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Finitely Generated Soluble Groups with an Engel Condition on Infinite Subsets.

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Dedicated to Professor Cesarina Tibiletti Marchionna for her 70th birthday

1. Introduction.

B. H. Neumann proved in [7] that a group G is centre-by-finite if and only if in every infinite subset X of G there exist two different elements that commute. This answered to a question posed by Paul Erdös. Extensions of problems of this type are studied in [1], [2], [4], [5], [6], and recently in [9].

For example in [6] J. C. Lennox and J. Wiegold studied the class $N(\infty)$ of groups G such that in every infinite subset X of G there are two elements x, y such that $\langle x, y\rangle$ is nilpotent, and proved that a finitely generated soluble group is in $N(\infty)$ if and only if it is finite-by-nilpotent.

We denote by $E(\infty)$ the class of groups G such that, for every infinite subset X of G, there exist different $x, y \in X$ such that $[x, {}_k y] = 1$ for some $k = k(x, y) \geqslant 1$.

If the integer k is the same for any infinite subset X of G, we say that G is in the class $E_k(\infty)$.

We prove the following

THEOREM 1. *Let G be a finitely generated soluble group. Then $G \in E(\infty)$ if and only if G is finite-by-nilpotent.*

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Moreover, if $R(G)$ denotes the characteristic subgroup of G consisting of all right 2-Engel elements of G, we show that

THEOREM 2. *Let G be a finitely generated soluble group. Then $G \in E_2(\infty)$ if and only if $G/R(G)$ is finite.*

Our notation and terminology follow [8]. In particular, if x and y are elements of a group G and k is a non-negative integer, the commutator $[x, {}_k y]$ is defined by the rules

$$[x, {}_0 y] = x \quad \text{and} \quad [x, {}_{k+1} y] = [[x, {}_k y], y].$$

2. Proofs.

We start with some preliminary Lemmas.

LEMMA 2.1. *Let $G \in E(\infty)$ and let A be an infinite normal abelian subgroup of G.*

Then, for every $x \in G$, there exists a subgroup $B \leqslant A$ (depending on x) of finite index in A and such that, for every $b \in B$, $[b, {}_{k(b)} x] = 1$ for some $k(b) \geqslant 1$ (depending on b).

PROOF. Let $x \in G$. If b is in G, call $(*)$ the following property:

$(*)$ *there exists an integer $k(b) \geqslant 1$ such that $[b, {}_{k(b)} x] = 1$.*

Put $B = \{b \in A / b$ *satisfies* $(*)\}$. For arbitrary $b, c \in B$ we have $[b, {}_n x] = 1 = [c, {}_m x]$ for suitable integers n, m. Write $d = \max\{m, n\}$, then $[b^{-1} c, {}_d x] = [b, {}_d x]^{-1} [c, {}_d x]$, since A is abelian and normal in G. Therefore B is a subgroup of A.

Assume by contradiction that $|A : B|$ is infinite. Then there exists a sequence $(a_i)_{i \in \mathbb{N}}$ of elements of A such that $a_i^{-1} a_j \notin B$ for any $i \neq j$. Thus the set $\{x a_i / i \in \mathbb{N}\}$ is infinite, and there exist an integer $k \geqslant 1$ and $i \neq j$ such that $[x a_i, {}_k (x a_j)] = 1$, since $G \in E(\infty)$.

Hence $[[x, a_j][a_i, x], {}_{k-1}(x a_j)] = 1$, and $[a_j^{-1} a_i, {}_k x] = 1$; therefore $a_j^{-1} a_i \in B$, a contradiction. ■

LEMMA 2.2. *Let $G \in E(\infty)$ be a finitely generated soluble group. Suppose that there exists an infinite normal abelian subgroup A of G with G/A polycyclic. Then $A \cap \zeta(G)$ is infinite.*

PROOF. We show that $A \cap \zeta(H)$ is infinite, for every normal subgroup H of G, with $H \geqslant A$ and H/A polycyclic.

Put $H/A = \langle h_1 A, \dots, h_s A\rangle$. By Lemma 2.1 there exists a subgroup B of A, with $|A:B|$ finite, and such that for every $b \in B$ there is a positive integer $k(b)$ for which $[b, {}_{k(b)}h_i] = 1$, for any $i \in \{1, \dots, s\}$.

Write l the derived length $l(H/A)$ of H/A and argue by induction on l.

If $l = 1$, then H/A is abelian and $[c, [x, y]] = 1$, for any $c \in A$ and $x, y \in H$. Thus $c^{xy} = c^{yx}$ and $[c, x, y] = (c^{-1}c^x)^{-1}(c^{-1}c^x)^y = (c^x)^{-1}c(c^y)^{-1}c^{xy} = (c^y)^{-1}c(c^x)^{-1}c^{yx} = [c, y, x]$ for every $c \in A$, $x, y \in H$.

Hence $[c, y, x] = [c, x, y]$ for any $c \in A$, $x, y \in H$. Now let $b \in B$ and put $n = s\ k(b)$. Let $h_{i_1}, \dots, h_{i_n}$ be arbitrary elements of $\{h_1, \dots, h_s\}$. Then, for any $a \in A$, $[a, h_{i_1}, \dots, h_{i_n}] = [a, h_{i_{\sigma(1)}}, \dots, h_{i_{\sigma(n)}}]$ for every permutation σ of $\{1, \dots, n\}$.

Furthermore at least k of the h_{i_j} must be equal to the same $h_i \in \{h_1, \dots, h_s\}$. Hence we get $[b, h_{i_1}, \dots, h_{i_n}] = [b, \underbrace{h_i, \dots, h_i}_{k(b)}, h_{j_{n-k(b)}}] = 1$.

That holds for any $h_{i_1}, \dots, h_{i_n} \in \{h_1, \dots, h_s\}$, so that $b \in \zeta_n(H)$, the n-th centre of H. Thus for every $a \in B$ there exists a positive integer m such that $a \in \zeta_m(H)$. Then $a^G \leq \zeta_m(H)$ since H is normal in G. But G satisfies Max$n$, the maximal condition on normal subgroups, because it is a finitely generated abelian-by-polycyclic group (see [8], part I, Theorem 5.34). Hence $B^G = b_1^G b_2^G \dots b_v^G$, for some finite subset $\{b_1, b_2, \dots, b_v\}$ of B. Therefore there exists a positive integer i such that $B^G \leq A \cap \zeta_i(H)$, and $A \cap \zeta_i(H)$ is infinite. From that we easily get that $A \cap \zeta(H)$ is infinite, as required.

Now assume $l > 1$. Then $H'A$ is normal in G, $(H'A)/A$ is polycyclic and $l((H'A)/A) < l$. Therefore, by induction, $A \cap \zeta(H'A)$ is infinite. Write $C = A \cap \zeta(H'A)$. Then, arguing as before, we get, for any $c \in C$, $[c, h_{i_1}, \dots, h_{i_t}] = [c, h_{i_{\sigma(1)}}, \dots, h_{i_{\sigma(t)}}]$ for any $t \geq 2$, $h_{i_1}, \dots, h_{i_t} \in \{h_1, \dots, h_s\}$, and for any permutation σ of $\{1, \dots, t\}$. Furthermore, with $D = B \cap C$, we have that D is infinite and for any $d \in D$ there exists a positive integer $m = m(d)$ such that $[d, h_{i_1}, \dots, h_{i_m}] = 1$, for any $h_{i_1}, \dots, h_{i_m} \in \{h_1, \dots, h_s\}$. Hence $d \in \zeta_m(H)$. As before, from $D^G = d_1^G \dots d_l^G$ for some finite subset $\{d_1, \dots, d_l\}$ of D, we get $D^G \leq \zeta_j(H) \cap A$ for a suitable j. Hence $\zeta_j(H) \cap A$ is infinite, and $\zeta(H) \cap A$ is infinite, as required. ■

PROOF OF THEOREM 1. Let $G \in E(\infty)$ be a finitely generated infinite soluble group. By induction on the derived length $l = l(G)$, we show that $\zeta(G)$ is infinite. From that the result will follow, since we get $G/\bar{\zeta}(G)$ finite, where $\bar{\zeta}(G)$ is the hypercentre of G, and $G/\zeta_i(G)$ finite for some $i \in \mathbb{N}$, since G is finitely generated. Then G is finite-by-nilpotent by a result of P. Hall (see [3]).

If $l = 1$, the result is trivial. Assume $l > 1$, and write $A = G^{(l-1)}$ the last non-trivial term of the derived series of G. Then by induction every infinite quotient of G/A has an infinite centre, so that G/A is finite-by-nilpotent and hence polycyclic.

If A is finite, then G is finite-by-nilpotent, and $G/\zeta_i(G)$ is finite for some $i \in \mathbb{N}$ (see [3]), so that $\zeta(G)$ is infinite.

If A is infinite, then Lemma 2.2 applies, and $A \cap \zeta(G)$ is infinite. Hence again $\zeta(G)$ is infinite.

Conversely, assume that G is a finitely generated finite-by-nilpotent soluble group.

Then, by a result of P. Hall (see [3]), there exists $k \in \mathbb{N}$ such that $G/\zeta_k(G)$ is finite. Hence, if X is an infinite subset of G, there exist $x, y \in X$ with $x \neq y$ and $x\zeta_k(G) = y\xi_k(G)$. Thus $y = xa$, with $a \in \zeta_k(G)$, and we get $1 = [a, {}_kx] = [xa, {}_kx] = [y, {}_kx]$, as required. ■

Notice that we have shown that *if a finitely generated soluble group G is in $E(\infty)$, then $G \in E_k(\infty)$ for some $k \geqslant 1$.*

Proof of Theorem 2. Suppose that $G \in E_2(\infty)$ is infinite. Then G is finite-by-nilpotent by Theorem 1, and $G/\zeta_i(G)$ is finite, for a suitable $i \in \mathbb{N}$. Thus $\zeta(G)$ is infinite. Furthermore G satisfies the maximal condition on subgroups. Let A be a subgroup of G maximal with respect to being normal, torsion-free and contained in some $\zeta_j(G)$, $j \in \mathbb{N}$.

Then $\zeta(G/A)$ is finite, and G/A is finite by Theorem 1.

We show that $\zeta(G/(A \cap R(G)))$ is finite, so that $G/(A \cap R(G))$ is finite by Theorem 1 and $G/R(G)$ is finite, as required.

Assume by contradiction that there exists $a(A \cap R(G)) \in \zeta(A/(A \cap \cap R(G)))$, $a(A \cap R(G))$ torsion-free.

Then $[a, b] \in A \cap R(G)$ for every $b \in G$. Hence $\langle [a, b] \rangle^G$ is abelian, $[a, b, a, a] = 1 = [a, b, b, b]$. Thus, by induction on i, it is easy to verify that, for any $i \in \mathbb{N}$,

1) $[a^i, b, a] = [a, b, a]^i$,

2) $[a^i, b] = [a, b]^i [a, b, a]^{i(i-1)/2}$,

3) $[a, b^i] = [a, b]^i [a, b, b]^{i(i-1)/2}$.

Furthermore we have

4) $[a, b, a, b] = [a, b, b, a]$.

For, from $[a, b] \in R(G)$ it follows $[a, b, a, b] = [a, b, b, a]^{-1}$, moreover, from $[a, b]^{ab} = [a, b]^{ba}$ it follows $[a, b, ab] = [a, b, ba]$ and $[a, b, b] \cdot [a, b, a][a, b, a, b] = [a, b, a][a, b, b][a, b, b, a]$, so that $[a, b, a, b] = [a, b, b, a]$ and $[a, b, b, a]^2 = 1$. Thus $[a, b, b, a] = 1$, since A is torsion-free.

Finally, from 4) and 2) we get easily

5) $[a^i, b, b] = [a, b, b]^i$, for any $i \in \mathbb{N}$.

Now consider the infinite set $\{a^i b / i \in \mathbb{N}\}$. Then there exist $i, j \in \mathbb{N}$, with $i \neq j$, such that

$$1 = [a^i b, a^j b, a^j b] = [[a^i, b]^b [b, a^j]^b, a^j b] =$$

$$= [[a^i, b][b, a^j], ba^j] = [a^i, b, ba^j][b, a^j, ba^j] =$$

$$= [a, b, a]^{ij} [a, b, b]^i [a, b, a]^{-j^2} [a, b, b]^{-j}.$$

Hence

$$(*) \qquad 1 = [a, b, a]^{ij} [a, b, b]^i [a, b, a]^{-j^2} [a, b, b]^{-j}.$$

Therefore $[a, b, a^{ij - j^2} b^{i-j}] = 1$.

Write $\alpha = ij - j^2$, $\beta = i - j$. Then $[a, b, a^\alpha b^\beta] = 1$, and

$$[a, a^\alpha b^\beta, a^\alpha b^\beta] = [a, b^\beta, a^\alpha b^\beta] = [[a, b]^\beta, a^\alpha b^\beta] = 1.$$

Hence, with $c = a^\alpha b^\beta$, we have $[a, c, c] = 1$.

Arguing on a and c as before on a and b, we get

$$[a, c, a]^{hk} [a, c, c]^h [a, c, a]^{-k^2} [a, c, c]^{-k} = 1 = [a, c, a]^{hk - k^2},$$

for some $h, k \in \mathbb{N}$, $h \neq k$.

Then $[a, c, a] = 1$, since A is torsion-free, so that $1 = [a, b^\beta, a] = [[a, b]^\beta, a] = [a, b, a]^\beta$, and $[a, b, a] = 1$, again since A is torsion-free.

Hence, by $(*)$, $[a, b, b]^{i-j} = 1$ and $[a, b, b] = 1$. That holds for every $b \in G$, so $a \in R(G)$.

From $a^s \in A$ for some $s \in \mathbb{N}$ it follows $a^s \in A \cap R(G)$, a contradiction since $a(A \cap R(G))$ is torsion-free.

Conversely, if $G/R(G)$ is finite, then $G \in E_2(\infty)$ arguing as in Theorem 1. ■.

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