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Infinitely Many Spacelike Periodic Trajectories on a Class of Lorentz Manifolds.

CARLO GRECO (*)

ABSTRACT - Let us consider $\boldsymbol{R}^4$ equipped with a Lorentzian tensor g with signature $(+,+,+,-)$. In this paper we prove, under suitable assumptions on g, the existence of infinitely many spacelike geodesics $z(s) = (x(s), t(s))$ with the periodicity conditions $x(s+1) = x(s)$, $t(s+1) = t(s) + T$ $(T > 0)$ on the Lorentz manifold $(\boldsymbol{R}^4, g)$.

1. Introduction.

Let us consider the manifold $(\boldsymbol{R}^4, g)$, where $g(z) = g(x, t)$ is a Lorentz tensor on $\boldsymbol{R}^4$, with signature $(+,+,+,-)$. Let $z(s) = (x(s), t(s))$ be a geodesic on $(\boldsymbol{R}^4, g)$, and suppose that $t(0) = 0$, and there exist σ, $T > 0$ such that $x(s+\sigma) = x(s)$, $t(s+\sigma) = t(s) + T$ for every $s \in \boldsymbol{R}$. Then we shall say that z is a σ-periodic T-trajectory on $(\boldsymbol{R}^4, g)$. Moreover, if z is a geodesic, there exists $E_z \in \boldsymbol{R}$ such that $g(z(s))[\dot{z}(s), \dot{z}(s)] \equiv E_z$, and z called spacelike, null or timelike if $E_z > 0$ or, respectively, $E_z = 0$, $E_z < 0$ (see [14], p. 69).

Suitable Lorentz manifolds are used in Relativity theory in order to describe the physical space-time. Then, timelike (or, respectively, null) periodic trajectories corresponds to periodics orbits of a particle of positive mass (or, respectively, of a light ray). Spacelike geodesics are not trajectories of particles, but they are important in order to study geometrical properties of a semiriemannian manifold.

Some multiplicity results for timelike periodic trajectories on $(\boldsymbol{R}^4, g)$ are given, for instance, in [5] and [9] under the assumption that

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the gravitational field vanish at infinity, so that g tends to the Minkowski metric at infinity (see Remark 1.3 below for further informations).

In this paper we consider a completely different behavior at infinity for g, and we are able to prove that, for any $T > 0$, there are infinitely many spacelike 1-periodic T-trajectories on the semiriemannian manifold $(\boldsymbol{R}^4, g)$.

Let $\{g_{ij}\}_{i,j=1,\dots,4}$ be the components of g. We suppose that g not depend to the time, $g_{ij} = g_{ji} \in C^1(\boldsymbol{R}^3, \boldsymbol{R})$, and $g_{i4} = 0$ for $i = 1, 2, 3$. We set, for simplicity, $\alpha = \{\alpha_{ij}\}_{i,j=1,2,3} = \{g_{ij}\}_{i,j=1,2,3}$, and $\beta = -g_{44}$, so that we have, for every $x \in \boldsymbol{R}^3$ and every $\begin{pmatrix}\xi\\ \tau\end{pmatrix} \in \boldsymbol{R}^4$:

$$g(x)\left[\begin{pmatrix}\xi\\ \tau\end{pmatrix}, \begin{pmatrix}\xi\\ \tau\end{pmatrix}\right] = \alpha(x)[\xi, \xi] - \beta(x)\,\tau^2 .$$

Moreover we assume that there exist α_0, α_1, $R > 0$, $p > 2$ and $q \in]0, p-2[$ such that for every x, $\xi \in \boldsymbol{R}^3$:

$$\alpha(x)[\xi, \xi] \geqslant \alpha_0 \, |\xi|^2 , \tag{1.1}$$

$$(q\alpha(x) - \alpha'(x)(x))[\xi, \xi] \geqslant \alpha_1 \, |\xi|^2 , \tag{1.2}$$

$$p\beta(x) \leqslant (\beta'(x)|x) \quad \text{if } |x| \geqslant R , \tag{1.3}$$

$$0 < \beta_0 \equiv \beta(0) = \min_{\boldsymbol{R}^3} \beta , \tag{1.4}$$

$$\lim_{|x| \to 0} \frac{\beta(x) - \beta_0}{|x|^2} = 0 , \tag{1.5}$$

$$\alpha(x) = \alpha(-x), \qquad \beta(x) = \beta(-x). \tag{1.6}$$

Then we have the following theorem.

THEOREM 1.1. *If* (1.1)-(1.6) *are satisfied, then, for every* $T > 0$, *there exist infinitely many spacelike* 1*-periodic* T*-trajectories on* $(\boldsymbol{R}^4, g)$.

REMARK 1.1. If $x_0 \in \boldsymbol{R}^3$ and $\beta'(x_0) = 0$, it is easy to check that $z(s) = (x_0, Ts)$ is a trivial periodic trajectory. We shall see later that the trajectories given by Theorem 1.1 are not trivial, and are geometrically distinct.

REMARK 1.2. Condition (1.3) is a sort of superquadraticity condition at infinity. It has been introduced by P. H. Rabinowitz in the theory of Hamiltonian systems. (1.4) implies that there exists $c_1 > 0$ such that, for every $x \in \boldsymbol{R}^3$, with $|x| \geqslant R$:

$$\beta(x) \geqslant c_1 |x|^p . \tag{1.7}$$

Condition (1.3) means that $\sum_{i,j=1}^{3} [q\alpha_{ij}(x) - (\alpha'_{ij}(x)|x)]\xi_i \xi_j \geqslant \alpha_1 |\xi|^2$; it is satisfied, for instance, if $\alpha(x) = \{\delta_{ij}\}_{i,j=1,2,3}$. Moreover, because of (1.3), there exists $c_2 > 0$ such that

$$\|\alpha(x)\| \leqslant c_2 |x|^q \tag{1.8}$$

for $|x| \geqslant 1$. Infact, let $x \in \boldsymbol{R}^3$ with $|x| \geqslant 1$. Since

$$d(t^{-q}\alpha(tx/|x|)[\xi, \xi])/dt \leqslant 0 ,$$

we have

$$|x|^{-q}\alpha(x)[\xi, \xi] \leqslant \alpha(x/|x|)[\xi, \xi] \leqslant c_2 |\xi|^2 \quad \text{where } c_2 = \max_{|y|=1} \|\alpha(y)\| .$$

REMARK 1.3. The problem of geodesics for a Lorentz manifold (M, g) has been recently studied by many authors (see [2]-[5], [7]-[12]). If particular, in the papers [5], [9], are given multiplicity results for timelike periodic trajectories on $(\boldsymbol{R}^4, g)$ under the assumption $\beta(x)$ bounded.

The main difficult in the variational approach of this kind of problems is that the action functional

$$\int g(z)[\dot{z}, \dot{z}] = \int \alpha(x)[\dot{x}, \dot{x}] - \int \beta(x)\dot{t}^2$$

is strongly indefinite, i.e. it is not of the form identity + compact, even «modulo compact perturbations». In ordert to avoid this difficult, we use the convexity of the functional with respecto to $\dot{t}$ and search for the critical points of a functional f depending only on x.

If $\beta(x)$ is bounded as in [9] (or it is subquadratic), the functional f is bounded from below, and satisfies easily the Palais-Smale compactness condition. In our case f is unbounded, so we need some linking argument; moreover more care is required in order to prove compactness conditions.

In Section 2 we expose the functional framework and we prove the compactness condition using assumptions (1.1)-(1.5). Then we prove Theorem 1.1 with a mountain pass argument by using (1.6).

2. Proof of the results.

In the following we assume that (1.1)-(1.5) hold. Let as consider a geodesic $z(s) = (x(s), t(s))$ on $(\boldsymbol{R}^4, g)$; then z satisfies the geodesic equations:

$$\frac{d}{ds}[\alpha(x)\dot{x}] = \frac{1}{2}(\alpha'(x)[\dot{x}, \dot{x}] - \beta'(x)\dot{t}^2),$$

$$\frac{d}{ds}[\beta(x)\dot{t}] = 0.$$

If z is a σ-periodic T-trajectory, we shall call the minimal period of x, the minimal period of z. Notice that, if $z_1 = (x_1, t_1)$ and $z_2 = (x_2, t_2)$ are σ-periodic T-trajectories on $(\boldsymbol{R}^4, g)$, with $z_1 \neq z_2$, then z_1 and z_2 are geometrically distinct.

In fact, if $z_2(s) = z_1(\varphi(s))$ for some reparametrization $\varphi(s)$, from geodesic equations we have $\varphi(s) = as + b$ for some $a, b \in \boldsymbol{R}$ (see [14], p. 69), so that $t_2(s) = t_1(as + b)$. Since $\dot{t}_1(s) \neq 0$ for any $s \in \boldsymbol{R}$, from $t_1(0) = = 0 = t_2(0) = t_1(b)$, we have $b = 0$, and from $t_1(as + a\sigma) = t_2(s + \sigma) = = t_2(s) + T = t_1(as) + T = t_1(as + \sigma)$, we have $a\sigma = \sigma$ and $a = 1$, which is impossible.

In particular, if z_1 and z_2 have not the same minimal period, then its are geometrically distinct.

REMARK 2.1. We observe now that, if $z(s) = (x(s), t(s))$ is a k^{-1}-periodic Tk^{-1}-trajectory, x and $\dot{t}$ are also 1-periodic and $t(s+1) = = t(s) + T$. Infact, it is easy to chek that $t(s+1) = t(s + (k-h)/h) + + Th/k$ for every $h = 1, \dots, k$; then z is a 1-periodic T-trajectory on $(\boldsymbol{R}^4, g)$, with minimal period less or equal to $1/k$. So, in order to prove Theorem 1.1, we can show that there exists $k_0 \in \boldsymbol{N}$ such that, for every $k \in \boldsymbol{N}$ with $k \geq k_0$, there exists a k^{-1}-periodic Tk^{-1}-trajectory $z(s) = = (x(s), t(s))$, with $\dot{x} \neq 0$.

Let $k \in \boldsymbol{N}$ be free for the moment, and let us consider the functional

$$I(x, \eta) = \int_0^{1/k} \alpha(x)[\dot{x}, \dot{x}]\, ds - \int_0^{1/k} \beta(x)(T/k + \eta)^2\, ds,$$

defined on $H^{1,2}(S^{1/k}, \boldsymbol{R}^3) \times L_0(S^{1/k}, \boldsymbol{R})$, where $H^{1,2}(S^{1/k}, \boldsymbol{R}^3)$ is the Sobolev space of k^{-1}-periodic functions $x\colon \boldsymbol{R} \to \boldsymbol{R}^3$ with

$x, \dot{x} \in L^2([0, 1/k])$, and

$$L_0(S^{1/k}, \boldsymbol{R}) = \left\{ \eta \in L^2(S^{1/k}, \boldsymbol{R}) \mid \int_0^{1/k} \eta \, ds = 0 \right\}.$$

It is easy to check that, if (x, η) is a critical point of I, then $z(s) = (x(s), t(s))$, where $t(s) = Ts/k + \int_0^s \eta \, ds$ is a critical point of the action functional

$$\int_0^{1/k} \alpha(x)[\dot{x}, \dot{x}] \, ds - \int_0^{1/k} \beta(x) \dot{t}^2 \, ds ;$$

so, it is a 1-periodic T-trajectory on $(\boldsymbol{R}^4, g)$, with minimal period less or equal to $1/k$ (see Remark 2.1).

Notice that, because of (1.4), for every $x \in H \equiv H^{1,2}(S^{1/k}, \boldsymbol{R}^3)$, the functional $\eta \mapsto \int_0^{1/k} \beta(x)(T/k + \eta)^2 \, ds$ is strictly convex, so it possess a unique minimum point $\eta_x \in L_0(S^{1/k}, \boldsymbol{R})$. Let $f: H \to \boldsymbol{R}$ be the functional

$$f(x) = \int_0^{1/k} \alpha(x)[\dot{x}, \dot{x}] \, ds - \int_0^{1/k} \beta(x)(T/k + \eta_x)^2 \, ds + \frac{\beta_0 T^2}{k^3} .$$

LEMMA 2.2. *The function $x \mapsto \eta_x$ is continuous from H to $L_0(S^{1/k}, \boldsymbol{R})$; moreover $f \in C^1(H, \boldsymbol{R})$ and*

$$\langle f'(x), y \rangle = \left\langle \frac{\partial I}{\partial x}(x, \eta_x), y \right\rangle,$$

so that, $x \in H$ is a critical point of f if and only if (x, η_x) is a critical point of I.

PROOF. The proof is contained in [9]. We recall it for the reader convenience. First of all we observe that $\int_0^{1/k} \beta(x)(T/k + \eta_x)\eta_x \, ds = 0$, because of η_x is a critical point of the functional $\eta \mapsto \int_0^{1/k} \beta(x)(T/k + \eta)^2 \, ds$.

So $\int_0^{1/k} \beta(x)\eta_x^2 ds = -(T/k)\int_0^{1/k} \beta(x)\eta_x ds$, and then

$$\|\eta_x\| \leqslant \frac{T\|\beta(x)\|_\infty}{k\beta_0}. \tag{2.1}$$

Now, let $x, y \in H$. Clearly

$$I(x, \eta_y) - I(y, \eta_y) \leqslant f(x) - f(y) \leqslant I(x, \eta_x) - I(y, \eta_x), \tag{2.2}$$

and $I(x, \eta_x) - I(y, \eta_x) \to 0$ as $y \to x$. Moreover, since

$$I(x, \eta_y) - I(y, \eta_y) =$$

$$= \int_0^{1/k} \alpha(x)[\dot{x}, \dot{x}] - \alpha(y)[\dot{y}, \dot{y}]\, ds - \int_0^{1/k} (\beta(x) - \beta(y))(T/k + \eta_y)^2 ds,$$

using (2.1) we get $I(x, \eta_y) - I(y, \eta_y) \to 0$ as $y \to x$, so f is continuous.

We prove now that $x \mapsto \eta_x$ is continuous. Infact, arguing by contradiction, we suppose that there exist $x \in H$, $(x_n) \subset H$ and $\varepsilon > 0$ such that $x_n \to x$ and $\|\eta_x - \eta_{x_n}\| \geqslant \varepsilon$. Since $\int_0^{1/k} \beta(x)(T/k + \eta)^2 ds$ is strictly convex, we have

$$\sup\{I(x, \eta) \mid \eta \in L_0(S^{1/k}, \boldsymbol{R}), \|\eta - \eta_x\| = \varepsilon/2\} \leqslant I(x, \eta_x) - \delta$$

for some $\delta > 0$. Let $\mu_n \in \partial B(\eta_x, \varepsilon/2) \cap \{\eta_x + \lambda(\eta_{x_n} - \eta_x) \mid \lambda \in [0, 1]\}$; since $I(x_n, \cdot)$ is concave, we have $I(x_n, \mu_n) \geqslant I(x_n, \eta_x)$, so that

$$I(x, \eta_x) - \delta \geqslant I(x, \mu_n) = I(x, \mu_n) -$$

$$- I(x_n, \mu_n) + I(x_n, \mu_n) \geqslant I(x, \mu_n) - I(x_n, \mu_n) + I(x_n, \eta_x).$$

Since (μ_n) is bounded and $x_n \to x$, we get $I(x, \mu_n) - I(x_n, \mu_n) \to 0$, and $I(x_n, \eta_x) \to I(x, \eta_x)$, and then we have a contradiction.

Finally, fix $x, y \in H$, and let $\tau > 0$. From (2.2) we have

$$\frac{I(x + \tau y, \eta_x) - I(x, \eta_x)}{\tau} \leqslant$$

$$\leqslant \frac{f(x + \tau y) - f(x)}{\tau} \leqslant \frac{I(x + \tau y, \eta_{x+\tau y}) - I(x, \eta_{x+\tau y})}{\tau}.$$

For $\tau \to 0$ we get $\langle f'(x), y\rangle = \langle \partial I(x, \eta_x)/\partial x, y\rangle$, so the lemma is proved. ■

REMARK 2.3. Notice that $\int_0^{1/k} \beta(x)(T/k + \eta_x)\eta\, ds = 0$ for every

$\eta \in L_0(S^{1/k}, \boldsymbol{R})$. In other words, there exists $c_x \in \boldsymbol{R}$ such that $\beta(x(s))(T/k + \eta_x(s)) = c_x$ for every $s \in \boldsymbol{R}$. Since $c_x \leq 0$ implies $T/k + \eta_x(s) \leq 0$, so $T/k^2 = \int_0^{1/k} (T/k + \eta_x)\, ds \leq 0$, we have $c_x > 0$, and then $T/k + \eta_x(s) > 0$ for every s. Moreover $\beta(x)(T/k + \eta_x)^2 = c_x(T/k + \eta_x)$, so $c_x = (k^2/T) \int_0^{1/k} \beta(x)(T/k + \eta_x)^2\, ds$.

LEMMA 2.4. *Fix* $\rho > 0$ *and* $x \in H$, *and set* $I = \{s \in [0, 1/k] \,|\, |x(s)| \leq \rho\}$. *Then, if* $|I| > 0$,

$$\int_0^{1/k} \beta(x)(T/k + \eta_x)^2\, ds \leq \frac{T^2 M}{k^4 |I|},$$

where $M = \max\{\beta(x) \,|\, |x| \leq \rho\}$, *and* $|I|$ *is the Lebesgue measure of* I.

PROOF. Let c_x be as in Remark 2.3, so that $T/k + \eta_x(s) = c_x/\beta(x(s))$ for every $s \in \boldsymbol{R}$. If $s \in I$, we have $c_x/M \leq T/k + \eta_x(s)$, and then $c_x^2/M \leq \beta(x(s))(T/k + \eta_x(s))^2$.

Integrating on I, we have:

$$\frac{c_x^2}{M} |I| \leq \int_0^{1/k} \beta(x)(T/k + \eta_x)^2\, ds = \frac{Tc_x}{k^2}.$$

Then $c_x \leq TM/k^2 |I|$, so that the lemma is proved. ∎

LEMMA 2.5. *Let* $0 < r < \rho$ *and* $(x_n) \subset H$ *be such that* $\operatorname{dist}(\operatorname{Im}(x_n), 0) \leq r$ *and* $\|x_n\|_\infty \geq \rho$. *Then*

$$\int_0^{1/k} \beta(x_n)(T/k + \eta_{x_n})^2\, ds \leq \frac{T^2 M}{k^4 (\rho - r)^2} \|\dot{x}_n\|_2^2,$$

where $M = \max\{\beta(x) \,|\, |x| \leq \rho\}$.

PROOF. Let $I_n = \{s \in [0, 1/k] \,|\, |x_n(s)| \leq \rho\}$; since $\|x_n\|_\infty \geq \rho$, $|I_n| > 0$, so that

$$\int_0^{1/k} \beta(x_n)(T/k + \eta_{x_n})^2\, ds \leq T^2 M/k^4 |I_n|$$

because of Lemma 2.4. Moreover, since $\mathrm{dist}(\mathrm{Im}(x_n), 0) \leq r$, we have $\rho - r \leq \int_{I_n} |\dot{x}_n|\, ds \leq \|\dot{x}_n\|_2 \, |I_n|^{1/2}$, and the lemma follows. ■

We say that a functional $f\colon H \to \boldsymbol{R}$ verifies the Palais-Smale-Cerami (PSC) condition (see [6]) if every sequence $(x_n) \subset H$ such that $f(x_n) \to \to c \in \boldsymbol{R}$ and $\langle f'(x_n), x_n \rangle \to 0$ as $n \to \infty$, possesses a convergent subsequence.

We have the following lemma.

LEMMA 2.6. *There exists $k_0 \in \boldsymbol{N}$ such that, for every $k \geq k_0$, the functional f satisfies the* PSC-*condition.*

PROOF. Let $M = \max\{\beta(x) \mid |x| \leq R + 1\}$ (R is defined in (1.3)), and let $k_0 \in \boldsymbol{N}$ be such that $\alpha_0 - T^2 M / k_0^4 > 0$. Fix $k \in \boldsymbol{N}$ with $k \geq k_0$, and let us consider a sequence $(x_n) \subset H$ such that $f(x_n) \to c \in \boldsymbol{R}$ and $\langle f'(x_n), x_n \rangle \to 0$ as $n \to \infty$. First of all, we prove that $(\|\dot{x}_n\|_2)$ is bounded modulo subsequences. Infact, we distinguish two cases:

1) case: for every $n \in \boldsymbol{N}$, $\mathrm{dist}(\mathrm{Im}(x_n), 0) > R$ (modulo subsequences). Then $p\beta(x_n(s)) \leq (\beta'(x_n(s)) \mid x_n(s))$ for every s (see (1.3)), so, from $f(x_n) \to c$ we get (setting $\eta_n \equiv \eta_{x_n}$):

$$p \int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x}_n]\, ds \leq pc + \int_0^{1/k} (\beta'(x_n) \mid x_n)(T/k + \eta_n)^2 ds + o(1).$$

Since $\langle f'(x_n), x_n \rangle \to 0$, we have

$$\int_0^{1/k} \alpha'(x_n)(x_n)[\dot{x}_n, \dot{x}_n]\, ds + 2 \int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x}_n]\, ds - $$

$$- \int_0^{1/k} (\beta'(x_n) \mid x_n) \left(\frac{T}{k} + \eta_n \right)^2 ds = o(1),$$

so that

$$\int_0^{1/k} (q\alpha(x_n) - \alpha'(x_n)(x_n))[\dot{x}_n, \dot{x}_n]\, ds \leq pc + o(1),$$

then $(\|\dot{x}_n\|_2)$ is bounded because of (1.2).

2) case: for every $n \in \boldsymbol{N}$, $\operatorname{dist}(\operatorname{Im}(x_n), 0) \leq R$ (modulo subsequences). Then, if $(\|x_n\|_\infty)$ is bounded, we have $\beta(x_n(s)) \leq M_1$ for $n \in \boldsymbol{N}$, $s \in \boldsymbol{R}$, so $\int_0^{1/k} \beta(x_n)(T/k + \eta_n)^2 ds \leq M_1 T^2/k^3$, and the claim follows from the fact that $f(x_n) \to c$ as $n \to \infty$. So, we can assume $\|x_n\|_\infty \to \infty$. Let $I_n = \{s \in [0, 1/k] \mid |x_n(s)| \leq R + 1\}$; from Lemma 2.5 (with $r = R$ and $\rho = R + 1$), we have

$$\int_0^{1/k} \beta(x_n)(T/k + \eta_n)^2 ds \leq \frac{T^2 M}{k^4} \|\dot{x}_n\|_2^2 .$$

Then, since $f(x_n) \to c$,

$$\int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x}_n] ds \leq \frac{T^2 M}{k^4} \|\dot{x}_n\|_2^2 + c + o(1),$$

so that (see (1.1)): $(\alpha_0 - T^2 M/k^4)\|\dot{x}_n\|_2^2 \leq c + o(1)$. Since $k \geq k_0$, the claim follows.

We set now $x_n = \xi_n + y_n$, where $\xi_n \in \boldsymbol{R}^3$, and $\int_0^{1/k} y_n(s) ds = 0$; we shall prove that (ξ_n) is bounded. In fact, we can assume that $y_n \to y$ weakly in $H^{1,2}$ and strongly in L^∞; then

$$|\xi_n| - (\|y\|_\infty + 1) \leq |x_n(s)| \leq |\xi_n| + (\|y\|_\infty + 1)$$

for n large enough, so that, since

$$\alpha(x_n(s))[\dot{x}_n(s), \dot{x}_n(s)] \leq c_2 |x_n(s)|^q |\dot{x}_n(s)|^2$$

(see (1.8)), we have $\int_0^{1/k} \alpha(x_n)[\dot{x}_x, \dot{x}_n] ds \leq c_3 |\xi_n|^q + c_4$ for some $c_3, c_4 > 0$. On the other hand, $\beta(x_n(s)) \geq c_1 |x_n(s)|^p$, then $\int_0^{1/k} \beta(x_n)(T/k + \eta_n)^2 ds \geq$ $\geq c_5 |\xi_n|^p + c_6$. Since $f(x_n) \to c$, we have

$$c_5 |\xi_n|^p + c_6 \leq \int_0^{1/k} \beta(x_n)(T/k + \eta_n)^2 ds =$$

$$= \int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x}_n] ds - c + o(1) \leq c_3 |\xi_n|^q + c_4 + c + o(1),$$

so (ξ_n) is bounded. Let us suppose $x_n \to x$ weakly in $H^{1,2}$ and strongly L^∞. Then

$$\langle f'(x_n), x - x_n \rangle = \int_0^{1/k} \alpha'(x_n)(x - x_n)[\dot{x}_n, \dot{x}_n]\, ds +$$

$$+2 \int_0^{1/k} \alpha(x_n)[\dot{x}_n, \dot{x} - \dot{x}_n]\, ds - \int_0^{1/k} (\beta'(x_n) | x - x_n)(T/k + \eta_n)^2\, ds\,;$$

because of (2.1) we have that (η_n) is bounded, so, the fact that $\langle f'(x_n), x - x_n \rangle = o(1)$ implies $\int_0^{1/k} \alpha(x_n)[\dot{x}_n \dot{x} - \dot{x}_n]\, ds = o(1)$. Then $\int_0^{1/k} |\dot{x} - \dot{x}_n|^2 ds \leqslant \alpha_0^{-1} \int_0^{1/k} \alpha(x_n)[\dot{x} - \dot{x}_n, \dot{x} - \dot{x}_n]\, ds = o(1)$, so that $x_n \to x$ strongly in H, and the lemma is proved. ∎

Let $H = H^{1,2}(S^{1/k}, \boldsymbol{R}^3) = \boldsymbol{R}^3 \times Y$, where

$$Y = \left\{ x \in H \,|\, \int_0^{1/k} x(s)\, ds = 0 \right\}.$$

As well-known (see e.g. [13], p. 9), for every $y \in Y$ we have $\|\dot{y}\|_2 \geqslant a\|y\|$, and $\|y\|_\infty \leqslant b\|\dot{y}\|_2$, where $a = 2k\pi(1 + 4k^2\pi^2)^{-1/2}$, and $b = (1/12k)^{1/2}$.

We have now the following lemma.

Lemma 2.7. *There exist $\delta, \rho > 0$ such that $f(y) \geqslant \delta$ for every $y \in Y$ with $\|y\| = \rho$. Moreover δ is independent of k.*

Proof. Fix $\varepsilon > 0$ such that $\alpha_0 - \varepsilon T^2/\sqrt{12} > 0$. (1.5) implies that there exists $\rho_1 > 0$ such that $\beta(x) \leqslant \beta_0 + \varepsilon|x|^2$ for $|x| \leqslant \rho_1$. Set $\rho = \rho_1/b$ and

$$\delta = \frac{4\pi^2}{1 + 4\pi^2}\left(\alpha_0 - \frac{\varepsilon T^2}{12}\right)\rho_1^2 12\,.$$

For $y \in Y$ with $\|y\| = \rho$, we have $\|y\|_\infty \leqslant b\|\dot{y}\|_2 \leqslant b\|y\| = b\rho = \rho_1$, so that $\beta(y(s)) \leqslant \beta_0 + \varepsilon|y(s)|^2 \leqslant \beta_0 + \varepsilon b^2\|\dot{y}\|_2^2$. Then

$$\int_0^{1/k} \beta(y)(T/k + \eta_y)^2 ds \leqslant (\beta_0 + \varepsilon b^2\|\dot{y}\|_2^2)\, T^2/k^3\,,$$

so

$$f(y) \geq \alpha_0 \|\dot{y}\|_2^2 - (\beta_0 + \varepsilon b^2 \|\dot{y}\|_2^2) T^2/k^3 + \beta_0 T^2/k^3 = (\alpha_0 - \varepsilon T^2 b^2/k^3) \|\dot{y}\|_2^2 \geq$$

$$\geq (\alpha_0 - \varepsilon T^2 b^2/k^3) a^2 \|y\|^2 = a^2 (\alpha_0 - \varepsilon T^2 b^2/k^3) \rho_1^2/b^2 .$$

Since $a^2(\alpha_0 - \varepsilon T^2 b^2/k^3)\rho_1^2/b^2 > \delta$ for every $k \in N$, the lemma is proved. ■

REMARK 2.8. Lemma 2.5 implies that the functional

$$x \mapsto \int_0^{1/k} \beta(x)(T/k + \eta_x)^2 \, ds ,$$

under assumption (1.3) is not superquadratic at infinity on finite-dimensional subspaces of H. This fact make not possible to apply the standard linking theorem of f. In order to avoid this difficult, we consider the subspace $E = \{x \in H \mid x(s + 1/2k) = -x(s)\}$. Clearly $E \subset Y$; moreover we have the following lemma.

LEMMA 2.9. *Let us suppose that* (1.6) *holds. Then, every critical point $x \in E$ of the functional $f_{|E}$ is a critical point of f.*

PROOF. Let $x \in E$ be a critical point of $f_{|E}$, and $z \in H$; we shall prove that $\langle f'(x), z \rangle = 0$. In fact, set $z_1(s) = z(s) - z(s + 1/2k)$, and $z_2(s) = z(s) - z_1(s)$, so that $z_1 \in E$, and $z = z_1 + z_2$. Since $\langle f'(x), z_1 \rangle = 0$, we have $\langle f'(x), z \rangle = \langle f'(x), z_2 \rangle$. From Remark 2.3, there exists $c_x > 0$ such that $\beta(x(s))(T/k + \eta_x(s)) = c_x$. Since β is even and $x \in E$, we have that $\eta_x(s + 1/2k) = \eta_x(s)$, and then it is easy to check, by using (1.6), that $\langle f'(x), z_2 \rangle = -\langle f'(x), z \rangle$, and the lemma is proved. ■

PROOF OF THEOREM 1.1. Let us suppose that (1.1)-(1.6) hold, let $k_0 \in N$ be as in Lemma 2.6, $\delta > 0$ as in Lemma 2.7, and fix $k \in N$ such that $k \geq k_0$ and $k\delta - \beta_0 T^2/k^2 > 0$. From Lemma 2.6, the functional $f_{|E}$ satisfies the PSC condition on E. Let $w(s) = r(\cos(2k\pi s), \sin(2k\pi s), 0)$; cleary $w \in E$, and since $\beta(w(s)) \geq ar^p + b$, we have

$$\int_0^{1/k} \beta(w)(T/k + \eta_w)^2 \, ds \geq (ar^p + b) T^2/k^3 ,$$

so that (see Remark 1.2) $f(w) \leq 4k\pi^2 c_2 r^{q+2} - (ar^p + b) T^2/k^3 + \beta_0 T^2/k^3$, and $f(w) < 0$ for r large enough (we recall that $q + 2 < p$). Set

$\Gamma = \{\gamma \in C([0,1], E) \mid \gamma(0) = 0, \gamma(1) = w\}$, and let

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} f(\gamma(t)).$$

Let ρ be as in Lemma 2.7; since $f(0) = 0$ and we can assume $\|w\| > \rho$, we have $\delta \leq c < +\infty$. From the mountain pass lemma (see [1]), we have that c is a critical value for the functional $f_{|E}$. From Lemma 2.9 we get a critical point $x \in H$ of f with $f(x) = c$. Since $c > 0$, we have $\dot{x} \neq 0$. Because of Remark 2.1, $z(s) = (x(s), t(s))$, where $t(s) = Ts/k + \int_0^s \eta_x(\tau)\,d\tau$, is a 1-periodic T-trajectory on $(\boldsymbol{R}^4, g)$.

Finally, in order to prove that z is spacelike, we observe that

$$E_z = \int_0^1 \alpha(x)[\dot{x}, \dot{x}]\,ds - \int_0^1 \beta(x)\dot{t}^2\,ds =$$

$$= k\left(\int_0^{1/k} \alpha(x)[\dot{x}, \dot{x}]\,ds - \int_0^{1/k} \beta(x)(T/k + \eta_x)^2\,ds\right) =$$

$$= k\left(f(x) - \frac{\beta_0 T^2}{k^3}\right) \geq k\delta - \frac{\beta_0 T^2}{k^2} > 0,$$

so that $E_z > 0$, and Theorem 1.1 is proved. ■

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