. We can then assume that the coefficients are bounded functions (with all their derivatives), and that (4) holds with a constant M fixed.

In order to simplify the exposition of the proof, we introduce some notations.

Given two absolutely continuous functions, Hölder continuous functions $\phi(x) \leqslant \psi(x)$ on $[-R, R]$, such that the set $\{x \colon \psi'(x) = 0\}$ has a boundary with measure 0 (or differs for a negligeable set from a set whose boundary has measure 0), and that the products $a(\phi, x)\phi'^2$, $a(\psi, x)\psi'^2$ are less then 1, we write

$$G(\phi, \psi) = \{(t, x) \colon \phi(x) < t < \psi(x)\}$$

and we call $G(\phi, \psi)$ a *regular domain for a*. Since $u(t, x)$ is 0 for $|x| \geqslant R$, we will not distinguish between this domain and the infinite strip obtained by continuing ϕ, ψ as constants outside $[-R, R]$.

Moreover, if with γ we denote the set of points

$$\gamma = \{(t, x) \colon t = \phi(x)\},$$

we shall write for j integer and $v(t, x)$ a C^∞ function

$$\|D^j v\|_\gamma = \sum_{|\alpha| \leqslant j} \|D^\alpha v\|_{L^\infty(\gamma)};$$

more generally, for any set A, we will write

$$\|D^j v\|_A = \sum_{|\alpha| \leqslant j} \|D^\alpha v\|_{L^\infty(A)}.$$

We begin with a technical lemma.

LEMMA 1. *Let $v(t, x)$ be a C^∞ function on $[0, T] \times \boldsymbol{R}_x$, vanishing for $|x| \geqslant R$, and let $\phi(x)$ be a continuous function on $[-R, R]$. Denote with γ the set $\{(t, x) \colon t = \phi(x)\}$. Then there exists a C^2 function $w(t, x)$, which is C^∞ outside γ, and such that, in the points of γ, $D^\alpha v = D^\alpha w$ for $|\alpha| \leqslant 2$. Moreover, w vanishes for $|x| \geqslant R + 1$, and the second derivatives of w can be estimated by the second derivatives of v along γ: for* *any compact set* K *there exists a constant* $c(K)$ *such that*

$$\|D^2 w\|_K \leqslant c(K)\|D^2 v\|_\gamma . \tag{5}$$

PROOF. The proof is a simple application of Whitney's extension theorem, for which we refer to [Fe], Th. 3.1.14. With the notations used there, we define functions $P_a(b)$ with $a = (\phi(\xi), \xi) \in \gamma$ and $b = (t, x)$ which are nothing but the Taylor series development of $v(t, x)$ in a, truncated to the second order and computed in the point b. It is easy to verify that the assumptions of Whitney's theorem are fulfilled. Moreover, the estimate of the second derivatives of w is a consequence of the explicit expression for w (see [Fe], p. 226), and w can be chosen to vanish for $|x| \geqslant R'$ for any $R' > R$ (it is sufficient to multiply by a suitable regular function of x alone with compact support). ∎

We can now prove the energy estimates, beginning with the regions where $a_t \geqslant -Ka$.

In the following Lemmas 2 and 3, we will assume that $v(t, x)$ is a C^∞ solution of the equation

$$v_{tt} = (a(t, x)v_x)_x + b_1(t, x)v_x + c_1(t, x)v_t + d_1(t, x)v + f_1(t, x) \tag{6}$$

where $a(t, x)$ is the same coefficient as in eq. (1), while the other coefficients may be different. In particular, we shall assume that

$$v(t, x) = f_1(t, x) = 0 \qquad \text{for } |x| \geqslant R, \tag{7}$$

$$b_1^2(t, x) \leqslant Ma(t, x). \tag{8}$$

For brevity, in the following computations we shall omit the index 1 from the coefficients.

LEMMA 2. *Let* $G = G(\phi, \psi)$ *be a regular domain for* a, *and suppose that* $a_t(t, x) \geqslant -Ka(t, x)$ *in* G *for some constant* K. *Denote with* γ *the lower boundary of* G, *and with* $D_t \subset \boldsymbol{R}_x$ *the set*

$$D_t = \{x: (t, x) \in G\} = \{x: \phi(x) < t < \psi(x)\}.$$

Then for $\inf \phi \leqslant t \leqslant \sup \psi$ *one has*

$$\int_{D_t} v^2(t, x)\,dx \leqslant C\cdot\left[\iint_G f^2 + \|D^2 v\|_\gamma\right] \tag{9}$$

where C *is a constant depending on* M, T, ϕ, ψ *and the coefficients.*

PROOF. Suppose firstly that v, v_t and v_x vanish along γ. Denote with G_τ the domain $G \cap (\boldsymbol{R}_x \times [0, \tau])$, and define for $(t, x) \in G_\tau$

$$V(t, x) = \int_t^{\tau \wedge \psi(x)} v(s, x)\, ds. \tag{10}$$

The function V is well defined and continuous on G_τ, and vanishes along the upper boundary of G_τ; moreover

$$V_t(t, x) = -v(t, x), \tag{11}$$

$$V_{xt}(t, x) = -v_x(t, x), \tag{12}$$

$$V_x(t, x) = \int_t^{\tau \wedge \psi(x)} v_x(s, x)\, ds + v(\tau \wedge \psi(x), x) \frac{d}{dx}(\tau \wedge \psi(x)), \tag{13}$$

where $(d/dx)(\tau \wedge \psi(x))$, defined a.e., means 0 where $\psi > \tau$ and ψ' elsewhere (indeed, the function $\tau \wedge \psi(x)$ is absolutely continuous and hence almost everywhere differentiable). Thus V_t and V_{tx} are C^∞ functions; note that V_x is the sum of a continuous function of (t, x), which is regular in t, and a function of x alone, which is integrable on $\boldsymbol{R}$.

Multiply now equation (6) by $Ve^{\theta t}$, where $\theta \geq 0$ is to be chosen, and integrate over G_τ. Consider the resulting terms one by one as in [Ol]; for brevity, we omit the domain of integration G_τ in the following integrals.

Thanks to the particular form of the set G_τ, the first term $\iint v_{tt} Ve^{\theta t}$ is easy to compute:

$$\begin{aligned} \iint v_{tt} \cdot Ve^{\theta t}\, dx\, dt &= -\iint v_t (e^{\theta t} V)_t = -\iint v_t \theta e^{\theta t} V + \iint v_t e^{\theta t} v = \\ &= \frac{1}{2} \iint (e^{\theta t} v^2)_t - \frac{\theta}{2} \iint e^{\theta t} v^2 + \iint \theta v (Ve^{\theta t})_t = \\ &= \frac{1}{2} \iint (e^{\theta t} v^2)_t - \frac{3\theta}{2} \iint v^2 e^{\theta t} + \theta^2 \iint vVe^{\theta t}; \end{aligned} \tag{14}$$

we have used the fact that $v = v_t = 0$ on γ, while $V = 0$ on the upper boundary $\{t = \psi(x)\}$.

The second and the third terms, $\iint (av_x)_x Ve^{\theta t} + \iint bv_x Ve^{\theta t}$, are more difficult, since it is necessary to integrate by parts in the variable x. To justify this passage, it is sufficient to observe that, at t fixed, the set $\{x: (x, t) \in G_\tau\}$ is open, then it consists of a countable union on intervals, and in each of these the integration by parts can be performed. Since v_x vanishes on γ, while V vanishes on the upper boundary

$\{t = \psi(x)\}$, no boundary terms appear. Thus we have:

$$(15) \quad \iint (av_x)_x \cdot Ve^{\theta t} + \iint bv_x \cdot Ve^{\theta t} = - \iint av_x V_x e^{\theta t} - \iint (bV)_x e^{\theta t} v;$$

note that, thanks to (13), the right hand member is well defined. To estimate the first integral at right in (15), consider the function $aV_x^2 e^{\theta t}$: thanks to (13) it can be differentiated w.r. to t, giving

$$(aV_x^2 e^{\theta t})_t = a_t V_x^2 e^{\theta t} - 2av_x V_x e^{\theta t} + \theta a V_x^2 e^{\theta t}$$

and hence, as $a_t \geqslant -Ka$,

$$(16) \qquad -av_x V_x e^{\theta t} \leqslant \frac{1}{2}(aV_x^2 e^{\theta t})_t - \frac{\theta}{2} aV_x^2 e^{\theta t} + \frac{K}{2} aV_x^2 e^{\theta t}.$$

On the other hand,

$$-(bV)_x e^{\theta t} v = -b_x Ve^{\theta t} v - bV_x e^{\theta} v \leqslant -b_x Ve^{\theta t} + \frac{1}{2} b^2 V_x^2 e^{\theta t} + \frac{1}{2} v^2 e^{\theta t}$$

and recalling the Levi condition (8),

$$(17) \qquad -(bV)_x e^{\theta t} v = -b_x Ve^{\theta t} v + \frac{M}{2} aV_x^2 e^{\theta t} + \frac{1}{2} v^2 e^{\theta t}.$$

Let now $\mu = M + |K|$. Choosing $\theta = \mu$ in (16) and (17) and summing, we get:

$$(18) \quad -av_x V_x e^{\mu t} - (bV)_x e^{\mu t} v \leqslant \frac{1}{2}(aV_x^2 e^{\mu t})_t - b_x Ve^{\mu t} v + \frac{1}{2} v^2 e^{\mu t}.$$

Now we integrate over G_τ. Firstly we integrate in t, at x fixed, between $t = \phi(x)$ and $t = \tau \wedge \psi(x)$ (when these limits coincide, the corresponding section of G_τ is empty); note in particular that

$$(19) \qquad \int_{\phi(x)}^{\tau \wedge \psi(x)} (aV_x^2 e^{\mu t})_t \leqslant aV_x^2 e^{\mu t} |_{t = \tau \wedge \psi(x)}$$

(since the function $aV_x^2 e^{\mu t}$ is nonnegative); but, by (13),

$$V_x(\tau \wedge \psi(x), x) = v(\tau \wedge \psi(x), x) \cdot (\tau \wedge \psi(x))'_x$$

and hence the function

$$aV_x^2 e^{\mu t} |_{t = \tau \wedge \psi(x)}$$

is equal to 0 when $\psi(x) > \tau$, while is equal to

$$a(\psi(x), x)\,\psi'(x)^2 v^2(\psi(x), x)\, e^{\mu\psi(x)} \leqslant v^2(\psi(x), x)\, e^{\mu\psi(x)} \tag{20}$$

when $\psi(x) \geqslant \tau$; we have used the assumption $a(\psi, x)\,\psi'^2 < 1$.

Thus we can integrate (18) also in x (note that we are forced to integrate *first* in time, *then* in space), to get

$$-\iint a v_x V_x e^{\mu t} - \iint (bV)_x e^{\mu t} \leqslant \tag{21}$$

$$\leqslant \frac{1}{2} \int_{\psi(x)} v^2(\psi(x), x)\, e^{\mu\psi(x)}\, dx - \iint b_x V e^{\mu t} + \frac{1}{2} \iint v^2 e^{\mu t}.$$

Now it is sufficient to observe that the functions in (21) are all integrable on G_τ, and to apply Fubini's theorem, to ensure that (21) does not depend on the order of integration and is thus valid in the ordinary Lebesgue sense.

The remaining terms are harmless; they give (recall that $\theta = \mu$)

$$\iint c v_t \cdot V e^{\mu t} = - \iint (c V e^{\mu t})_t v = - \iint c_t v V e^{\mu t} + \tag{22}$$

$$+ \iint c v^2 e^{\mu t} - \iint \theta c V v e^{\mu t} \leqslant C(\|c\|_\infty, \|c_t\|_\infty) \iint (v^2 + V^2)\, e^{\mu t};$$

$$\iint dv \cdot V e^{\mu t} \leqslant C(\|d\|_\infty) \iint (v^2 + V^2)\, e^{\mu t}; \tag{23}$$

$$\iint f \cdot V e^{\mu} \leqslant \frac{1}{2} \iint f^2 e^{\mu t} + \frac{1}{2} \iint V^2 e^{\mu t}. \tag{24}$$

We can now sum (14), (21), (23), (24) obtaining

$$\frac{1}{2} \iint (e^{\mu t} v^2)_t \leqslant \frac{3\mu}{2} \iint e^{\mu t} v^2 - \mu^2 \iint e^{\mu t} v V + \tag{25}$$

$$+ \frac{1}{2} \int_{\psi(x) \leqslant \tau} v^2(\psi(x), x)\, e^{\mu\psi(x)}\, dx + C \cdot \iint (v^2 + V^2)\, e^{\mu t} + \frac{1}{2} \iint f^2 e^{\mu t}$$

where the constant C depends on the ∞ norms of a, b_x, c, c_t, d.

Now the left hand member of (25) gives

$$\frac{1}{2} \iint (e^{\mu t} v^2)_t = \frac{1}{2} \int_{\phi(x) < \tau} e^{\mu(\tau \wedge \psi(x))} v^2(\tau \wedge \psi(x), x)\, dx \tag{26}$$

since v vanishes along γ, while directly from def. (10) it follows that

$$\iint V^2 e^{\mu t} \leqslant T \iint v^2 e^{\mu t}; \tag{27}$$

introducing (26) and (27) in (25) we find

$$\int\limits_{\phi(x) < \tau} v^2(\tau \wedge \psi(x), x)\, e^{\mu(\tau \wedge \psi(x))}\, dx - \int\limits_{\psi(x) \leqslant \tau} v^2(\psi(x), x)\, e^{\mu\psi(x)}\, dx \leqslant$$

$$\leqslant C_1 \cdot \left[\iint v^2 e^{\mu t} + \iint f^2 e^{\mu t}\right].$$

Rearranging the first term, since

$$D_\tau = \{x\colon \phi(x) < \tau < \psi(x)\} = \{\phi(x) < \tau\} \setminus \{\psi(x) \leqslant \tau\},$$

we have

$$\int\limits_{D_\tau} v^2(\tau, x)\, e^{\mu\tau}\, dx \leqslant C_1 \cdot \left[\iint v^2 e^{\mu t} + \iint f^2 e^{\mu t}\right]. \tag{28}$$

The constant C_1 depends on M, T, R and the supremum norms of a, b_x, c, c_t, d (recall that we are assuming that v has compact support).

To conclude, define

$$y(\tau) = \iint\limits_{G_\tau} v^2(t, x)\, e^{\mu t}\, dx\, dt \equiv \int\limits_0^\tau \left(\int\limits_{D_\sigma} v^2(\sigma, x)\, e^{\mu\sigma}\, dx \right) d\sigma;$$

from this it follows that

$$y'(\tau) = \int\limits_{D_\tau} v^2(\tau, x)\, e^{\mu\tau}\, dx.$$

Thus (28) implies

$$y'(\tau) \leqslant \widetilde{C}_1 \cdot \left[y(\tau) + \iint\limits_{G_\tau} f^2 \right]. \tag{29}$$

Applying now Gronwall's lemma, we conclude the proof in the special case of v, v_t, v_x vanishing on the initial curve (the boundedness of y implies the boundedness of y' from eq. (29) itself).

To deal with the general case of v not vanishing along γ, we reduce ourself to the preceding one by a straightforward application of Lemma 1: we consider the function $w(t, x)$ furnished by Lemma 1 and apply the above estimate to the difference $v - w$; this difference satisfies eq. (6) with $\tilde{f}$ replaced by another function, whose L^2 norm can be estimated by $\|\tilde{f}\|^2$ and the values of v and its first and second derivatives along γ (by (5)). ∎

We now consider the regions where the reverse inequality holds, $a_t \leqslant Ka$. Let v denote again a solution of (6), such that (7), (8) hold.

LEMMA 3. *Let $G = G(\psi, \chi)$ be a regular domain for a, and assume that $a_t \leqslant Ka$ on G for some constant K. Let as above $D_t = \{x\colon \psi(x) < t < \chi(x)\}$, and consider the energy of v, defined as*

$$E(t) = \int_{D_t} (a|v_x(t, x)|^2 + |v_t(t, x)|^2 + |v(t, x)|^2)\, dx .$$

Then, denoting with γ the lower boundary of G, for $\inf \psi \leqslant t \leqslant \sup \chi$ *one has*

$$E(t) \leqslant C \cdot \left[\iint_G f^2 + \|D^2 v\|^\gamma \right], \tag{30}$$

where C is a constant depending on M, T, ψ, χ and the supremum norms of some derivatives of the coefficients.

PROOF. We need a formula for the derivative of an integral whose domain varies with time.

More precisely, let $F(x)$ be a C^1 function supported on $[-R, R]$, let $\beta(x)$ be an absolutely continuous function on $[-R, R]$, and consider the integral

$$I_\beta(t) = \int_{B_t^\beta} F(x)\, dx$$

where

$$B_t^\beta = \{x \in [-R, R]\colon \beta(x) > t\}.$$

In general, the function $I_\beta(t)$ is not even continuous (consider for instance the case $F \equiv 1$, $\beta = \text{const.}$). But if we assume that the set

$$Y_1 = \{x \in [-R, R]\colon \beta'(x) = 0\}$$

has Lebesgue measure zero, then we claim that $I_\beta(t)$ is absolutely continuous, and its derivative is given for almost any t by the formula

$$(31) \qquad I_\beta'(t) = -\sum_{x \in \partial B_t^\beta} \frac{F(x)}{|\beta'(x)|};$$

indeed, we shall prove that the set

$$\partial B_t^\beta \equiv \{x\colon \beta(x) = t\}$$

is *finite* for almost any t, and thus the sense of (31) is clear.

To prove the claim, we shall use some basic tools from geometric measure theory. The standard reference is [Fe].

The derivative of $\beta(x)$ is defined almost everywhere, let us denote with Y_2 the set where the derivative of β does not exist. We define then the set

$$Z = \beta(Y_1) \cup \beta(Y_2)$$

which is of course a negligible set.

For a measurable set $M \subset \boldsymbol{R}^n$, the *essential boundary* is defined as follows:

$$\partial^* M = \left\{x\colon \limsup_{\rho \to 0^+} \frac{|B(x, \rho) \cap M|}{\rho^n} > 0,\ \limsup_{\rho \to 0^+} \frac{|B(x, \rho) \setminus M|}{\rho^n} > 0\right\}$$

(where as usual $B(x, \rho)$ is the open ball centered in x with radius ρ, while $|M|$ denotes the n-dimensional Lebesgue measure of the set M).

We remark that, since $\beta'(x) \neq 0$ almost everywhere, the following holds:

$$(32) \qquad \partial^* B_t^\beta = \partial B_t^\beta \quad \text{almost everywhere}$$

(this means that the two sets coincide, apart from a negligible set). In fact, it is well known that the derivative of an absolutely continuous function $\beta(x)$ is given, where it exists, by the usual limit

$$\lim_{h \to 0} \frac{1}{h}(\beta(x + h) - \beta(x)),$$

and assuming that this limit is different from 0 at some point $x \in \partial B_1^\beta$, taking h small enough it is easy to see that

$$\limsup_{\rho \to 0^+} \frac{|B(x, \rho) \cap B_t^\beta|}{\rho} = \limsup_{\rho \to 0^+} \frac{|B(x, \rho) \setminus B_t^\beta|}{\rho} = \frac{1}{2}.$$

But in fact, if $t \notin Z$, we have instead of (32)

$$\partial^* B_t^\beta \equiv \partial B_t^\beta \tag{33}$$

since, for such values of t, ∂B_t^β consists *exclusively* of points where $\beta'(x)$ *exist and is different from 0.* This fact will play an important role in the following.

We shall need a special case of the so-called *Fleming-Rishel formula* (or *coarea* formula); for the general result see [Fe 4.5.9] or [Gi]. It states that if $g(x)$ is a nonnegative Borel function on $\boldsymbol{R}$ (or bounded Borel), and $f(x)$ is absolutely continuous on the bounded open set $\Omega \subset \boldsymbol{R}$, then

$$\int_\Omega g(x)\,|f'(x)|\,dx = \int_{-\infty}^{+\infty} \left(\int_{\partial^*\{x:\ f(x) > t\} \cap \Omega} g(x)\, d\mathcal{H}^0 \right) dt$$

where $d\mathcal{H}^0$ is the Hausdorff zero-dimensional measure on $\boldsymbol{R}$, which is nothing but the *counting measure* on $\boldsymbol{R}$. (To be precise, the usual coarea formula holds for $f(x)$ Lipschitz continuous; but in dimension 1 this assumption can be relaxed to $W^{1,1}$ since there exists ([Fe] 3.1.16) a sequence of compact sets $K_k \subset \Omega$ such that $\Omega \setminus \cup K_k$ is negligible and $f(x)$ restricted to K_k is Lipschitz).

Note the following consequence of the formula: given a function $f \in W^{1,1}$, the set $\partial^*\{x:\ f(x) > t\}$ is finite for almost any t (a fact that is not immediately intuitive).

We apply now the coarea formula with

$$f(x) = \beta(x), \qquad g(x) = \frac{F(x)}{|\beta'(x)|}$$

(recall that $\beta' \neq 0$ a.e.) and, fixed $t_2 > t_1$,

$$\Omega = \{x:\ t_2 > \beta(x) > t_1\}.$$

We thus obtain

$$\int_{\{t_2 > \beta(x) > t_2\}} F(x) = \int_{t_1}^{t_2} \int_{\partial^* B_t^\beta} \frac{F(x)}{|\beta'(x)|}\, d\mathcal{H}^0\, dt\,;$$

but recalling (33), the observation following it and the definition of

$I_\beta(t)$, this can be rewritten as

$$I_\beta(t_1) - I_\beta(t_2) = \int_{t_1}^{t_2} \sum_{x \in \partial B_t^\beta} \frac{F(x)}{|\beta'(x)|}\, dt\,. \tag{34}$$

This means that $I_\beta(t)$ is absolutely continuous, and that (31) holds.

An immediate consequence of (31) is the following more general formula, valid for any $F(t, x)$ of class C^1 (say) and vanishing for x outside some compact set:

$$\frac{d}{dt} \int_{B_t^\beta} F(t, x)\, dx = \int_{B_t^\beta} F_t(t, x)\, dx - \sum_{x \in \partial B_t^\beta} \frac{F(x)}{|\beta'(x)|}\,. \tag{35}$$

We shall now apply (35) to prove an energy estimate for solutions of (6).

Suppose $v(t, x)$ is a solution of (6) vanishing with its derivatives up to the second order along γ. The function $\tilde{v}(t, x)$ defined as

$$\tilde{v}(t, x) = \begin{cases} v(t, x) & \text{if } t \geqslant \psi(x), \\ 0 & \text{if } t \leqslant \psi(x), \end{cases}$$

is also C^2 (but is a solution of (6) only for $\psi(x) \leqslant t \leqslant T$).

Now we shall construct a sequence $\chi_k(x)$ of absolutely continuous functions such that, on $[-R, R]$, $\chi_k(x) \geqslant \chi(x)$, $\chi_k'(x) \neq 0$ almost everywhere, $\chi_k \to \chi$ uniformly on $[-R, R]$, and eventually

$$a(\chi_k(x), x)\, \chi_k'(x)^2 < 1\,.$$

To this end, we take a sequence of positive numbers $\varepsilon_k \downarrow 0$ with the property that the sets $\{x\colon \chi_k'(x) = \pm \varepsilon_k\}$ have measure 0. Moreover, we recall that the set $\{\chi' = 0\}$ coincides a.e. with a set A whose boundary has measure 0. Let C be the closure of $\boldsymbol{R} \setminus A$; C and A have the same boundary. Moreover, the function $\rho(x) = \operatorname{dist}(x, C)$ is Lipschitz continuous, and by the properties of C it follows that $\rho' = 0$ a.e. on C and $\rho' = \pm 1$ elsewhere. Then the functions $\chi_k(x) = \chi(x) + \varepsilon_k \cdot \rho(x)$ have all the required properties.

Let now $B_t^k = \{x \in [-R, R]\colon 0 \leqslant t < \chi_k(x)\}$, and consider

$$E_k(t) = \int_{B_t^k} (a|\tilde{v}_x|^2 + |\tilde{v}_t|^2 + |\tilde{v}|^2)\, dx\,.$$

Note that $E_k(t)$ is an integral of the form $I_{\chi_k}(t)$, fulfilling the assumptions relative to formula (35). Applying (35) we get, for almost every t,

$$E_k'(t) = \int_{B_t^k} \frac{\partial}{\partial t}(a|\tilde{v}_x|^2 + |\tilde{v}_t|^2 + |\tilde{v}|^2)\,dx - \sum_{x \in \partial B_t^k} \frac{(a|\tilde{v}_x|^2 + |\tilde{v}_t|^2 + |\tilde{v}|^2)|_{(\chi_k(x),x)}}{|\chi_k'(x)|}$$

(note that in the last sum $\chi_k(x) = t$). The first integral $I_k(t)$ gives

$$I_k(t) \leqslant \int_{B_t^k} a_t|\tilde{v}_x|^2 + 2\int_{B_t^k}(a\tilde{v}_x\bar{\tilde{v}}_{xt} + \tilde{v}_{tt}\bar{\tilde{v}}_t) + 2\int_{B_t^k}\tilde{v}\bar{\tilde{v}}_t \leqslant$$

$$\leqslant \int_{B_t^k} a_t|\tilde{v}_x|^2 + 2\int_{B_t^k}(\tilde{b}\tilde{v}_x + \tilde{c}\tilde{v}_t + \tilde{d}\tilde{v} + \tilde{f})\bar{\tilde{v}}_t + 2\int_{B_t^k}(a\tilde{v}_x\bar{\tilde{v}}_t)_x + E_k(t)$$

(we have integrated by parts and used equation (6); note that since $\tilde{v}_t$ vanishes in the region between $t = 0$ and $t = \psi(x)$ the above inequality is correct, even though $\tilde{v}$ is not a solution of (6) there); estimating the lower order terms with E_k, and using the Levi condition, we get

$$I_k(t) \leqslant \int_{B_t^k} a_t|\tilde{v}_x|^2 + C\cdot E_k(t) + \int_{B_t^k}|\tilde{f}|^2 + \sum_{x \in \partial B_t^k}(2av_x\bar{v}_t)|_{(\chi_k(x),\,x)}$$

(recall the above expression for ∂B_t^k) where the constant C depends on the supremum norms of $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ and on M, R, T. Now, since $a(\chi_k(x), x)\chi_k'(x)^2 \leqslant 1$, we have

$$(2a\tilde{v}_x\bar{\tilde{v}})|_{(\chi_k(x),\,x)} \leqslant \sqrt{a}(a|\tilde{v}_x|^2 + |\tilde{v}_t|^2)|_{(\chi_k(x),\,x)} \leqslant \frac{(a|\tilde{v}_x|^2 + |\tilde{v}_t)|_{(\chi_k(x),\,x)}}{|\chi_k'(x)|}$$

and summing up

$$E_k'(t) \leqslant \int_{B_t^k} a_t|\tilde{v}_x|^2 + C\cdot E_k(t) + \int_{B_t^k}|\tilde{f}|^2.$$

Integrating between 0 and t (as evidently $E_k(0) = 0$) we have

$$E_k(t) \leqslant \int_0^t\left(\int_{B_t^k} a_t|\tilde{v}_x|^2 + C\cdot E_k(s) + \int_{B_t^k}|\tilde{f}|^2\right)ds.$$

We can now pass to the limit $k \to \infty$ observing that

$$B_t^k \downarrow B_t = \{x \in [-R, R]:\ \chi(x) > t\},$$

and that $E_k(t) \downarrow E(t)$ (in fact we have

$$E(t) = \int_{B_t} (a|\tilde{v}_x(t, x)|^2 + |\tilde{v}_t(t, x)|^2 + |\tilde{v}(t, x)|^2)\, dx \equiv$$

$$\equiv \int_{D_t} (a|v_x(t, x)|^2 + |v_t(t, x)|^2 + |v(t, x)|^2)\, dx$$

since $\tilde{v}$ vanishes below $\psi(x)$) we get

$$E(t) \leqslant C \cdot \int_0^t E(s)\, ds + \int_0^t \int_{B_t} a_t |\tilde{v}_x|^2\, dx\, ds + \int_0^t \int_{B_t} |\tilde{f}|^2\, dx\, ds\,.$$

Since $a_t \leqslant Ka$ in the region G under consideration, the second term is not greater than $|K| \int_0^t E(s)\, ds$. Thus we get

$$E(t) \leqslant C \int_0^t E(s)\, ds + \int_0^t \int_{B_t} |\tilde{f}|^2\, dx\, ds$$

and applying Gronwall's lemma we obtain the estimate

$$E(t) \leqslant C \cdot \iint_G |\tilde{f}|^2 \tag{36}$$

with C depending on M, R, T, ν_0 and the supremum norms of the coefficients.

In the general case, when v does not vanish up to the second order along γ, we apply as above Lemma 1. ■

We can now conclude the proof of Theorem 1.

Let u be a solution with support in $[0, T] \times [-R, R]$. Suppose that in the first region $G_R^{\frac{1}{k}}$ (with lower boundary the initial line) $a_t \geqslant -Ka$ (the other case is of course similar). Then Lemma 2 allows to estimate u in the norm $L_t^\infty(L_x^2)$ and thus in the norm of $L^2(G_R^{\frac{1}{k}})$.

Now differentiate eq. (1) with respect to x. It is easy to verify that the equation thus obtained is again of the type (6), with (7) and (8) fulfilled. In fact, b is replaced by a_x, for which a condition analogous to (8) holds, namely the well known inequality $(a_x)^2 \leqslant c\|a_{xx}\|_\infty \cdot a$; moreover, $\tilde{f} + f_x + d_x u$, which is a sum of terms for which L^2 estimates are already available. Similarly, proceeding by induction, one can estimate all k-derivatives of u in the L^2 norm on the first region. Now derivate with respect to t once, and any number of times with respect to x; applying again the Lemmas we can estimate all the derivatives of u of the form $D_t D_x^j$ in term of the initial values and of the derivatives already estimated. Proceed by induction. At the end, we obtain a L^2 estimate for all the derivatives of u in the first region. This implies also the boundedness in L^∞ of all the derivatives of u by Sobolev immersion; indeed, we remark that the embedding theorems for Sobolev spaces holds also for our regular domains, with different indexes depending on the Hölder regularity of the boundary (see e.g. [Ma]). In particular, we have L^∞ estimates for the value of u and its derivatives along the upper boundary of the first region, which is the lower boundary of the second region. Thus one can start again the method in the second region, applying Lemma 1 or 2 according to the type of the region.

In a finite number of steps, we reach the line $t = T$.

To conclude, suppose arbitrary C^∞ data u_0, u_1 are given. Using a partition of unity, reduce the problem to the case in which the data u_0, u_1 and the function $f(t, x)$ vanish for $|x| > R$. If we approximate u_0, u_1 with Gevrey data with compact support (in the topology of C^∞), well known theorems furnish us a sequence of approximate solutions (see [Ni2]). They have compact support, since the finite speed of propagation property holds for weakly hyperbolic equations in the class of Gevrey functions (see the Appendix of [D]); thus we can apply the a priori estimate obtained above, obtaining the C^∞ convergence of the approximate solutions to a solution of the original problem.

Finally, we can sum together the various solutions corresponding to the compactified data: the sum will be locally finite at every $t > 0$ since the slope of the domain of influence depends only on the supremum norm of $a(t, x)$, which is bounded by assumption.

This concludes the proof of Theorem 1. ■

3. Proof of Theorem 2.

Let $a(t, x)$ be a nonnegative, real analytic function on $\boldsymbol{R}^2$. We shall prove that the rectangle $G_R = [0, T] \times [-R, R]$ can be partitioned in a finite number of regions satisfying assumption (A).

We shall firstly prove that (A) holds locally, *i.e.* in the neighbourhood of each point (t_0, x_0). It is not restrictive to assume that $(t_0, x_0) = (0, 0)$.

We shall need the following properties of the function $a(t, x)$, which are proved in Lemma 2.2 of [Ni1]. By Weierstrass' preparation theorem the set

$$\{(t, x) \in \boldsymbol{C} \times \boldsymbol{R} \colon a(t, x) = 0\}$$

can be described near $(0, 0)$ as the union of a finite number of curves, $t_1(x), \ldots, t_m(x)$, $m \geqslant 0$, and, possibly, of the line $\{x = 0\}$. The curves $t = t_j(x)$ have the following properties: $t_j(0) = 0$ for all j,

$$\operatorname{Re} t_j(x) \leqslant \operatorname{Re} t_{j+1}(x), \qquad 1 \leqslant j < m,$$

and more precisely the first l functions $t_1(x), \ldots, t_l(x)$ are real valued; moreover $t_j(x)$ is analytic for $x \neq 0$, Hölder continuous, and for $x \to 0$ the estimates

$$t_j(x) = O(|x|^\sigma), \qquad t_j'(x) = O(|x|^{\sigma - 1})$$

hold, for some rational $\sigma > 0$ (depending on j). Writing

$$\widehat{t}(t) = 2\left(\sum_{j=1}^{m} |t_j(x)|^2\right)^{1/2}$$

we have near $(0, 0)$, for some constants C, K,

$$\sup_{-\widehat{t}(x) \leqslant t \leqslant \widehat{t}(x)} a(t, x) \leqslant C|x|^2 \tag{37}$$

(this is an immediate consequence of (2.5) in [Ni]) and

$$a_t(t, x) \geqslant Ka(t, x) \qquad \text{if } t \geqslant \widehat{t}(x) \tag{38}$$

(this is exactly (2.6) of [Nil], together with the argument following it); one has analogously

$$a_t(t, x) \leqslant K' a(t, x) \qquad \text{if } t \leqslant -\widehat{t}(x) \tag{39}$$

$(t \to -t)$.

Suppose now that $a_t(t, x)$ vanishes along the (real) curve $t = \psi(x)$ such that $\psi(0) = 0$. By possibly restricting the neighbouhood, we can assume that one of the following alternatives holds: either $|\psi(x)| \leqslant \widehat{t}(x)$, or $|\psi(x)| \geqslant \widehat{t}(x)$. In the first case, it is easy to see that

$$a(\psi(x), x)\, \psi'(x)^2 \to 0 \qquad \text{for } x \to 0; \tag{40}$$

this follows from (37) and the fact that

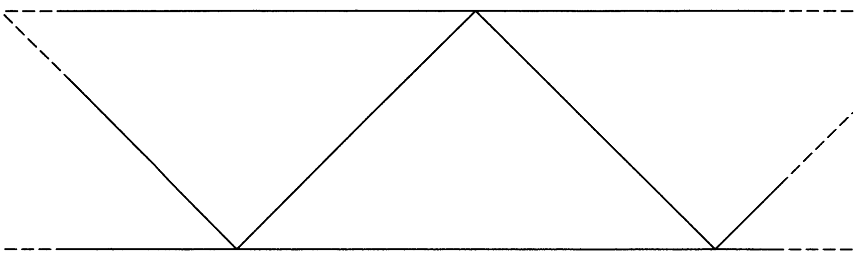
$$|\psi'(x)| \leqslant C|x|^{\sigma - 1}$$

for some rational $\sigma > 0$. The same argument shows that

$$a(\widehat{t}(x), x)\, \widehat{t}'(x)^2 \to 0 \qquad \text{for } x \to 0. \tag{41}$$

Now is it possible to construct the curves required in ass. (A) in a neighbourhood of $(0, 0)$. Restrict the neighbourhood so that all the zeroes of $a(t, x)$, $a_t(t, x)$ can be described as above, and each real zero of $a_t(t, x)$ verifies one of the inequalities $|\psi(x)| \geq \hat{t}(x)$ or $\leq \hat{t}(x)$. Then the curves $\phi_j(x)$ will be the real zeroes of $a_t(x)$ lying between $-\hat{t}(x)$ and $\hat{t}(x)$, *plus* the two curves $t = -\hat{t}(x)$ and $t = \hat{t}(x)$, arranged in increasing order. In fact, A.1 is evidently satisfied, A.2 follows from (40) and (41), possibly restricting the neighbourhood; finally, A.3 is obvious in the regions bounded by the real zeroes of a_t (since a_t has constant sign on them) and follows from (38) and (39) in the remaining upper and lower regions.

Now to conclude the proof, divide the rectangle G_R in horizontal strips of height ε, and divide each strip in triangular cells according to the pattern shown in the picture:



The slope α of the oblique sides is given by

$$\alpha^2 = \frac{1}{\|a\|_{L^\infty(G_R)} + 1} .$$

With this choice of α, if a piece of one oblique side is used in order to construct one of the curves $\phi_j(x)$, ass. A.2 will be automatically satisfied along it.

Since as $\varepsilon \to 0$ the diameter of the triangles can be made arbitrarily small, we can apply the above local result to each cell; thus we can assume that each cell can be partitioned according to assumption (A).

Now it is sufficient to orderly patch pieces of the horizontal lines, pieces of the oblique sides, and the curves into each cell, to obtain the global curves required in ass. (A).

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Manoscritto pervenuto in redazione il 10 agosto 1992.