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On Homomorphic Images of Locally Graded Groups.

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A group G is said to be locally graded if every nontrivial, finitely generated subgroup of G has a nontrivial finite image. It is evident that the class of locally graded groups is quite extensive. Indeed, it may be considered one of the most natural classes to consider if one wishes to avoid the presence of finitely generated, infinite simple subgroups. The reader is referred to the paper [1] for an example of a substantial theorem which holds for locally graded groups but not for all groups.

It is usually desirable to know whether a given class of groups is closed under some operation or other. The class $\mathfrak{X}$ of locally graded groups is, trivially, L-closed. It is easily seen to be closed under forming subgroups and cartesian products and (therefore) a group which is residually $\mathfrak{X}$ is itself an $\mathfrak{X}$-group. P'-closure (extension closure in the strongest sense) is also easily verified. Certainly $\mathfrak{X}$ is not closed under forming homomorphic images, and the purpose of this note is to establish some results which indicate a form of «partial Q-closure».

Now if G is a residually finite group and H is a normal subgroup of G then G/H is again residually finite provided that H is either (i) finite or (ii) a maximal normal abelian subgroup of G or (iii) the centre of G. These facts are well-known and straightforward to prove. Here we consider homomorphic images of locally graded groups by (normal) subgroups which possess properties related to those described in (i), (ii) and (iii). It will be seen that there are reasonably satisfactory theorems concerning hypercentral and certain generalized soluble subgroups, but that the full picture is not clear in the latter case (even with regard to abelian subgroups). Further, with a certain restriction imposed, one may say something concerning factor groups by hyperfinite and FC-

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central subgroups. Related to (i) and (ii), then, we present two specific questions which are, it is hoped, of some interest.

Our first result, an easy one, will shortly be generalized and will be found useful on one or two further occasions.

LEMMA 1. Let G be a locally graded group and H a subgroup of the centre of G. Then G/H is locally graded.

PROOF. Let G, H be as stated and suppose, for a contradiction, that G/H is not locally graded. Then, for some finitely generated subgroup F not contained in H, $F/F \cap H$ has no nontrivial finite image. We may as well assume that $F = G$. Then it is easy to see that $G = HG'$ and thus $G = HK$, for some finitely generated subgroup K of G'. Thus we have $K \leqslant G' = K'$ and hence $G' = K$. Certainly $G' \neq 1$ and so there is a normal subgroup N of G' such that G'/N is finite and nontrivial. But this implies that $G = HN$ and thus $G' = N'$, a contradiction which completes the proof.

Before proceeding, we note that the reduction to the case where G is finitely generated and G/H is nontrivial, with no nontrivial finite images, is valid for subgroups H other than those contained in the centre of G. We may also note here the well-known fact that the corresponding result is not (of course) true in the case of residually finite groups (that is, one cannot in general factor by an arbitrary subgroup of the centre and retain residual finiteness). To see this, one need only consider the additive group of p-adic rationals, for some prime p. (The same example shows that one cannot delete «maximal» from condition (ii) above.)

The above lemma enables us to prove the following.

THEOREM 2. Let G be a locally graded group and H a G-invariant subgroup of the hypercentre of G. Then G/H is locally graded.

PROOF. Supposing the hypotheses satisfied, let α be the least ordinal such that $H \leqslant Z_\alpha = Z_\alpha(G)$, the α-th term of the upper central series of G. If $\alpha = 0$ there is nothing to prove. For a contradiction, we may choose the pair (G, H) with $\alpha > 0$ minimal such that G/H is not locally graded. If α is not a limit ordinal then $G/H \cap Z_{\alpha-1}$ is locally graded and the result follows from Lemma 1. Suppose, then, that $Z_\alpha = \bigcup_{\beta < \alpha} Z_\beta$ and let F be a finitely generated subgroup of G such that $F/F \cap H$ is nontrivial but has no nontrivial finite images. Then $F = (F \cap H)F'$ and so $F = (F \cap Z_\beta)F'$, for some $\beta < \alpha$. It follows that $F/Z_\beta(F)$ is perfect and thus has upper central series of length at most one. Hence $F \cap H \leqslant$

$\leqslant Z_\alpha(F) = Z_{\beta+1}(F)$. By Lemma 1, therefore, $F/(F \cap H)Z_\beta(F)$ is locally graded and hence, by hypothesis, trivial. This implies that $F/Z_\beta(F)$ is abelian and therefore trivial, giving $F \cap H = H \cap Z_\beta(F)$. But $F/H \cap Z_\beta(F)$ is locally graded and we have the desired contradiction.

We next consider images of locally graded groups by abelian normal subgroups. The following result is basic and will be used in the proof of Theorem 4.

LEMMA 3. Let G be a locally graded group and A a normal, periodic abelian subgroup of G. Then G/A is locally graded.

PROOF As in the proof of Lemma 1 we may assume (for a contradiction) that G is finitely generated and G/A is nontrivial and has no nontrivial finite images. In particular, we have $G = AG'$ and hence $G/G' \cong A/A \cap G'$, giving G/G' finite and G' finitely generated. Since $G' \neq 1$, there exists a G-invariant subgroup $N(\neq G')$ of finite index in G'. But then $G = AN$ and $G' \leqslant N$, a contradiction.

THEOREM 4. Let G be a locally graded group and A a normal abelian subgroup of G. Then G/A is locally graded if either of the following conditions is satisfied.

(*a*) $\langle a \rangle^G$ is finitely generated, for all $a \in A$.

(*b*) A has finite torsion-free rank.

PROOF (*a*) With the hypotheses satisfied, suppose that F is a finitely generated subgroup of G such that FA/A is nontrivial and has no nontrivial finite images. Let a be an arbitrary element of A and let C denote the centralizer of $\langle a \rangle^G$ in FA. Since $\langle a \rangle^G$ has a G-invariant series of subgroups of finite index which intersect in the identity, we see that FA/C is residually finite and hence trivial. Thus $[A, F] = 1$ and Lemma 1 gives us a contradiction.

(*b*) Assume that A has finite torsion-free rank. By Lemma 3 we may assume A is torsion-free and, as before, we may suppose G is finitely generated etc. Let $C = C_G(A)$. Then G/C embeds in $GL(n, \boldsymbol{Q})$, for some finite n, and is therefore residually finite. The result now follows as for part (*a*).

Corollary 6 will provide a considerable improvement to part (*b*) above, but first we require a rather technical result, which will also be of use when we are discussing images by subgroups of the FC-centre.

PROPOSITION 5. Let G be a finitely generated, locally graded group and suppose that H is a normal subgroup of G such that G/H is nontrivial and has no nontrivial finite images. Suppose further that H is the union of an ascending series of G-invariant subgroups K_λ such that each G/K_λ is locally graded. Then G has a homomorphic image $\bar{G} = G/N$ satisfying the following properties.

(i) $\bar{G}$ is residually finite.

(ii) $\bar{H} = HN/N$ is the direct product of infinitely many finite simple groups $\bar{A}_1, \bar{A}_2, \ldots$, each normal in $\bar{G}$.

(iii) There is a descending series $G = N_0 > N_1 > N_2 > \ldots$ of normal subgroups N_j of G such that $\bigcap_{j=0}^{\infty} N_j = N$ and, for each $j = 1, 2, \ldots, N_{j-1}/N_j$ is isomorphic to $\bar{A}_j$ and $\bar{G} = \bar{A}_1 \times \ldots \times \bar{A}_j \times \bar{N}_j$.

(iv) $\bar{G}/\bar{H}$ is nontrivial (and has no nontrivial finite images).

PROOF. Let G and H be as given. Then G is countable and $H \neq K_\lambda$, for any λ. By relabelling if necessary, we may assume that $H = \bigcup_{i=1}^{\infty} K_i$. Write $G = N_0$ and choose N_1 to be any maximal normal subgroup of G such that G/N_1 is finite—such exists since G is finitely generated and locally graded (and $G \neq 1$). By hypothesis, we have $G = N_1 H$ and hence $G = N_1 H_1$, where $H_1 = K_{i_1}$, for some i_1. Using bars (temporarily) to denote factor groups modulo $N_1 \cap H_1$, we thus have $\bar{G} = \bar{H}_1 \times \bar{N}_1$, where $\bar{H}_1$ is finite simple. Next, let N_2 be a maximal normal subgroup of N_1 such that $N_1 \cap H_1 \leqslant N_2$ and N_1/N_2 is finite. (Here we have used the fact that G/H_1 is locally graded). Then we may write $G = N_2 H_2$, where $H_2 = K_{i_2}$, say. Certainly $H_1 < H_2$, else $N_1 = N_1 \cap N_2 H_1 = N_2$, a contradiction. Note also that $N_1 \cap H_1 \leqslant N_2 \cap H_2$ and $H_2 = H_1(N_1 \cap H_2)$. Using bars this time to denote factor groups modulo $N_2 \cap H_2$, we have $\bar{G} = \bar{H}_2 \times \bar{N}_2 = \bar{H}_1 \times \overline{N_1 \cap H_2} \times \bar{N}_2$ and $\overline{N_1 \cap H_2} \cong N_1/N_2$, which is finite simple. Assume that, for some $j \geqslant 2$, we have chains of subgroups $1 = H_0 < H_1 < \ldots < H_j$, $G = N_0 \rhd N_1 \rhd \ldots N_j$, where, for each $t = 1, \ldots, j$:

(a) $H_t = K_{i_t}$, for some i_t.

(b) N_{t-1}/N_t, is finite simple.

(c) $G = N_t H_t$ and $N_t \lhd G$.

(d) $N_{t-1} \cap H_{t-1} \leqslant N_t$

and such that, if bars denote factor groups modulo $N_j \cap H_j$, we also have

(e) $\bar{G} = \bar{H}_1 \times \overline{N_1 \cap H_2} \times \ldots \times \overline{N_{j-1} \cap H_j} \times \bar{N}_j$.

(Note that

$$\overline{N_{t-1} \cap H_t} \cong (N_{t-1} \cap H_t)/(N_j \cap H_t) \cong (N_{t-1} \cap H_t)N_t/N_t = N_{t-1}/N_t,$$

for each t, using the inclusions $N_1 \cap H_1 \leqslant N_2 \cap H_2 \leqslant \dots$.) Choose N_{j+1} to be a maximal normal subgroup of N_j such that $N_j \cap H_j \leqslant N_{j+1}$ and N_j/N_{j+1} is finite. Then $G = N_{j+1}H_{j+1}$, where $H_{j+1} = K_{i_{j+1}}$, say. As before, we see that $H_j < H_{j+1}$ and, using bars to denote factor groups modulo $N_{j+1} \cap H_{j+1}$, we have

$$\bar{G} = \bar{H}_1 \times \overline{N_1 \cap H_2} \times \dots \times \overline{N_j \cap H_{j+1}} \times \overline{N_{j+1}}.$$

Also, N_j/N_{j+1} is finite simple and $N_{j+1} \lhd G$. Inductively, then, we have an infinite sequence $\{H_i\}$ of subgroups whose union is H and an infinite descending chain $G = N_0 > N_1 > N_2 > \dots$ such that properties (a) to (e) are satisfied at each stage. Write $N = \bigcap_{j=0}^{\infty} N_j$. Then G/N is finitely generated, infinite and residually finite. Also, for each $j \geqslant 0$ we have $N_j \cap H_j \leqslant N_{j+1} \cap H_{j+1} \leqslant \dots$ and so $N_j \cap H_j \leqslant N$, that is, $N_j \cap H_j = N \cap H_j$. Now write $A_j = N_{j-1} \cap H_j$, for each $j \geqslant 1$. Then $A_jN/N \cong (N_{j-1} \cap H_j)/(N_j \cap H_j) \cong N_{j-1}/N_j$, which is finite simple. Further, $H_j = A_jA_{j-1} \dots A_1$ and so $H = \langle A_j \colon j = 1, 2, \dots \rangle$. Modulo N, H is in fact the direct product of the A_j. For, supposing this is not the case, there exist j, k with $k > j$ such that $A_j \leqslant NA_kA_{k-1} \dots \widehat{A_j} \dots A_1$, where the Λ denotes that the factor A_j is omitted. Then $A_j \leqslant$ $\leqslant N_kA_kA_{k-1} \dots \widehat{A_j} \dots A_1$ and, using $N_tA_t = N_{t-1}$ (for each t), we obtain $A_j \leqslant N_jA_{j-1} \dots A_1 = B_j$, say. But, modulo $N_j \cap H_j$, G is the direct product of A_j and B_j, and we have $A_j \leqslant N_j$ and thus $N_j = N_{j-1}$, a contradiction.

To conclude the proof of the theorem, we need only show that $G \neq HN$. But this follows immediately from the fact that HN/N is not finitely generated.

The first consequence of the proposition that we note is the following which improves on Theorem 4.

Corollary 6. Let G be a locally graded group and H a normal subgroup of G. Suppose that H has a G-invariant, ascending series of subgroups with factors which are abelian and of finite torsion-free rank. Then G/H is locally graded.

PROOF. We may assume that G is finitely generated and argue as in the proof of Theorem 2 (using Theorem 4(b) in place of Lemma 1) to reduce to the case where H is a union of G-invariant subgroups K_λ, where each G/K_λ is locally graded and G/H has no nontrivial finite images (but $G \neq H$). By Proposition 5, we may suppose that H is a direct product of G-invariant, finite simple groups. But this implies that H is abelian and periodic, and Lemma 3 may now be used to obtain a contradiction.

In view of Lemma 1, it is reasonable to ask whether one may factor by a subgroup H of the FC-centre and retain local gradedness. It is easy to see that all is well if H is finite (see below) and it is evident that Proposition 5 could be useful in our investigations. Let us make the following elementary observation.

LEMMA 7. Let G be a locally graded group and H a finitely generated, G-invariant subgroup of the FC-centre of G. Then G/H is locally graded.

PROOF. Since a finitely generated FC-group is centre-by-finite, we may apply Theorem 4(a) to reduce to the case where H is finite. It suffices to show that CH/H is locally graded, where $C = C_G(H)$. But this follows from Lemma 1.

Now suppose that G is a locally graded group and that H is a G-invariant subgroup of the FC-centre of G. In order to try to establish that G/H is locally graded we may assume that G is finitely generated (etc.) and apply Lemma 7 and Proposition 5 to reduce to the case where H is a direct product of G-invariant, finite simple groups A_i. Assuming that G/H has no finite images, we see that each abelian A_i is central in G and thus, by Lemma 1, may be assumed trivial. Since G is finitely generated, it cannot be mapped homomorphically onto S^n, all n, for any finite simple group S [2]. Thus there are infinitely many isomorphism types among the A_i. Let K denote the direct product of any infinite set of pairwise nonisomorphic A_i's. Write $H = K \times L$ and let $C = C_G(K)$. Then G/C is residually finite and C contains L. Also $KC/C \cong K$ and so we may factor by C and retain our hypotheses on G. (Note that $G \neq KC$, since K is not finitely generated.) We may as well assume that $K = H$. Now suppose $N \lhd G$ and G/N is finite and nontrivial. Then $G = NH$ and we may choose $J = A_{i_1} \times \ldots \times A_{i_k}$ (say) of minimal order subject to $G = NJ$. Thus $N \cap J = 1$. We see that our proposed counterexample G is residually (finite non-abelian simple). Finally, G

must be perfect, since every finite nontrivial G/N is isomorphic to some J as above. Our considerations have led us to the following problem.

QUESTION 1. *Does there exist a finitely generated (perfect) subgroup G of the cartesian product of infinitely many, distinct, nonabelian, finite simple groups A_i such that G contains the direct product H of the A_i and G/H has no nontrivial finite images?*

A construction due to B. H. Neumann [3] provides us with a finitely generated subgroup G of the cartesian product of infinitely many (distinct) finite alternating groups such that G contains their direct product H. In this case, G/H is isomorphic to an extension of the (finitary) alternating group on $\aleph_0$ symbols by an infinite cyclic group. Based on this construction is an example due to James Wiegold (unpublished) in which the resulting image G/H is perfect (but has many finite images). There seems some hope that Question 1 may have an affirmative answer.

Before posing our second problem, we may as well pause to salvage something positive from our deliberations on subgroups of the FC-centre. Indeed, appealing to Proposition 5 once more and arguing as in the proof of Corollary 6, we have the following.

THEOREM 8. Let G be a locally graded group and H a G-invariant subgroup of the FC-hypercentre of G. If H has no section isomorphic to the direct product of infinitely many, nonabelian, finite simple groups then G/H is locally graded. (In particular, the conclusion of Theorem 8 holds if H has no infinite elementary abelian 2-sections!)

Our second problem is concerned with abelian subgroups. The question as to whether factoring by an arbitrary abelian normal subgroup retains local gradedness is easily reduced to the following, in view of Lemma 3.

QUESTION 2. *Does there exist a finitely generated, locally graded group G which contains a normal, torsionfree abelian subgroup A such that G/A is nontrivial but has no nontrivial finite images?*

Any group G satisfying the above conditions cannot, of course, be residually finite. We conclude with the obvious remark that it is because the class of locally graded groups is so extensive that counterexamples may be expected (in general) to be elusive.

REFERENCES

[1] B. BRUNO - R. E. PHILLIPS, *On minimal conditions related to Miller-Moreno type groups*, Rend. Sem. Mat. Padova, **69** (1983), pp. 153-168.
[2] P. HALL, *The Eulerian functions of a group*, Quart. J. Math. (Oxford), **8** (1936), pp. 134-151.
[3] B. H. NEUMANN, *Some remarks on infinite groups*, J. London Math. Soc., **12** (1937), pp. 120-127.

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