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Definitions of Standard Stress and Standard Heat Flux, in Simple Bodies, Treated According to Mach and Painlevé.

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SOMMARIO - *Nozioni di stress standard e flusso di calore standard, nei corpi semplici, trattati alla Mach-Painlevé.* Il presente lavoro riguarda i fondamenti in fisica classica della teoria termodinamica dei mezzi continui ordinari, e costituisce la naturale estensione sia della Parte 2 di [5] che di [14]. Per generali corpi semplici nel caso puramente meccanico, in [5] le forze di contatto sono definite tramite le forze a distanza e qualche altra nozione cinematica. Lo scopo è raggiunto mediante i seguenti due passi: (i) si dimostra un teorema di unicità stretta per la funzione (ordinaria) dello stesso di un corpo iperelastico, che implicitamente permette di definire le forze di contatto per questi corpi—v. [5, Part 1]; quindi (ii) questa unicità viene estesa al funzionale standard dello stress di un generico corpo semplice, considerando la possibilità di opportuni esperimenti di taglio e contatto di questo con corpi iperelastici—[5, Part 2]. L'articolo [14] estende alla termodinamica [5, Part 1], in quanto in esso sono dimostrati teoremi di unicità per tutte le funzioni di risposta (ordinaria) di un qualunque corpo termoelastico. Il presente lavoro estende [14] al caso termodinamico allo stesso modo in cui, nel caso puramente meccanico [5, Part 2], estende [5, Part 1]. Infatti, qui si definiscono i funzionali standard dello stress e del flusso di calore di un generale corpo semplice, e l'unicità delle funzioni di risposta (ordinaria) di un corpo termoelastico viene estesa a tali

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funzionali postulando la possibilità di opportuni esperimenti di taglio e contatto con corpi termoelastici. Ciò permette di definire le corrispondenti nozioni di *stress standard* e *flusso di calore standard*, per generali corpi semplici, anche in termodinamica.

SUMMARY - This work concerns the foundations of the thermodynamic theory of ordinary continuous media within classical Physics; it constitutes the natural extension of both the papers [5, Part 2] and [14]. In [5] we define contact forces, in the purely mechanical case for general simple bodies, in terms of forces at a distance and some kinematic notions. The aim is attained by two steps: (i) by proving a strict uniqueness theorem for the (ordinary) response function for the stress of a hyper-elastic body, which allows us to implicitly define contact forces for these bodies—see [5, Part 1]; and then (ii) this uniqueness is extended to the standard functional for the stress of any simple body by taking into account the possibility of suitable experiments, which roughly consist in cutting parts of the simple body and putting them in contact with some hyper-elastic bodies—see [5, Part 2]. Paper [14] constitutes the thermodynamic extension of [5, Part 1], in that in it uniqueness theorems are proved for all the (ordinary) response functions of a thermo-elastic body. The present paper extends [14] in the thermodynamic case, in the same way as [5, Part 2] extends [5, Part 1] in the purely mechanical case. Indeed, by defining the standard functionals and by postulating the possibility of suitable experiments of cutting the body and putting it in contact with some thermo-elastic bodies, here we extend the uniqueness of the response functions of a thermo-elastic body to the standard functionals for the stress and for the heat flux in a generic simple body. This allows us to define the corresponding notions of stress and heat flux for general simple bodies in thermodynamics too.

1. Introduction.

In [5] contact forces are defined in terms of forces at a distance and some kinematic notions, within classical physics and for purely mechanic continuous media that are simple bodies in a wide sense—see [5, Defs. 8.1, 8.2 and 12.3]. In part 1 of [5]

(BM.1) *generalized and ordinary functions for the stress are defined for any purely mechanical elastic body* $\mathcal{B}$.

A *generalized response function for the stress* is a smooth enough function $\widehat{\boldsymbol{T}}$: $R^3 \times \mathrm{Lin}^+ \to \mathrm{Lin}$ such that, under suitable regularity conditions for the motion $\hat{x}$ of $\mathcal{B}$ and for the body force, the balance laws are satisfied by the field

$$\boldsymbol{T}(\boldsymbol{x}, t) = \hat{\boldsymbol{T}}(\boldsymbol{X}, \mathrm{GRAD}\,\hat{\boldsymbol{x}}(\boldsymbol{X}, t)), \qquad (\boldsymbol{x} = \hat{\boldsymbol{x}}(\boldsymbol{X}, t)),$$

where, according to § 2 below, t denotes time, $\boldsymbol{X} = \varphi(X)$ denotes the reference position of the body point X, $\mathcal{B} = \varphi(\mathcal{B})$ is the position of $\mathcal{B}$ in the reference configuration φ and $\hat{\boldsymbol{x}}$ represents the motion of $\mathcal{B}$ with respect to the reference position $\mathcal{B}$—see [5, § 1 and Def. 3.2]. Incidentally note that, in this way, the balance laws are simply regarded as conditions in the field $\boldsymbol{T}(\boldsymbol{x}, t)$. An *ordinary response function for the stress* is a generalized response function, according to the previous definition, which also satisfies the boundary condition [5, (6.α)]: the normal stress determined by it at the boundary of $\mathcal{B}$ necessarily vanishes wherever this boundary «touches» an empty space region. For more details see § 1 and Def. 6.1 in [5].

Incidentally, note that a very general notion of physical possibility is used in the present work, as well as in [5]. Following Bressan in [1], the concept of physical possibility is regarded here as a primitive one. To say that the process p is physically possible is not equivalent to say that it is compatible with dynamical laws. Rather, p is *physically possible* is meant here as p can (or could) be carried out by ideal experimenters. See Bressan [1] and [3] for a detailed characterization and discussion of the notion of physical possibility in classical physics.

Then [5] proves that

(BM.2) *for any hyper-elastic body $\mathcal{B}_h$ the ordinary response function for the stress is uniquely determined by the validity of the first balance law along any possible process for $\mathcal{B}_h$, whereas any two generalized functions for the stress differ by a constant Eulerian pressure*—see [5, Theor. 5.1].

This uniqueness theorem, which incidentally has been proved in [12] by a different procedure, provides the afore mentioned definition of contact forces in elastic bodies.

In the paper [14], which constitutes the thermodynamic extension of [5, Part 1],

(M.1) *generalized and ordinary sets of response functions are defined in connection with any thermo-elastic body.*

A *generalized set of response function for the thermo-elastic body* $\mathcal{B}_e$ is a set of response functions for the stress, heat flux, internal energy and entropy, respectively, for which the balance laws and the

Clausius-Duhem inequality are necessarily satisfied under suitable regularity conditions for the possible process of $\mathcal{B}_e$, for the body force, and for the heat supply—cf [14, Def. 3.1]. An *ordinary function for the stress in* $\mathcal{B}_e$ is defined as in the purely mechanical case—see (BM1) above. To discuss the uniqueness of generalized sets of response functions, [13] proves that

(M.2) *there is an indetermination in the generalized functional for the heat flux of any general simple body, which is physically unobservable by means of usual experiments, that is, experiments in which cuts of the body are not taken into account.*

With the aim of selecting, among the generalized functions for the heat flux in $\mathcal{B}_e$, a physically privileged one, to be called *ordinary*, in [13, § 7]

(M.3) *the possibility of isolating, by means of cuts, any portion of* $\mathcal{B}_e$, *and of putting it in a suitable contact with some (heat-)nonconducting body is postulated. An ordinary function for the heat flux for* $\mathcal{B}_e$ *is a generalized one for which the balance laws hold along such processes.*

Lastly, an *ordinary set of response functions* for $\mathcal{B}_e$ is a generalized one in which both the functions for the stress and for the heat flux are ordinary. Then, also by developing methods used in the proof of some theorems in [12], in [14]

(M.4) *uniqueness theorems are proved for* (i) *the generalized and ordinary functions of the stress,* (ii) *the ordinary functions of the heat flux, and* (iii) *the functions of the internal energy and entropy that belong to any generalized set of response functions, in which the function for the stress is ordinary.*

Therefore, roughly speaking,

(M.5) *with regard to thermo-elastic bodies, besides the contact forces, also the normal heat flux and the specific internal energy and entropy, are implicitly defined by the validity of the balance laws and of the Clausius-Duhem inequality.*

In [5, Part 2], in considering purely mechanical simple bodies,

(BM.3) *it is assumed that we can isolate any part of a simple* *body $\mathcal{B}_s$ by means of cuts, and we can put this part in a suitable contact with some hyper-elastic body $\mathcal{B}_h$.*

A *standard response functional for the stress in $\mathcal{B}_s$* is a response functional $\bar{\boldsymbol{T}}_s$ for the stress which satisfies the balance laws along both usual processes and the processes in which $\mathcal{B}_s$ has been cut and put in contact with some hyper-elastic bodies. Furthermore, on the basis of suitable axioms, the uniqueness of the ordinary response function $\hat{\boldsymbol{T}}_h$ for the stress of $\mathcal{B}_h$ (proved in [5, Part 1]) is extended to $\bar{\boldsymbol{T}}_s$ by using processes of the kind mentioned in (BM.3)—see [5, Theor. 11.1]. Lastly, the *standard stress* in $\mathcal{B}_s$ is determined through the (unique) standard functional $\bar{\boldsymbol{T}}_s$. Therefore

(BM.4) *contact forces can be defined also in connection with purely mechanical general simple bodies.*

The thermodynamic theory of this paper extends the one in [14] in the same way in which the mechanical theory in [5, Part 2] extends the one in [5, Part 1]. In more detail:

(1.α) *In* [5, Part 1] *the strict uniqueness of the ordinary function for the stress $\hat{\boldsymbol{T}}_{\mathrm{ord}}$ is proved in mechanics for any hyper-elastic body $\mathcal{B}_h$.*

(1.β) *In* [5, Part 2] *the notion of standard stress is introduced, and the uniqueness of the standard functional for the stress $\hat{\boldsymbol{T}}_{\mathrm{st}}$ is proved in mechanics, for any general simple body $\mathcal{B}_s$; the aim is reached by using cuts of $\mathcal{B}_s$, suitable contacts with some hyper-elastic body $\mathcal{B}_h$, and by employing in an essential way the strict uniqueness of $\hat{\boldsymbol{T}}_{\mathrm{ord}}$ for $\mathcal{B}_h$*—see (BM.1-4) and (1.α).

The main consequence of (1.β) is explained in assertion (BM.5).

(1.α′) *In* [14], *which extends to thermodynamics* [5, Part 1], *the uniqueness of each function belonging to an ordinary set of response functions* $\{\bar{\boldsymbol{T}}_{\mathrm{ord}}, \hat{\boldsymbol{q}}_{\mathrm{ord}}, \hat{e}_{\mathrm{ord}}, \hat{\eta}_{\mathrm{ord}}\}$, *connected with any thermo-elastic body $\mathcal{B}_e$, is proved*—see (M.1-5).

The main consequence of (1.α′) is explained in assertion (M.5).

(1.β′) *In the present paper, by introducing the notions of standards stress, standard heat flux and standard internal energy for any general simple body $\mathcal{B}_s$, we extend to thermodynamics the results of* [5, Part 2]; *and the uniqueness of the functionals $\bar{\boldsymbol{T}}_{\mathrm{st}}$ and $\bar{\boldsymbol{q}}_{\mathrm{st}}$, for the standard stress and heat flux respectively, is proved*; *this aim is reached* (i) *by* *postulating cuts of* $\mathcal{B}_s$ *and suitable contacts, with some thermo-elastic body* $\mathcal{B}_e$, *and* (ii) *by using in an essential way the uniqueness of* $\hat{\boldsymbol{T}}_{\mathrm{ord}}$ *and* $\hat{\boldsymbol{q}}_{\mathrm{ord}}$ *for* $\mathcal{B}_e$—see (1.α') and (1.β). *Then the uniqueness of* $\bar{\boldsymbol{T}}_{\mathrm{st}}$ *and* $\bar{\boldsymbol{q}}_{\mathrm{st}}$ *easily implies the uniqueness of the response functional for the standard internal energy.*

The main consequence of (1.β′) is that, in this work,

(M.6) *with regard to general simple bodies, the definition of contact forces is extended to thermodynamics by means of the standard notion of stress. Furthermore, through the introduction of the notion of standard heat flux, and the uniqueness theorem regarding the corresponding functional, also the notion of normal heat flux can be regarded as defined for these bodies.*

* * *

To summarize all the results of the various paper mentioned above, we note that to any general simple body $\mathcal{B}_s$ one can associate sets of response functionals of various kinds:

generalized $\{\hat{\boldsymbol{T}}_{\mathrm{gen}}, \hat{\boldsymbol{q}}_{\mathrm{gen}}, \hat{e}_{\mathrm{gen}}, \hat{\eta}_{\mathrm{gen}}\}$,

ordinary $\{\hat{\boldsymbol{T}}_{\mathrm{ord}}, \hat{\boldsymbol{q}}_{\mathrm{ord}}, \hat{e}_{\mathrm{ord}}, \hat{\eta}_{\mathrm{ord}}\}$,

standard $\{\hat{\boldsymbol{T}}_{\mathrm{st}}, \hat{\boldsymbol{q}}_{\mathrm{st}}, \hat{e}_{\mathrm{st}}, \hat{\eta}_{\mathrm{st}}\}$.

The functional $\hat{\boldsymbol{q}}_{\mathrm{gen}}$ is not unique in general—see (M.2).

Given any elastic body $\mathcal{B}_e$, (e.1) *for the function* $\hat{\boldsymbol{T}}_{\mathrm{gen}}$ *a* (*non-strict*) *uniqueness theorem holds*—see (1)—, and (e.2) a *mathematical characterization of the maximal indetermination class for* $\hat{\boldsymbol{q}}_{\mathrm{gen}}$ *is given*—see [13, Theors. 5.1, 5.2]. Furthermore (e.3) *no uniqueness theorem for* $\hat{e}_{\mathrm{gen}}$ *and* $\hat{\eta}_{\mathrm{gen}} \in \{\hat{\boldsymbol{T}}_{\mathrm{gen}}, \hat{\boldsymbol{q}}_{\mathrm{gen}}, \hat{e}_{\mathrm{gen}}, \hat{\eta}_{\mathrm{gen}}\}$ *are proved*; in more detail the indetermination of $\hat{\boldsymbol{T}}_{\mathrm{gen}}$, due to the Eulerian pressure, is strictly related with an indetermination in $\hat{e}_{\mathrm{gen}}$; and the uniqueness theorems for $\hat{e}_{\mathrm{gen}}$ and $\hat{\eta}_{\mathrm{gen}}$ can be proved only with regard to sets of the kind $\{\hat{\boldsymbol{T}}_{\mathrm{ord}}, \hat{\boldsymbol{q}}_{\mathrm{gen}}, \hat{e}_{\mathrm{gen}}, \hat{\eta}_{\mathrm{gen}}\}$—see at the end of § 3. That is, if for $i = 1, 2$ $\{\hat{\boldsymbol{T}}_i, \hat{\boldsymbol{q}}_i, \hat{e}_i, \hat{\eta}_i\}$ is a set of the kind $\{\hat{\boldsymbol{T}}_{\mathrm{ord}}, \hat{\boldsymbol{q}}_{\mathrm{gen}}, \hat{e}_{\mathrm{gen}}, \hat{\eta}_{\mathrm{gen}}\}$ for $\mathcal{B}_e$, then

(1) This program is called by Truesdell the *first problem of thermodynamics*, and may be phrased as follows—see [16, p. 121]. *In an assigned class of thermokinetic processes* $\boldsymbol{x}(\cdot)$, $\theta(\cdot)$ *and an assigned class of functionals* $\boldsymbol{T}$, $\boldsymbol{h}$, ψ, η, *to determine all constitutive functionals, namely, those that satisfy the Clausius-Duhem inequality identically.*

$\hat{T}_1 \equiv \hat{T}_2$, and furthermore $\hat{e}_1 - \hat{e}_2$ and $\hat{\eta}_1 - \hat{\eta}_2$ are constant at each material point—see [14, Theors. 5.1, 6.1]. Lastly, (e.4) *each element of the ordinary set* $\{\hat{T}_{\mathrm{ord}}, \hat{q}_{\mathrm{ord}}, \hat{e}_{\mathrm{ord}}, \hat{\eta}_{\mathrm{ord}}\}$ *has certain uniqueness properties corresponding to physical expectations*—see [14].

If $\mathcal{B}_s$ is not purely elastic, e.g. it is a body of differential type or one with a fading memory, then (s.1) *uniqueness theorems fail to be at disposal both for the sets* $\{\hat{T}_{\mathrm{gen}}, \hat{q}_{\mathrm{gen}}, \hat{e}_{\mathrm{gen}}, \hat{\eta}_{\mathrm{gen}}\}$ *and* $\{\hat{T}_{\mathrm{ord}}, \hat{q}_{\mathrm{ord}}, \hat{e}_{\mathrm{ord}}, \hat{\eta}_{\mathrm{ord}}\}$; and (s.2) *the indetermination of* $\hat{q}_{\mathrm{gen}}$ *still holds*; obviously the same holds for $\hat{T}_{\mathrm{gen}}$.

Some theorems of strict uniqueness are proved for the standard functionals $\hat{T}_{\mathrm{st}}$ and $\hat{q}_{\mathrm{st}}$ of any general simple body $\mathcal{B}_s$. If one defines the *standard functionals* $\hat{e}_{\mathrm{st}}$ and $\hat{\eta}_{\mathrm{st}}$ *for internal energy* and *entropy, respectively*, simply as functionals belonging to a set $I_{\mathrm{st}} = \{\hat{T}_{\mathrm{st}}, \hat{q}_{\mathrm{st}}, \hat{e}_{\mathrm{st}}, \hat{\eta}_{\mathrm{st}}\}$ in which the functionals for the stress and the heat flux are standard, without additional specific physical criteria to select physically privileged internal energies and entropies, then also a uniqueness theorem for the functional $\hat{e}_{\mathrm{st}}$ holds, whereas (s.3) *the uniqueness of the functional* $\hat{\eta}_{\mathrm{st}}$ *cannot be proved.*

Assertion (s.3) is a consequence of the well known paper [8] of Day, in which an infinite number of distinct entropy functionals, all compatible with the Clausius-Duhem inequality, are defined in connection with a particular simple material. It is for this that

(M.7) *in the present paper the existence of an entropy functional is not assumed*; *furthermore the Clausius-Duhem inequality, or any other dissipation inequality, is not postulated.*

Therefore

(M.8) *all the results established here also hold in many thermodynamic theories for simple bodies, whichever the assumptions on the entropy notion and the dissipation inequality may be* (2).

Now, in analogy with the thermo-elastic case—see (M.4)—, with regard to $\mathcal{B}_s$ one might ask whether or not *some uniqueness theorems for the ordinqary response functionals* $\hat{T}_{\mathrm{ord}}$, $\hat{q}_{\mathrm{ord}}$ *and* $\hat{e}_{\mathrm{ord}}$ *hold.*

(2) In particular, the theory developed here also holds in a Green-Naghdi theory, that is one in which (i) besides the functionals for the entropy, a further response functional for the (specific) internal rate of entropy production is assumed to exist; (ii) the Clausius-Duhem inequality is replaced by a balance entropy equality; and (iii) an unusual form of the second law of thermodynamics is assumed—see [9].

In other words: to attain (i) the uniqueness of the above functionals, and consequently (ii) the definitions of contact forces and normal heat flux,

— *are the standard notions of stress and heat flux really necessary?*

— *are the corresponding ordinary notions enough to attain this aim?*

The present paper leaves these questions unanswered, which I believe worthy of further deepening.

* * *

A contribution to a program considered by C. Truesdell in [17] is given here, as in [5] and [14]—see § 1 and (1) (2) in [14]. This program is called by Truesdell the first problem of thermodynamics, and may be phrased as follows—see [17, p. 121].

In an assigned class of thermokinetic processes $\boldsymbol{x}(\cdot)$, $\theta(\cdot)$ *and an assigned class of functionals* $\boldsymbol{T}$, $\boldsymbol{h}$, ψ, η, *to determine all constitutive functionals, namely, those that satisfy the Clausius-Duhem inequality identically.*

This contribution differs from others especially for the use of the *possibility of cutting bodies*—see (1.β), (1.α′) and (1.β′). Incidentally note that, by the nature of the indetermination in the response functional for the heat flux ascertained in [13], and by the way used in [14], to postulate the possibility of cutting bodies appears to be a necessary tool to single out a physical privileged such functional.

This contribution also differs from others in that *possible indeterminations in the response functionals are considered, at least a priori* here and in [5], [14]. Such indeterminations, admittedly of a very different nature, were taken into account only in piezo-elasticity. For instance, in [2, Chap. 9], where materials with memory and axiomatic foundations are dealt with within general relativity, the existence of all constitutive equations needed in that theory is assumed; and the problem of determining all admissible choices of those equations (possibly up to simple functions such as the constants functions) is left completely open. Incidentally this also occurs for piezo-elastic materials. However formula (36.9) in [2] shows e.g. how, for every such material, any admissible constitutive equation for stress changes in connection with any change of the expression of the electromagnetic tensor in terms of the electric and magnetic fields and inductions. For more details on this point one can read the introduction of both [5] and [14].

As explained above, one of the aims of this paper is to show that the notion of standard stress, defined in [5, Part 2] in the purely mechanical case, can be introduced in thermodynamics too in connection with any general simple body. There, to treat this matter with the greatest degree of generality, the most general simple bodies that reasonably can exist, namely piecewise regular simple bodies, along C^0-motions which are piecewise C^2, are considered. Unlikely in [5], in the present theory *we limit our considerations to regular simple bodies along C^1 processes*—see Defs. 2.1(a) and 3.1(a); indeed the extension to those general bodies is a straightforward matter, whose difficulty is only of technical kind, whereas here we are mainly interested in showing that the aforementioned extension is possible.

2. Preliminaries.

The notions of *body*, *configuration of a body* and *material point* are regarded as known here. A possible configuration φ to be used as a reference one is fixed; and the body $\mathcal{B}$ can be identified with the closed region $\mathcal{B}_\varphi = \varphi(\mathcal{B})$ of the ambient space $\mathbb{R}^3$ in the sense that, if $\boldsymbol{X} \in \mathcal{B}_\varphi$, then «the material point $\boldsymbol{X}$» means «the material point X that occupies the geometric point $\boldsymbol{X}$ in φ»—for precise definitions see e.g. [4] and [5, § 1 and Def. 3.2].

We assume that $\mathcal{B}_\varphi$ is a regular domain (the closure of its own interior) in $\mathbb{R}^3$, whose boundary $\partial\mathcal{B}\varphi$ is piecewise smooth. Subbodies (or parts) $\mathcal{P}, \mathcal{Q}, \ldots$ of $\mathcal{B}$ are identified with the regular subdomains $\mathcal{P}_\varphi, \mathcal{Q}_\varphi, \ldots$ of $\mathcal{B}_\varphi$.

Euclidean coordinates in the inertial ambient space $\mathbb{R}^3$ [in the reference space $\mathcal{B}_\varphi$] are denoted by $\{x^a\}[\{X^A\}]$.

Differently from [5, Part 2], where more general processes are used, here we only consider processes that are smooth enough; moreover physically possible processes, instead of conceivable ones, are used here—see (3).

(3) Following Bressan in [1], the concept of physical possibility is regarded here as primitive. To say that the process p is physically possible is not equivalent to say that it is compatible with dynamical laws. Rather, p is *physically possible* is meant here as p can (or could) be carried out by ideal experimenters. See [1] and [3] for a detailed characterization and discussion of the notion of physical possibility in classical physics.

DEF. 2.1. *(a) We say that* $\hat{p} = (\hat{\boldsymbol{x}}, \hat{\theta})$, *where*

$$\hat{\boldsymbol{x}}: \mathcal{B}_\varphi \times \mathbb{R} \to \mathbb{R}^3 \quad \textit{and} \quad \hat{\theta}: \mathcal{B}_\varphi \times \mathbb{R} \to \mathbb{R}^+, \tag{2.1}$$

is a (conceivable, thermo-kinematic) process for the body $\mathcal{B}$ *(described with respect to the configuration* φ*), if*

(i) $\hat{p}$ *is a function of class* C^1;

(ii) *for each* $t \in \mathbb{R}$ *the restriction of* $\hat{\boldsymbol{x}}(\cdot, t)$ *to the interior* $\mathring{\mathcal{B}}_\varphi$ *of* $\mathcal{B}$ *is a homeomorphism;*

(iii) $\det \operatorname{Grad} \hat{\boldsymbol{x}}(\cdot, t) > 0$ *on* $\mathcal{B}_\varphi$; *and*

(iv) $\hat{\theta}(\boldsymbol{X}, t) > 0$ *for each* $(\boldsymbol{X}, t) \in \mathcal{B}_\varphi \times \mathbb{R}$.

(b) We say that $\hat{p}$ *is* C^2 *at* $(\boldsymbol{X}, t)$ *if it is* C^2 *on a neighborhood of* $(\boldsymbol{X}, t)$ *in* $\mathcal{B}_\varphi \times \mathbb{R}$.

DEF. 2.2. *(a) We say that the conceivable process* $\hat{p} = (\hat{\boldsymbol{x}}, \hat{\theta})$ *is the actual process for the body* $\mathcal{B}$ *if*

$$\boldsymbol{x} = \hat{\boldsymbol{x}}(\boldsymbol{X}, t) \quad \textit{and} \quad \theta = \hat{\theta}(\boldsymbol{X}, t) \tag{2.2}$$

are the position and the temperature, respectively, of the material point $\boldsymbol{X}$ *at time* t *for each* $(\boldsymbol{X}, t) \in \mathcal{B}_\varphi \times \mathbb{R}$.

(b) Any coinceivable process $\hat{p}$ *for* $\mathcal{B}$ *is (obviously) said to be physically possible if it is physically possible for* $\hat{p}$ *to be the actual process for* $\mathcal{B}$.

Following [4], where *(inertial) mass* is defined for continuous media, we have that

(2.α) *if a mass unit is chosen, then for some choice of* φ *there is a unique* C^1 *scalar function* $\hat{\rho}: \mathcal{B}_\varphi \to \mathbb{R}^+$, *such that*

$$\hat{m}(\mathcal{P}) = \int_{\mathcal{P}_t} \rho \, dv \tag{2.3}$$

$(\boldsymbol{x} = \hat{\boldsymbol{x}}(\boldsymbol{X}, t),\ \rho = [\hat{\rho}_\varphi(\boldsymbol{X})]_{\boldsymbol{X} = \hat{\boldsymbol{x}}^{-1}(\boldsymbol{x}, t)},\ \mathcal{P}_t = \hat{\boldsymbol{x}}(\mathcal{P}, t))$ *is the mass of any part* $\mathcal{P}$ *of* $\mathcal{B}$ *along the actual process*—see (2.1)-(2.2).

Let $\mathcal{V}$ be any finite-dimensional linear space on the real field $\mathbb{R}$, and let $f(\cdot)$ be a function from (a, b) to $\mathcal{V}$, where $-\infty \leqslant a < b \leqslant +\infty$; for $t \in (a, b)$ the function $f^t(\cdot): [0, \alpha_t) \to \mathcal{V}$ $[f^t_p(\cdot): (0, \alpha_t) \to \mathcal{V}]$ defined by

$$f^t(s) = f(t - s) \quad [f^t_p(s) = f(t - s)], \tag{2.4}$$

where $s \in [0, \alpha_t)$ $[s \in (0, \alpha_t)]$ and $\alpha_t = t - a \in \mathbb{R}^+ \cup \{+\infty\}$, is called the *total* [*past*] *history of* $f(\cdot)$ *up to time* t. In connection with materials with memory, for the sake of simplicity it is usually assumed $(a, b) \in \mathbb{R}$.

3. Simple bodies.

Following [5, Part 2] a *simple body* $\mathcal{B}$ is a body constituted by a material which is simple in the sense that, given a suitably smooth reference configuration φ for $\mathcal{B}$, to every matter point X of $\mathcal{B}$ a set of functionals

$$(3.1) \qquad \bar{\boldsymbol{T}}_{\boldsymbol{X}}(\cdot), \quad \bar{\boldsymbol{q}}_{\boldsymbol{X}}(\cdot), \quad \bar{e}_{\boldsymbol{X}}(\cdot), \quad \bar{\eta}_{\boldsymbol{X}}(\cdot) \quad (\boldsymbol{X} = \varphi(X)),$$

for the stress, heat-flux, internal energy and entropy, respectively, is associated. The domain $\mathcal{D}_{\boldsymbol{X}}$ of these functionals is a suitable subset of $([0, a) \to \mathcal{V})$, equipped with a suitable topology, where $a \in \mathbb{R}^+ \cup \{+\infty\}$ and $\mathcal{V} = \mathrm{Lin} \times \mathbb{R} \times \mathbb{R}^3$ (4).

Let $PP_{\mathcal{B}, \varphi}$ be the set of the possible φ-processes for $\mathcal{B}$—see Defs. 2.1-2.2; and, for any $\hat{p} = (\hat{\boldsymbol{x}}, \hat{\theta})$ among them, let us consider the tensors

$$(3.2) \qquad \begin{cases} \boldsymbol{F}_{\boldsymbol{X}}(t) = [F^a{}_A(\boldsymbol{X}, t)] = \mathrm{Grad}\, \hat{\boldsymbol{x}}(\boldsymbol{X}, t) & \left(F^a{}_A(\boldsymbol{X}, t) = \dfrac{\partial \hat{x}^a}{\partial X^A}(\boldsymbol{X}, t)\right), \\ \boldsymbol{G}_{\boldsymbol{X}}(t) = [G_A(\boldsymbol{X}, t)] = \mathrm{Grad}\, \hat{\theta}(\boldsymbol{X}, t) & \left(G_A(\boldsymbol{X}, t) = \dfrac{\partial \hat{\theta}}{\partial X^A}(\boldsymbol{X}, t)\right). \end{cases}$$

Now we consider the *domain* DS$(= \mathrm{DSRF}_{\mathcal{B}, \varphi})$ *of any* (*conceivable*) *simple response φ-functional for* $\mathcal{B}$:

(3.3) $DS = \text{closure}\,\{(\boldsymbol{X}, \Lambda(\cdot)) \,|\, \boldsymbol{X} \in \mathring{\mathcal{B}}_\varphi;$ for some $\tau \in \mathbb{R}$ and $\hat{p} \in PM_{\mathcal{B}, \varphi}$, $\hat{p}$ is C^2 near $(\boldsymbol{X}, \tau)$ and $\Lambda(\cdot) = (\boldsymbol{F}^\tau(\boldsymbol{X}, \cdot), \theta^\tau(\boldsymbol{X}, \cdot), \boldsymbol{G}^\tau(\boldsymbol{X}, \cdot))\}$—see (2.4), (3.2).

In order to be compatibe with many thermodynamic theories, next we define simple bodies in a way that, on the one hand, allows us to attain the aims (1.β′) of the present paper and, on the other hand, is considerably unspecified. For instance, no notion of entropy is involved in the definition below—see § 1, (M.7-8).

(4) Different mathematical versions of the principle of fading memory are used in various theories of simple materials—see e.g. [6], [7] and [18]. In spite of this the present work does not use such a principle; hence it complies with any of those theories.

DEF. 3.1. *Assume that $\mathcal{B}$ is a body and*

(3.α) *the reference configuration φ of $\mathcal{B}$ renders* (2.3) *true for some* (*smooth*) *choice of* $\hat{\rho}$.

Then (*a*) [(*b*)] *we say that* $\{\bar{\boldsymbol{T}}, \bar{\boldsymbol{q}}, \bar{e}\}$ *is a* [*frame-indifferent*] *generalized set of simple response functionals for $\mathcal{B}$, connected with φ*—see (M.7-8)—*in case conditions* (i) *and* (ii) [(i) *to* (iii)] *below hold.*

(i) $\bar{\boldsymbol{T}} \in C^0(DS, \mathrm{Lin})$, $\bar{\boldsymbol{q}} \in C^0(DS, \mathbb{R}^3)$ *and* $\bar{e} \in C^0(DS, \mathbb{R})$—see (3.3);

(ii) *the functionals $\bar{\boldsymbol{T}}$, $\bar{\boldsymbol{q}}$ and $\bar{e}$ are frame-indifferent;*

(iii) *necessarily the conditions* (*balance laws*)

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho \boldsymbol{v}\, dV = \int_{\mathcal{P}_t} \rho \boldsymbol{b}\, dV + \int_{\partial\mathcal{P}_t} \boldsymbol{T}\boldsymbol{n}\, dA\,, \tag{3.4}$$

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho \boldsymbol{r} \times \boldsymbol{v}\, dV = \int_{\mathcal{P}_t} \rho \boldsymbol{r} \times \boldsymbol{b}\, dV + \int_{\partial\mathcal{P}_t} \boldsymbol{r} \times \boldsymbol{T}\boldsymbol{n}\, dA\,(^5)\,, \tag{3.5}$$

and

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho\left(e + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}\right) dv = \int_{\mathcal{P}_t} \rho(\boldsymbol{b}\cdot\boldsymbol{v} + r)\, dv + \int_{\partial\mathcal{P}_t} (\boldsymbol{v}\cdot\boldsymbol{T}\boldsymbol{n} - \boldsymbol{q}\cdot\boldsymbol{n})\, da \tag{3.6}$$

hold for every part $\mathcal{P}$ of $\mathcal{B}$ in case $\hat{p} = (\hat{\boldsymbol{x}}, \hat{\theta})$ is the actual process of $\mathcal{B}$ (so that $\hat{p} \in PM_{\mathcal{B},\varphi}$), the fields

$$\boldsymbol{b} = \hat{\boldsymbol{b}}(\boldsymbol{x}, t)\,, \qquad r = \hat{r}(\boldsymbol{x}, t) \tag{3.7}$$

are of class C^0, and represent the (*actual*) *densities per unit mass of the body force and heat supply respectively at the material point $\boldsymbol{X}$ and at the instant t; and in addition, for each $(\boldsymbol{X}, t) \in \mathcal{B}_\varphi \times \mathbb{R}$,*

$$\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{x}, t) = \bar{\boldsymbol{T}}(\boldsymbol{X}, \Lambda_{\boldsymbol{X}}^t(\cdot))\,, \qquad \boldsymbol{q} = \boldsymbol{q}(\boldsymbol{x}, t) = \bar{\boldsymbol{q}}(\boldsymbol{X}, \Lambda_{\boldsymbol{X}}^t(\cdot))\,, \tag{3.8}$$

$$e = e(\boldsymbol{x}, t) = \bar{e}(\boldsymbol{X}, \Lambda_{\boldsymbol{X}}^t(\cdot))\text{—see (2.2) and (3.2-3)}.$$

(*c*) [(*d*)] *Furthermore we say that $\mathcal{B}$ is a* [*frame-indifferent*] *simple body in case it has a* [*frame-indifferent*] *generalized set of simple response functionals.*

(5) Let «×» [«·»] denote the cross product [inner product] between vectors, let $\boldsymbol{r}$ be the position vector and let $\boldsymbol{n} = \hat{\boldsymbol{n}}(\boldsymbol{x}, t)$ be the unit outward normal at $\boldsymbol{x}$ to $\partial\mathcal{P}_t$.

The magnitudes $\boldsymbol{T}$, $\boldsymbol{q}$ and e in (3.8) represent the Cauchy stress tensor, heat flux vector and specific internal energy, respectively, expressed in Eulerian form. The scalar product $\boldsymbol{q}\cdot\boldsymbol{n}$ is the rate of flux of heat energy by conduction from $\mathcal{P}_t$ to its exterior across the boundary $\partial\mathcal{P}_t$ at $\boldsymbol{x}\in\partial\mathcal{P}_t$.

The above condition (ii) of material frame-indifference *will not be used* to prove the uniqueness theorems in § 5; thus also in the present framework the principle of material frame-indifference can be stated e.g. in the usual Noll's form—see e.g. pages 162 and 171-172 in [16]—, useful to show its physical content. For the sake of brevity one can state it by requiring every simple body to be frame-indifferent—see Def. 3.1(*c*)-(*d*).

For the sake of simplicity and for the reasons explained at the end of § 1, we have not defined (s,φ)-*regular simple bodies* $(s = 0, 1)$ in the sense of Def. 8.2 in [5, Part 2]. However, if one would enunciate the thermodynamic analogues of this Def. 8.2, then the simple bodies defined here would appear as particular $(0,\varphi)$-*regular simple bodies.*

Note that in any thermoelastic body there is a twofold indetermination for the generalized response functions of the stress and the internal energy. Indeed assume that both $\widehat{I} = \{\widehat{\boldsymbol{T}}_{\text{gen}}, \widehat{\boldsymbol{q}}_{\text{gen}}, \widehat{e}_{\text{gen}}, \widehat{\eta}_{\text{gen}}\}$ and $\widetilde{I} = \{\widetilde{\boldsymbol{T}}_{\text{gen}}, \widetilde{\boldsymbol{q}}_{\text{gen}}, \widetilde{e}_{\text{gen}}, \widetilde{\eta}_{\text{gen}}\}$ are two generalized sets of response functions for the thermo-elastic body $\mathcal{B}_{\text{e}}$, connected with the same reference configuration. By subtracting the energy balance law

$$\rho\dot{e} = \boldsymbol{T}\cdot\operatorname{grad}\boldsymbol{v} + \rho r - \operatorname{div}\boldsymbol{q} \tag{3.9}$$

written using $\widehat{I}$ with the same written using $\widetilde{I}$, we find

$$\rho\dot{E} = \boldsymbol{Z}\cdot\operatorname{grad}\boldsymbol{v} - \operatorname{div}\boldsymbol{Q}\,, \tag{3.10}$$

where $\widehat{E} = \widehat{e}_{\text{gen}} - \widetilde{e}_{\text{gen}}$, $\widehat{\boldsymbol{Z}} = \widehat{\boldsymbol{T}}_{\text{gen}} - \widetilde{\boldsymbol{T}}_{\text{gen}}$, $\widehat{\boldsymbol{Q}} = \widehat{\boldsymbol{q}}_{\text{gen}} - \widetilde{\boldsymbol{q}}_{\text{gen}}$ and E, $\boldsymbol{Z}$, $\boldsymbol{Q}$ are the corresponding Eulerian fields. As $\operatorname{grad}\boldsymbol{v} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$, for $\dot{\theta} = 0$ and $\dot{\boldsymbol{F}} = 0$ equality (3.10) yields $\operatorname{div}\boldsymbol{Q} = 0$, and thus $\rho\dot{E} = \boldsymbol{Z}\cdot\operatorname{grad}\boldsymbol{v}$ too. The material counterpart of the last (space-time) equality is

$$\rho_\kappa\dot{E} = \boldsymbol{S}\cdot\dot{\boldsymbol{F}}\,, \qquad \text{where} \qquad \widehat{\boldsymbol{S}} = \widehat{\boldsymbol{P}}_{\text{gen}} - \widetilde{\boldsymbol{P}}_{\text{gen}}\,, \tag{3.11}$$

and $\widehat{\boldsymbol{P}}_{\text{gen}}$ [$\widetilde{\boldsymbol{P}}_{\text{gen}}$] is the first Piola function for the stress associated with $\widehat{\boldsymbol{T}}_{\text{gen}}$ [$\widetilde{\boldsymbol{T}}_{\text{gen}}$]. By [5, Theorem 4.1 and (4.7)] we have

$$\boldsymbol{S} = \widehat{\boldsymbol{S}}(\boldsymbol{F}) = J\omega\boldsymbol{F}^{-T} \qquad \text{for some } \omega\in\mathbb{R} \qquad (J = \det\boldsymbol{F})\,. \tag{3.12}$$

Thus by the first equality below, the second yields the third equality below

$$(3.13)\qquad \frac{dJ}{dt} = JF^{-T}, \qquad V = \widehat{V}(F) = \omega J, \qquad S = \partial_F V.$$

Now by (3.12)-(3.13) equality (3.11) rewrites as $\rho_\kappa \dot{E} = \partial_F V \cdot \dot{F} = \dot{V}$, which implies that

$$(3.14)\qquad E = V + \widehat{f}(X) \text{ for some scalar function } \widehat{f}(X)\text{—see } (3.13)_2.$$

Therefore the indetermination of $\widehat{P}_{\text{gen}}$, due to the Piola-transform of the Eulerian pressure ωI—see (3.12)—, induces a corresponding indetermination for $\widehat{e}_{\text{gen}}$ given by (3.13)-(3.14).

This twofold indetermination disappears by replacing generalized functions for the stress with ordinary ones. Indeed in this case $\omega = 0$—see [14, Theorems 5.1-6.1]—, and thus (3.12)-(3.14) yield $\widehat{S} \equiv 0$, $\widehat{V} \equiv 0$, $E = \widehat{f}(X)$.

In order to prove the strict uniqueness of the response functionals for the stress and the heat flux in any simple body with fading memory, the notions of *ordinary stress* and *ordinary heat flux* do not suffice. This is the reason why in the next Sections we introduce the notions of *standard stress* and *standard heat flux.*

4. Possible processes for simple bodies fit to define standard stress and standard heat flux within these.

Following [5, Part 2, § 9] we assume that

(i) *$\mathcal{B}$ is a simple body and assertion* (3.α) *holds for it*—see Def. 3.1;

(ii) $\mathcal{B}_\varphi = \varphi(\mathcal{B})$ *and* $X \in \overset{\circ}{\mathcal{B}}_\varphi$;

(iii) *N is a reference unit vector, Π_N is the plane through X normal to N, with the orientation induced by N*; and

(iv) *for every $r > 0$, $\mathcal{H}(X, N, r)$ is the* (*hemisphere*) *intersection of the closed ball $\bar{B}(X, r)$ with the half-space determined by Π_N* (*which contains $X + N$*).

Then it is obvious that, for some $R > 0$ and some ψ-regular thermoelastic body $\mathcal{B}_e$, where ψ is a reference configuration for it,

(v) $\bar{B}(X, R) \subset \mathcal{B}_\varphi$ and $\psi(\mathcal{B}_e) = \mathcal{H}(X, -N, R)$,

and that by suitable cuts we can isolate a part $\mathcal{P}$ of $\mathcal{B}$ for which

(vi) $\varphi(\mathcal{P}) = \mathcal{H}(X, N, R)$.

Furthermore we assume that

(vii) $(\boldsymbol{X}, \Lambda(\cdot)) \in DS$—see (3.3).

Then it is (reasonably) possible—i.e. in some idealized experiments we can have—that

(viii) *$\hat{p}_1 = (\hat{\boldsymbol{x}}_1, \hat{\theta}_1)$ and $\hat{p}_2 = (\hat{\boldsymbol{x}}_2, \hat{\theta}_2)$ are the actual φ- and ψ-processes of $\mathcal{P}$ and $\mathcal{B}_e$, respectively,*

where conditions (ix) to (xi) below hold

(ix) *for $i = 1, 2$ the process $\hat{p}_i$ is C^1;*

(x) *$\Lambda(\cdot) = \Lambda_1^\tau(\cdot)$, where $\Lambda_1^\tau(\cdot)$ is $(\boldsymbol{F}^\tau(\boldsymbol{X}, \cdot), \theta^\tau(\boldsymbol{X}, \cdot), \boldsymbol{G}^\tau(\boldsymbol{X}, \cdot))$ in process $\hat{p}_1$*—see (3.2)-(3.3) and (2.4);

(xi) *there is a neighborhood (τ_1, τ_2) of τ such that for $i = 1, 2$ the motion $\hat{\boldsymbol{x}}_i$ is C^2 in $B(\boldsymbol{X}, R) \times (\tau_1, \tau_2)$ and at any $t \in (\tau_1, \tau_2)$, for some surfaces $\Sigma_i = \Sigma_i(t)$ $(i = 1, 2)$ we have*

$$(4.1) \quad \boldsymbol{X} \in \mathring{\Sigma}_1 \cap \mathring{\Sigma}_2, \quad \Sigma_1 \cup \Sigma_2 \subseteq \Sigma = \bar{B}(\boldsymbol{X}, R) \cap \Pi_N, \quad \hat{\boldsymbol{x}}_1(\Sigma_1, t) = \hat{\boldsymbol{x}}_2(\Sigma_2, t),$$

where for instance $\mathring{\Sigma}_1$ is Σ_1 deprived of its bounding curve, and

$$(4.2) \quad \hat{\boldsymbol{x}}_1(\boldsymbol{X}, \tau) = \hat{\boldsymbol{x}}_2(\boldsymbol{X}, \tau), \qquad \operatorname{Grad} \hat{\boldsymbol{x}}_1(\boldsymbol{X}, \tau)\boldsymbol{M} = \operatorname{Grad} \hat{\boldsymbol{x}}_2(\boldsymbol{X}, \tau)\boldsymbol{M}$$

$$\textit{for all } \boldsymbol{M} \perp \boldsymbol{N}.$$

Obviously the kinematic conditions (x) and (xi) allow mutual sliding of $\mathcal{P}$ and $\mathcal{B}_e$ through some neighborhoods $\Sigma_1 \subset \partial\varphi(\mathcal{P})$ and $\Sigma_2 \subset \partial\psi(\mathcal{B}_e)$ of $\boldsymbol{X}$ in Σ. They are compatible with $\mathcal{P}$ and $\mathcal{B}_e$ being two bodies, we mean compatible with $\mathcal{P}$ not being attached to $\mathcal{B}_e$ (through e.g. Σ). This keeps holding even when the dynamic features of the motion $\hat{\boldsymbol{x}}_1 \cup \hat{\boldsymbol{x}}_2$, described in (viii) through (xi), are taken into account, only in case

(xii) *the mutual actions of $\mathcal{P}$ and $\mathcal{B}_e$ through $\Sigma = \hat{\boldsymbol{x}}_1[\Sigma_1(t), t]$ at t have a pointwise pressure character.*

However, in the opposite case, $\mathcal{P}$ and $\mathcal{B}_e$ have to be mutually attached through some material surface $\subseteq \Sigma$. Therefore it is convenient to strengthen (xi) into the condition (xiii) below, which is again kinematic.

(xiii) *For some neighborhood (τ_1, τ_2) of τ (possibly with $\tau_1 = -\infty$) $\hat{\boldsymbol{x}}_1(\boldsymbol{Y}, t) = \hat{\boldsymbol{x}}_2(\boldsymbol{Y}, t)$ for all $\boldsymbol{Y} \in \Sigma = \bar{B}(\boldsymbol{X}, R) \cap \Pi_N$, and all $t \in (\tau_1, \tau_2)$.*

Next we state Axiom 4.1 to summarize rigorously the part of this section ending with (xiii).

AXIOM 4.1. *Conditions* (i) *through* (iv) *and* (vii) *imply that, for some* $R > 0$, *there is (or there can be) a* ψ*-regular thermo-elastic body* $\mathcal{B}_e$, *where* ψ *is a reference configuration for it, a part* $\mathcal{P}$ *of* $\mathcal{B}$, *and some (possible) processes* $\hat{p}_1$ *and* $\hat{p}_2$ *such that, first, conditions* (v) *to* (vi) *hold, and second, for some* $\tau \in R$ *it is possible for conditions* (viii) *to* (xi) *to take place together with either* (xii) *or* (xiii).

Note that by Axiom 4.1 the existence of a simple body implies that *some thermo-elastic bodies can exist.* Furthermore one can show that the last possibility assertion in Axiom 4.1 is reasonable in a way quite similar with what is done at the end of § 9 in [5] for Axiom 9.1.

5. Functionals for standard stress and heat flux in simple bodies. Existence axioms and uniqueness theorems.

Assume conditions (i) through (v) and (vii) in § 4; then by Axiom 4.1 for some $R > 0$ and for a thermo-elastic body $\mathcal{B}_e$ referred to the reference configuration ψ, conditions (viii) through (xi) and (vi), together with either (xii) or (xiii) in § 4, hold.

Now, let $\bar{\boldsymbol{T}}$ and $\bar{\boldsymbol{q}}$ be generalized functionals for the stress and for the heat flux, respectively, in $(\mathcal{B}, \varphi)$; and let $\hat{\boldsymbol{T}}$ and $\hat{\boldsymbol{q}}$ be ordinary functions for the stress and heat flux, respectively, in $(\mathcal{B}_e, \psi)$. Write the energy balance law (3.6) [the linear momentum balance law (3.6)] first for $\mathcal{B}_e \cup \mathcal{P}$, second for $\mathcal{B}_e$ and third for $\mathcal{P}$; hence subtract the sum of the last two inequalities from the first equality; we obtain

$$(5.1) \qquad \int_S (\boldsymbol{v} \cdot \bar{\boldsymbol{T}}\boldsymbol{n} - \bar{q} \cdot \boldsymbol{n})\, da + \int_S (\boldsymbol{v} \cdot \hat{\boldsymbol{T}}\boldsymbol{n}_1 - \hat{\boldsymbol{q}} \cdot \boldsymbol{n}_1)\, ds_1 = 0 \quad (\boldsymbol{n}_1 = -\boldsymbol{n}),$$

$$[(5.2) \qquad \int_S \bar{\boldsymbol{T}}\boldsymbol{n}\, da + \int_S \hat{\boldsymbol{T}}\boldsymbol{n}_1\, da_1 = 0 \qquad (\boldsymbol{n}_1 = -\boldsymbol{n}).]$$

where $S = \partial\varphi(\mathcal{P}) \cap \partial\psi(\mathcal{B}_e) = B(\boldsymbol{X}, R) \cap \Pi_N$—see (iii) through (vi)—, and $\hat{\boldsymbol{T}}$, $\hat{\boldsymbol{q}}$ are ordinary response functions of the stress and the heat flux, respectively, for $(\mathcal{B}_e, \psi)$. Equality (5.1) [(5.2)] implies the local equality

$$(5.3) \quad \boldsymbol{v} \cdot \bar{\boldsymbol{T}}(\boldsymbol{x}, t)\boldsymbol{n} - \bar{\boldsymbol{q}}(\boldsymbol{x}, t) \cdot \boldsymbol{n} = \boldsymbol{v} \cdot \hat{\boldsymbol{T}}(\boldsymbol{x}, t)\boldsymbol{n} - \hat{\boldsymbol{q}}(\boldsymbol{x}, t) \cdot \boldsymbol{n} \quad (\boldsymbol{x} = \hat{\boldsymbol{x}}(\boldsymbol{X}, t)).$$

$$[(5.4) \qquad \bar{\boldsymbol{T}}\boldsymbol{n}\, da + \hat{\boldsymbol{T}}\boldsymbol{n}_1\, da_1 = 0 \qquad (\boldsymbol{x} = \hat{\boldsymbol{x}}(\boldsymbol{X}, t)).]$$

By choosing the frame in the ambient space such that $\boldsymbol{v} = \boldsymbol{0}$ at $(\boldsymbol{x}, t)$, equality (5.3) [(5.4)] yields

$$\bar{\boldsymbol{q}}(\boldsymbol{x}, t)\cdot\boldsymbol{n} = \hat{\boldsymbol{q}}(\boldsymbol{x}, t)\cdot\boldsymbol{n} \qquad (\boldsymbol{x} = \hat{\boldsymbol{x}}(\boldsymbol{X}, t)). \tag{5.5}$$

[(5.6) $\quad \bar{\boldsymbol{T}}(\boldsymbol{x}, t)\boldsymbol{n} = \hat{\boldsymbol{T}}(\boldsymbol{x}, t)\boldsymbol{n} \qquad (\boldsymbol{x} = \hat{\boldsymbol{x}}(\boldsymbol{X}, t)).$]

Note that equality (5.5) [(5.6)] holds along any process described above and for any generalized response function for the heat flux [stress] in the thermo-elastic body $\mathcal{B}_e$.

A generalized response functional $\bar{\boldsymbol{q}}$ for the heat flux [stress] in $(\mathcal{B}, \varphi)$ is said to be *standard*, if equality (5.5) [(5.6)] holds for it at any $(\boldsymbol{X}, t)$ in which the simple body $\mathcal{B}$ has been cut and put in contact with a thermo-elastic body $\mathcal{B}_e$, as it is explained above in detail, and where in equality (5.5) [(5.6)] $\hat{\boldsymbol{q}}\,[\hat{\boldsymbol{T}}]$ is the (unique) ordinary response function for the heat flux [stress] in the thermo-elastic body $\mathcal{B}_e$, connected with ψ. Therefore we give the following definition.

DEF. 5.1 [5.2]. *Assume that $\mathcal{B}$ is a simple body and that $\{\bar{\boldsymbol{T}}, \bar{\boldsymbol{q}}, \bar{e}\}$ is a generalized set of response functionals for it, referred to the configuration φ. We say that $\bar{\boldsymbol{T}}$ $[\bar{\boldsymbol{q}}]$ is a standard functional for the stress [heat flux] in $\mathcal{B}$ in case, for all $\boldsymbol{X} \in \mathring{\mathcal{B}}_\varphi$, unit vector $\boldsymbol{N}$, $\tau \in \mathbb{R}$, if $\mathcal{B}_e$ is a regular thermo-elastic body having the reference configuration ψ, $\mathcal{P}$ is a part of $\mathcal{B}$, and for some history $\Lambda(\cdot)$ conditions* (iii) *to* (xi) *together with either* (xii) *or* (xiii) *in* § 4 *hold, then we have the equality*

$$\bar{\boldsymbol{T}}(\boldsymbol{X}, \Lambda_1^\tau(\boldsymbol{X}, \cdot))\boldsymbol{n} = \hat{\boldsymbol{T}}(\boldsymbol{X}, \Lambda_2(\boldsymbol{X}, \tau))\boldsymbol{n}, \tag{5.7}$$

[(5.8) $\quad \bar{\boldsymbol{q}}(\boldsymbol{X}, \Lambda_1^\tau(\boldsymbol{X}, \cdot))\boldsymbol{n} = \hat{\boldsymbol{q}}(\boldsymbol{X}, \Lambda_2(\boldsymbol{X}, \tau))\boldsymbol{n},$]

where

(α) [(β)] $\hat{\boldsymbol{T}}$ $[\hat{\boldsymbol{q}}]$ *is the (unique—*see [14] *ψ-regular ordinary response function for the Cauchy stress [heat flux] in $\mathcal{B}_e$,*

$$\Lambda_i = (\operatorname{Grad}\hat{\boldsymbol{x}}_i(\boldsymbol{X}, t), \hat{\theta}_i(\boldsymbol{X}, t), \operatorname{Grad}\hat{\theta}_i(\boldsymbol{X}, t)) \qquad (i = 1, 2), \tag{5.9}$$

and $\boldsymbol{n}$ is the unit normal at $\boldsymbol{x} = \hat{\boldsymbol{x}}(\boldsymbol{X}, t)$ for the image $\hat{\boldsymbol{x}}(\Sigma, t)$ of any surface Σ through $\boldsymbol{X}$ perpendicular to $\boldsymbol{N}$ at $\boldsymbol{X}$:

$$|\boldsymbol{n}| = 1, \qquad \boldsymbol{n} \perp \boldsymbol{F}_i(\boldsymbol{X}, t)\boldsymbol{M} \quad \textit{for all } \boldsymbol{M} \textit{ tangent to } \Sigma \textit{ at } \boldsymbol{X}. \tag{5.10}$$

Next we prove that the uniqueness of the ordinary response functions for the stress and the heat flux in thermo-elastic bodies implies the uniqueness of the standard response functionals for the stress and the heat flux in any simple body. The proof uses the possibility of suitable experiments of cutting the body and putting it in contact with some thermo-elastic bodies.

THEOREM 5.1 [5.2]. *Let both $\bar{T}$ and $\bar{T}'$ [$\bar{q}$ and $\bar{q}'$] be standard functionals for the stress [heat flux] in $(\mathcal{B}, \varphi)$. Then $\bar{T} = \bar{T}'$ [$\bar{q} = \bar{q}'$].*

PROOF. Let $\bar{T}$ and $\bar{T}'$ [$\bar{q}$ and $\bar{q}'$] be standard functionals, for the Cauchy stress [heat flux] in $\mathcal{B}$, connected with φ. Hence, by Defs. 3.1(*a-b*), (*a*) *both $\bar{T}$ and $\bar{T}'$ have DS as domain*—see (3.3). Now (*b*) *choose* $(X, \Lambda(\cdot)) \in DS$. Hence (vii) and (ii) in § 4 holds; and assume (iii)-(iv) in § 4. Then, by Axiom 4.1, for some $R > 0$ there can be a ψ-regular thermo-elastic body $\mathcal{B}_e$, where ψ is a reference configuration for it, and a part $\mathcal{P}$ of $\mathcal{B}$ such that, first, conditions (v) to (vi) hold, and second, for some $\tau \in R$ conditions (viii) to (xi) together with either (xii) or (xiii) can take place. Since both $\bar{T}$ and $\bar{T}'$ [$\bar{q}$ and $\bar{q}'$] are standard functionals of the stress [heat flux] for $(\mathcal{B}, \varphi)$, by Def. 5.1 [Def. 5.2] they satisfy equality (5.7) in $\hat{T}$ [(5.8) in $\hat{q}$], where conditions (α) [(β)] in Def. 5.1 and (5.9)-(5.10) hold. Hence the equality

$$(5.11)\ [(5.12)] \qquad \bar{T}(X,\Lambda(\cdot))\,n = \bar{T}'(X,\Lambda(\cdot))\,n \qquad [\bar{q}(X,\Lambda(\cdot))\,n = \bar{q}'(X,\Lambda(\cdot))\,n]$$

holds with $F_1N/|F_1N| = n = F_2N/|F_2N|$—see (x) in § 4.

Hence by the arbitrariness of the unit vector N—see (iii) in § 4—, equality (5.11) [(5.12)] implies the equality $\bar{T}(X, \Lambda(\cdot)) = \bar{T}'(X, \Lambda(\cdot))$ [$\bar{q}(X, \Lambda(\cdot)) = \bar{q}'(X, \Lambda(\cdot))$]. By the arbitrariness of $(X, \Lambda(\cdot))$ in *DS*—see (*a*) and (*b*) above—this equality yields the thesis. q.e.d.

Note that *the condition of frame-indifference involved in the condition* (i) *of* Def. 4.1 *has not been used*; *thus this condition is not necessary to attain the aims of this paper*; and therefore it was written between parentheses.

Next we postulate the existence of a standard functional for the stress and of one for the heat flux. Hence Theorems 5.1-5.2 imply the uniqueness of such functionals—see Corollaries 5.1-5.2 below.

AXIOM 5.1 [5.2]. *If $\mathcal{B}$ is a regular simple body*—see Def. 3.1—, *then there is some standard response functional for the Cauchy stress* [*heat flux*] *in $\mathcal{B}$*—see Defs. 5.1-5.2.

Now Theorem 5.1 [Theorem 5.2] obviously implies the following

COROLLARY 5.1 [5.2]. *Each simple body has a unique standard functional* $\bar{\boldsymbol{T}}$ [$\bar{\boldsymbol{q}}$] *for the stress* [*heat flux*], *which is connected with a given reference configuration.*

Next we show that if

(5.A) *both* $\{\bar{\boldsymbol{T}}, \bar{\boldsymbol{q}}, \bar{e}\}$ *and* $\{\bar{\boldsymbol{T}}', \bar{\boldsymbol{q}}', \bar{e}'\}$ *are generalized sets of response functionals for the simple body* $\mathcal{B}$, *connected with the same reference configuration, where* $\boldsymbol{T}$, $\boldsymbol{T}'$ *and* $\bar{\boldsymbol{q}}$, $\bar{\boldsymbol{q}}'$ *are standard functionals for the stress and the heat flux, respectively, then the difference* $\bar{e} - \bar{e}'$ *between the functionals for the internal energy is a function of the material point;*

that is, if the functionals for the stress and the heat flux are standard, then the functional for the internal energy satisfies a physically expected uniqueness property.

COROLLARY 5.3. *Assume* (5.A) *and let the functionals be of class* C^1. *Then* $\bar{e} - \bar{e}'$ *is a function of the material point.*

PROOF. Assume (5.A). Write the local energy law

$$\rho\dot{e} = \boldsymbol{T}\cdot\operatorname{grad}\boldsymbol{v} + \rho r - \operatorname{div}\boldsymbol{q}$$

first using the set $\{\bar{\boldsymbol{T}}, \bar{\boldsymbol{q}}, \bar{e}\}$, then using the set $\{\bar{\boldsymbol{T}}', \bar{\boldsymbol{q}}', \bar{e}'\}$ and lastly subtract the two equalities which are obtained; we find

$$\rho\dot{E} = \boldsymbol{Z}\cdot\operatorname{grad}\boldsymbol{v} - \operatorname{div}\boldsymbol{Q}\,, \tag{5.10}$$

where $\hat{E} = \bar{e} - \bar{e}'$, $\hat{\boldsymbol{Z}} = \bar{\boldsymbol{T}} - \bar{\boldsymbol{T}}'$, $\hat{\boldsymbol{Q}} = \bar{\boldsymbol{q}} - \bar{\boldsymbol{q}}'$ and E, $\boldsymbol{Z}$, $\boldsymbol{Q}$ are the corresponding Eulerian fields. The material counterpart of the space-time equality (5.10) is

$$\rho_\varphi\dot{E}_\varphi = \boldsymbol{S}\cdot\dot{\boldsymbol{F}} - \operatorname{DIV}\boldsymbol{Q}_\varphi\,, \tag{5.11}$$

where $\boldsymbol{S}$ and $\boldsymbol{Q}_\varphi$ are the Piola-transform of $\boldsymbol{Z}$ and $\boldsymbol{Q}$, respectively. By Corollaries 5.1-5.2 we have $\boldsymbol{S} = \boldsymbol{0}$ and $\boldsymbol{Q}_\varphi = \boldsymbol{0}$. Hence equality (5.11) yields $\dot{E}$v3.004 0 (along any possible process), which implies the thesis of the Lemma. q.e.d.

Now we say that a functional $\bar{e}$ for the internal energy is *standard* if it belongs to a generalized set of response functionals $\{\bar{\boldsymbol{T}}, \bar{\boldsymbol{q}}, \bar{e}\}$ where both the functionals $\bar{\boldsymbol{T}}$ and $\bar{\boldsymbol{q}}$ are standard. Furthermore in this case we say that $\{\bar{\boldsymbol{T}}, \bar{\boldsymbol{q}}, \bar{e}\}$ is a *standard set of functionals*. By Corollaries 5.1-5.3 the standard functionals of any simple body satisfy physically satisfactory uniqueness properties.

6. On primitive notions in thermodynamics.

In the definition of simple body, Def. 3.1, the balance laws (3.4)-(3.6) are not postulated; instead they are used as conditions imposed on the stress, internal energy and heat flux. Indeed *no notions of contact forces, internal energy and heat flux are considered as primitive here* (for the notion of entropy read (M.7-8) in § 1 again). This is the reason why the balance laws cannot be postulated here. Furthermore we postulate Axiom 4.1. By using this axiom we can prove physically satisfactory uniqueness properties for the standard functionals of the stress, heat flux and internal energy. Note that *the notion of forces at a distance* (*body forces*) *used here can be borrowed from papers* [4] *and* [11], *where within theories of various kinds it is defined on purely kinematic notions.*

Therefore this work (implicitly) conforms with the Mach-Painlevè point of view. By assertion (M.5) in § 1, *also the entropy notion can be regarded as defined in the thermo-elastic case.*

In [15], with regard to a certain class of differential materials of complexity one, it is proved that the response function for entropy satisfies the same uniqueness property proved in [14] for thermo-elastic bodies; thus *the entropy notion can be regarded as defined also for some class of dissipative materials.*

By the above italcized assertions we can state that *the present paper shows that, with regard to simple bodies, the number of primitive notions in thermodynamics can be considerably reduced with respect to the usual theories. In more details, to set up a thermodynamic theory for simple bodies, besides the usual purely kinematic notions (described e.g. in* [4]) *and the notion of forces at a distance, which can be regarded as defined, one needs the primitive notion of absolute temperature. For certain materials of the differential type also the notion of entropy can be regarded as defined.*

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