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A Further Glance at Classifiable 1-ary Functions.

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SUMMARY - Si affronta il problema di caratterizzare le teorie complete di k-uple di funzioni 1-arie classificabili secondo Shelah nel caso $k \geqslant 2$.

In [T1] and [T2] we studied classification theory for a unique 1-ary function; in [MPT] this analysis was extended to structures with an endomorphism, while in [T3] we considered functions with arity $\geqslant 2$. So it remains to examine the case of a nonempty set with a k-uple of 1-ary functions where $k \geqslant 2$; our aim here is to characterize the classifiable structures in this context. However there do exist classes of structures for which one can reasonably agree that no classifiability characterization is possible. This is the case, for instance, of binary relations, or, more particularly, of irreflexive symmetric binary relations, e.g. graphs. There are good reasons to believe that to characterize classifiable graphs is as difficult as to characterize all classifiable structures (see [Mr], for example). Then consider any axiomatizable class K of structures, and suppose that there exists a function F mapping any graph (X, R) into a structure of K such that, for all graphs (X, R), (X', R'),

(a) $(X, R) \simeq (X', R')$ iff $F(X, R) \simeq F(X', R')$,

(b) $(X, R) \equiv (X', R')$ iff $F(X, R) \equiv F(X', R')$,

(c) $\mathrm{Th}(X, R)$ is classifiable iff $\mathrm{Th}(F(X, R))$ is.

Say that $\mathrm{Th}(K)$ interprets the theory of graphs if such a function F exists. One can assume that, if $\mathrm{Th}(K)$ interprets the theory of graphs, then no characterization of classifiable structures in K can be expected. Hence the aim of this paper is to consider k-uples $(f_0, \ldots, f_{k-1})$ of 1-ary

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functions with $k \geqslant 2$, and either to characterize classifiable k-uples, or to show that this characterization cannot be given, just because the corresponding theory interprets the theory of graphs. We start from a lucky situation, when $f_0, \ldots, f_{k-1}$ are 1-1; in this case, any complete theory is bounded, hence classifiable. This holds, more generally, if, for any $i < k$, the set of preimages in f_i of any element is finite of bounded power. But we show that a very slight weakening of the previous assumption forbids classification.

We refer to [B] for stability theory. As usual we assume that, for every complete theory T, all models of T are elementary substructures of some big saturated model U. Finally, in order to simplify our notation, we deal only with pairs of 1-ary functions f, g. The results below can be easily extended to any $k > 2$.

1. – This section is devoted to show

THEOREM 1. Let T be a complete theory of two 1-1 1-ary functions f, g. Then T is classifiable.

PROOF. Let C denote the connection relation, namely the transitive closure ot the binary relation $= \cup f \cup f^{-1} \cup g \cup g^{-1}$. C is an equivalence relation and determines a partition of any model M of T into equivalence classes, called connected components of M. As f and g are 1-1, for all $a \in M$, $C(a, M) \subseteq \operatorname{dcl}(a)$ (= the definable closure of a). In particular $C(a, M) = C(a, U)$. Let $a, b \in U - M$ satisfy $\operatorname{tp}(a|\emptyset) = \operatorname{tp}(b|\emptyset)$. Then there exists an automorphism of U mapping a into b; as $a, b \notin M$, both $C(a, U)$ and $C(b, U)$ are disjoint from M, hence we can assume that the previous automorphism fixes M pointwise. Consequently $\operatorname{tp}(a|M) = \operatorname{tp}(b|M)$. It follows that any nonalgebraic 1-type over M is uniquely determined by its restriction to $\emptyset$. This shows that, for every model M of T, $|S_1(M)| \leqslant \max\{|M|, 2^{\aleph_0}\}$, hence T is superstable. Moreover any nonalgebraic 1-type over a model M of T is not orthogonal to $\emptyset$, namely T is bounded. Consequently T is classifiable. ■

REMARK. The previous theorem still holds under the weaker assumption

(*) there is $N \in \omega$ such that, for any $a \in U$, $|\{b \in U\colon f(b) = a \text{ or } g(b) = a\}| \leqslant N$.

For, one defines as before the connection relation and one notices that, if (*) holds, then, for all $a \in C$, $C(a, U) \subseteq \operatorname{acl}(a)$ (= the algebraic closure of a; we cannot expect now $C(a, U) \subseteq \mathrm{dcl}(a)$). This lets us apply the previous machinery. ■

Notice that, among the theories satisfying the assumptions of Theorem 1, one can find also the theory T_n of free algebras over $\{f, g\}$ with $n \leqslant \omega$ indecomposable elements (where «a indecomposable» means «$a \notin \mathrm{im}\, f \cup \mathrm{im}\, g$»). T_n can be axiomatized first by specifying n, and then by imposing

(1) f, g are 1-1;

(2) $\forall v(s(v) \neq t(v)) \in T_n$ for all terms s, t of $\{f, g\}$ with $s \neq t$.

Even in this simple case $|S_1(\emptyset)| = 2^{\aleph_0}$, and so T_n is not ω-stable. For, take any $s \in \{f, g\}^\omega$ and consider the following set of formulas

$$p(s) = \{v \in \mathrm{im}\, s(0) \cdot \ldots \cdot s(m) \colon m \in \omega\}$$

($s(0) \cdot \ldots \cdot s(m)$ denotes here the composition of $s(0), \ldots, s(m)$). $p(s)$ is consistent and can be enlarged to a type over $\emptyset$; but, if $s, s' \in \{f, g\}^\omega$ and $s \neq s'$, then $p(s) \cup p(s')$ is contradictory. ■

2. – Replace now the assumptions (1), (2) above with (1′), (2) where

(1′) g is 1-1.

So the equivalence relation $E(f)$ such that, for all $a, b \in U$, $(a, b) \in E(f)$ iff $f(a) = f(b)$ may admit classes of power $\geqslant 2$. We claim that, even in this case, the theory of pairs of functions f, g interprets the theory of graphs.

LEMMA 1. The theory of structures (M, E, h) where

- ⋆ E is an equivalence relation whose classes have power $\geqslant 2$,
- ⋆ h is a 1-1 1-ary function,
- ⋆ for all $a \in U$ and $k, l \in \omega$ with $k \neq l$, $(h^k(a), h^l(a)) \notin E$,

interprets the theory of graphs.

PROOF. Let (X, R) be a graph, so R is an irreflexive symmetric binary relation in X. Define a structure $F(X, R) = (\bar{X}, E, h)$ in the following way: let * be a bijection of X onto a copy in some new world, and put

- ⋆ $X_0 = X \cup \{(x, y), (x, y^*) \colon x, y \in X, (x, y) \in R\}$, $\bar{X} = X_0 \times \omega \times 2$;

★ E is the equivalence relation such that, for all $x \in X$,

$$E((x, 0, 0), \bar{X}) =$$

$$= \{(x, 0, e), (x, y, 0, e), (x, y^*, 0, e) \colon y \in X,\ (x, y) \in R,\ e \in 2\}.$$

while, for $n \in \omega$, $n \neq 0$, $y \in X$, $(x, y) \in R$,

$$E((x, n, 0), \bar{X}) = \{(x, n, e) \colon e \in 2\},$$

$$E((x, y, n, 0), \bar{X}) = \{(x, y, n, e) \colon e \in 2\},$$

$$E((x, y^*, n, 0), \bar{X}) = \{(x, y^*, n, e) \colon e \in 2\}.$$

★ h is the 1-ary function such that, for all $x, y \in X$ and $n \in \omega$, $e \in 2$,

$$h(x, n, e) = (x, n + 1, e),$$

while, for $(x, y) \in R$,

$$h(x, y^*, n + 1, e) = (x, y^*, n, e),$$

$$h(x, y, n, e) = (x, y, n + 1, e),$$

$$h(x, y^*, 0, e) = (y, x, 0, e).$$

Clearly any E-class has power $\geqslant 2$, h is 1-1 and, for all $a \in \bar{X}$ and $k, l \in \omega$ with $k \neq l$, $(h^k(a), h^l(a)) \notin E$. If X is infinite, then $|X| = |\bar{X}|$. Furthermore (X, R) can be recovered from $(\bar{X}, E, h)$. In fact X is given by the set of E-classes of elements in $\bar{X} - \operatorname{im} h$ so identifying any element $x \in X$ with the E-class of $(x, 0, 0)$ and, for $x, y \in X$,

$$(x, y) \in R \Leftrightarrow (\bar{X}, E, h) \models \exists v(E(v, (x, 0, 0)) \wedge E(h(v), (y, 0, 0))).$$

This obviously implies

(*a*) for all graphs (X, R), (X', R'), $(X, R) \simeq (X', R')$ iff $F(X, R) \simeq F(X', R')$.

We claim that the same happens with respect to $\equiv$, i.e.

(*b*) for all graphs (X, R), (X', R'), $(X, R) \equiv (X', R')$ iff $F(X, R) \equiv F(X', R')$.

($\Leftarrow$) follows from the fact that (X, R) is interpretable inside $F(X, R)$. ($\Rightarrow$) needs the following preliminary remark. For any partial isomorphism p of (X,R) in (X',R'), define a map $\bar{p}$ by putting

$$\operatorname{dom}\bar{p} = \{(x, n, e), (x, y, n, e), (x, y^*, n, e): \\ x, y \in \operatorname{dom} p,\ (x, y) \in R,\ n \in \omega,\ e \in 2\},$$

$$\operatorname{im}\bar{p} = \{(x', n, e), (x', y', n, e), (x', y'^*, n, e): \\ x', y' \in \operatorname{im} p,\ (x', y') \in R',\ n \in \omega,\ e \in 2\}$$

and, for $x, y \in \operatorname{dom} p$, $(x, y) \in R$ (so $(p(x), p(y)) \in R'$), $n \in \omega$, $e \in 2$,

$$\bar{p}(x, n, e) = (p(x), n, e),$$

$$\bar{p}(x, y, n, e) = (p(x), p(y), n, e),$$

$$\bar{p}(x, y^*, n, e) = (p(x), p(y)^*, n, e);$$

then $\bar{p}$ is a partial isomorphism of $F(X,R)$ in $F(X',R')$. Now assume $(X, R) \equiv (X', R')$; equivalently, there is a decreasing sequence $I = (I_m)_{m \in \omega}$ of sets of partial isomorphisms of (X,R) in X$',R')$ such that $I: (X, R) \simeq_\omega (X', R')$ (see [F]). Put $\overline{I_m} = \{\bar{p}: p \in I_m\}$ for all $m \in \omega$, $\bar{I} = (\overline{I_{2m}})_{m \in \omega}$. Then $\bar{I}$: $F(X, R) \simeq_\omega F(X', R')$, and $F(X, R) \equiv F(X', R')$.

Owing to (*b*), for every complete theory $T = \operatorname{Th}(X,R)$ of graphs, we can define a complete theory $F(T) = \operatorname{Th}(F(X, R))$; any structure $F(X',R')$ with $(X', R') \models T$ is a model of $F(T)$. Conversely any model (M, E, h) of $F(T)$ decomposes as the disjoint union of $F(X,R)$ for a suitable graph (X,R) defined inside (M,E,h) in the way we suggested before, and (possibly) some copies of $(\boldsymbol{Z} \times 2, E_0, h_0)$ where, for all $n, n' \in \boldsymbol{Z}$ and $e, e' \in 2$,

$$((n, e), (n', e')) \in E_0 \Leftrightarrow n = n', \qquad h_0(n, e) = (n+1, e).$$

At this point it is fairly clear that, for any cardinal λ, T has (up to isomorphism) less than 2^λ models of power $\leqslant \lambda$ iff $F(T)$ does. Equivalently

(*c*) T is classifiable iff $F(T)$ is.

The class of triples (M,E,h) defining a graph in this way is axiomatizable within the class of all triples in the statement of the lemma. So these triples interpret the theory of graphs. ■

LEMMA 2. The theory of two 1-ary functions f, g satisfying (1′), (2) interprets the theory of structures (M,E,h) where

⋆ E is an equivalence relation with classes of power $\geqslant 2$;

★ h is a 1-1 1-ary function;

★ for all $k, l \in \omega$ with $k \neq l$, for all $a \in U$, $(h^k(a), h^l(a)) \notin E$.

PROOF. For every (M, E, h), build a new structure $F(M, E, h) = (\bar{M}, f, g)$ in the following way.

★ $\bar{M} = M \cup \mathcal{F}(M/E)$ where $\mathcal{F}(M/E)$ denotes the $\{f, g\}$-algebra free over M/E (so the elements of M/E are $\{f, g\}$-indecomposable in $\mathcal{F}(M/E)$).

★ f restricted to M is the canonical projection of M onto M/E, and g restricted to M is h; f and g are defined in the obvious way in $\mathcal{F}(M/E)$.

$(\bar{M}, f, g)$, satisfies (1′) and (2). (1′) is clear, so let us consider (2). Let $a \in \bar{M}$, s, t be terms of $\{f, g\}$ with $s \neq t$; we claim $s(a) \neq t(a)$. If f occurs in both s and t, then we can restrict ourselves to $s = fg^k$, $i = fg^l$ where $k \neq l$, and the assumptions on h imply our claim; the case that f occurs in s but does not occur in t, or viceversa, is quite obvious; finally, if f occurs neither in s nor in t, then the claim again follows from the assumptions on h. (M, E, h) can be recovered from $(\bar{M}, f, g)$ by putting

$$M = \{a \in \bar{M}: \exists w(w \neq a \wedge f(w) = f(a))\},$$

$$E = E(f) \cap M^2,$$

and $h =$ the restriction of g to M. This implies

(*a*) For all (M, E, h), (M', E', h') as above, $(M, E, h) \simeq (M', E', h')$ iff $F(M, E, h) \simeq F(M', E', h')$.

The same happens with respect to elementary equivalence.

(*b*) For all (M, E, h), (M', E', h'), $(M, E, h) \equiv (M', E', h')$ iff $F(M, E, h) \equiv F(M', E', h')$.

($\Leftarrow$) holds because (M, E, h), (M', E', h') are 0-definable in the same way inside $F(M, E, h)$, $F(M', E', h')$ respectively. ($\Rightarrow$) can be shown by using the same techniques of Lemma 1. In fact, for any partial isomorphism p of (M, E, h) in (M', E', h'), one can define $\bar{p}$ as follows:

$$\operatorname{dom} \bar{p} = \operatorname{dom} p \cup \mathcal{F}(\operatorname{dom} p/E),$$

$$\operatorname{im} \bar{p} = \operatorname{im} p \cup \mathcal{F}(\operatorname{im} p/E),$$

the restriction of $\bar{p}$ to $\operatorname{dom} p$ is p, for all $a \in \operatorname{dom} p$ and s, t $\{f, g\}$-terms, $\bar{p}(s(a/E)) = s(p(a)/E')$. $\bar{p}$ is a partial isomorphism of $F(M, E, h)$ in $F(M', E', h')$. Now assume $(M, E, h) \equiv (M', E', h')$, so there is a decreasing sequence $I = (I_m)_{m \in \omega}$ of sets of partial isomorphisms such that $I\colon (M, E, h) \simeq_\omega (M', E', h')$; put $\overline{I_m} = \{\bar{p}\colon p \in I_m\}$ for all $m \in \omega$, $\bar{I} = (\overline{I_{2m}})_{m \in \omega}$. Clearly $\bar{I}\colon (\bar{M}, f, g) \simeq_\omega (\bar{M}', f', g')$, so $F(M, E, h) \equiv F(M', E', h')$ just as claimed.

At this point, for every complete theory $T = \mathrm{Th}(M, E, h)$, one can define a complete theory $F(T) = \mathrm{Th}(F(M, E, h))$; any structure $F(M', E', h')$ with $(M', E', h') \models T$ is a model of $F(T)$, and any model of $F(T)$ decomposes as the union of a structure $F(M', E', h')$ and (possibly) some copies of the $\{f, g, f^{-1}, g^{-1}\}$-algebra free over one generator. Moreover, if M is infinite, then $|M| = |\bar{M}|$. Thus, just as in Lemma 1, one obtains that T is classifiable iff $F(T)$ is (hence (c)).

The class of pairs of 1-ary functions defining structures (M, E, h) as in Lemma 1 in the above way is axiomatizable. This accomplishes the proof of Lemma 2. ■

Lemmas 1 and 2 imply

THEOREM 2. The theory of two 1-ary functions f, g satisfying (1′), (2) interprets the theory of graphs.

3. – In the previous section we have considered structures with an equivalence relation E and a 1-ary function h; we continue here that analysis. First we give a negative result.

PROPOSITION 1. The theory of structures (M, E, h) where

⋆ E is an equivalence relation,

⋆ h is an endomorphism of (M, E)

interprets the theory of graphs.

PROOF. Let (X, R) be a graph, put

$$D = \{(x, x)\colon x \in X\},$$

$$U = \{\{x, y\}\colon x, y \in X,\ (x, y) \in R\}$$

(= the set of edges of (X, R)). Define a new structure $F(X, R) = (\bar{X}, E, h)$ by setting:

⋆ $\bar{X} = X \cup R \cup D \cup U$;

⋆ E is the equivalence relation whose classes are $\{(x, x), (x, y)\colon y \in X,\ (x, y) \in R\}$ for any $x \in X$, and $X \cup U$;

$\star$ $h(x, x) = x$ for all $x \in X$, $h(x, y) = \{x, y\}$ for all $(x, y) \in R$, h is identical anywhere else.

Thus h is an endomorphism of $(\bar{X}, E)$. Furthermore (X, R) can be interpreted in $(\bar{X}, E, h)$, since X can be recovered as the set of classes $E(a, \bar{X})$ with a h-indecomposable (namely by identifying any $x \in X$ with $E((x, x), \bar{X})$) and, for all $x, y \in X$,

$$(x, y) \in R \Leftrightarrow (\bar{X}, E, h) \models$$

$$\models \exists v \exists w (h(v) = h(w) \wedge E(v, (x, x)) \wedge E(w, (y, y))) \wedge (x, x) \neq (y, y).$$

Just as in Lemma 1 one can show that, for all graphs (X, R), (X', R'),

(*a*) $(X, R) \simeq (X', R')$ iff $F(X, R) \simeq F(X', R')$;

(*b*) $(X, R) \equiv (X', R')$ iff $F(X, R) \equiv F(X', R')$.

(*b*) allows to define, for any complete theory $T = \mathrm{Th}(X, R)$ of graphs, a new theory $F(T)$ such that the models of $F(T)$ are just the structures $F(X', R')$ where $(X', R') \models T$. So

(*c*) T is classifiable iff $F(T)$ is.

As the class of structures (M, E, h) interpreting graphs in this way is axiomatizable, the claim follows. ■

A positive result can be shown about automorphisms of equivalence relations.

PROPOSITION 2. Let T be a complete theory of structures (M, E, h) where

$\star$ E is an equivalence relation,

$\star$ h is an automorphism of (M, E);

then T is classifiable.

PROOF. For every model (M, E, h) of T, put $\bar{M} = M/E$ and, for all $a \in M$, $\bar{h}(E(a, M)) = E(h(a), M)$. Then $\bar{h}$ is a bijection of $\bar{M}$ onto $\bar{M}$. Furthermore the isomorphism type of (M, E, h) fully determines the isomorphism type of $(\bar{M}, \bar{h})$. When (M, E, h) ranges over the models of T, the structures $(\bar{M}, \bar{h})$) obtained as before are elementarily equivalent. Let $\bar{T}$ denote their theory. We know that $\bar{T}$ is classifiable (see [T1], for instance); any model of $\bar{T}$ is a union of orbits of $\bar{h}$. So, given a model (M, E, h) of T of power $\leqslant \lambda = \aleph_\alpha$, the first invariant of the isomorphism type of (M, E, h) is just the isomorphism type of $(\bar{M}, \bar{h})$ (where

$|\bar{M}| \leqslant \lambda$). Hence the problem reduces to: given a model of $\bar{T}$ of power $\leqslant \lambda$, how many models $(M, \underline{E}, \underline{h})$ of T of power $\leqslant \lambda$ determine (up to isomorphism) a structure $(\bar{M}, \bar{h})$ isomorphic to it? Notice that, for all $a \in M$ and $n \in \boldsymbol{Z}$,

$$|E(a, M)| = |E(h^n(a), M)|.$$

So it is sufficient to specify, for every isomorphism type γ of orbits in $(\bar{M}, \bar{h})$ and for every cardinal $\kappa \leqslant \lambda$, how many elements $a \in M$ satisfy

$$|E(a, M)| = \kappa, \qquad (\{E(h^n(a), M): n \in \boldsymbol{Z}\}, \bar{h}) \in \gamma.$$

One gets at most $|\alpha + \omega|^{|\alpha + \omega|}$ possibilities. ■

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