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Harnack's Inequality for Quasilinear Elliptic Equations with Coefficients in Morrey Spaces.

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1. Introduction.

In recent years many papers have been devoted to the study of the regularity properties of the solutions to linear second order elliptic equations under very general assumptions on the lower order terms (see e.g. [1], [3], [4], [5], [9]).

These Authors study an equation of the form:

$$Lu + Vu = 0 \tag{1.1}$$

where $Lu = -(a_{ij} u_{x_i})_{x_j}$ for some uniformly elliptic measurable matrix $A = (a_{ij}(x))$ (indeed in [1], [5], [9] $a_{ij}(x) = \delta_{ij}$) and V belongs to the Stummel-Kato class or some closely related Morrey space (see [4] or [12] for comparisons).

We stress that these assumptions on the coefficient V are weaker than those used by [6] and [10] in their study of (1.1).

In this paper we begin to study the same kind of problem for a quasilinear equation of the form:

$$\operatorname{div} A(x, u, u_x) = B(x, u, u_x) . \tag{1.2}$$

Precisely, under structure conditions which are the same as in the classical work [8], we prove the local boundedness and an Harnack's inequality for the weak solutions of (1.2).

The pattern of the proof, which we only sketch, follows Serrin's work very closely, the only novelty consisting in the spaces in which we take the coefficients. These spaces are classical Morrey spaces which *properly* contain the L^p spaces in Serrin's work.

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This improvement on [8] has been made possible by some imbedding inequalities which we establish in Section 2 (essentially relying on a celebrated Fefferman's inequality). Also in Section 2 some comparison with Serrin's hypotheses may be found. Section 3 and 4 are devoted to the sketch of the proof of the local boundedness and Harnack's inequality respectively.

2. Structure hypotheses and preliminary results.

Let Ω be a bounded open subset of $\mathbb{R}^n$. Let

$$A(x, u, p) \colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

and

$$B(x, u, p) \colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

two continuous functions such that

$$\begin{cases} |A(x, u, p)| \leq a|p|^{\alpha-1} + b|u|^{\alpha-1}, \\ |B(x, u, p)| \leq c|p|^{\alpha-1} + d|u|^{\alpha-1}, \\ p \cdot A(x, u, p) \geq |p|^{\alpha} - d|u|^{\alpha}, \end{cases} \tag{2.1}$$

for a.e. $x \in \Omega$ $\forall u \in \mathbb{R}$, $\forall p \in \mathbb{R}^n$. Here α is a given number in $]1, n[$, a is a positive constant, and the functions $b(x)$, $c(x)$ and $d(x)$ are such that:

$$\begin{cases} b(x) \in L^{q,n-\alpha} & q > \dfrac{\alpha}{\alpha - 1} > 1, \\ c(x) \in L^{q,n-1} & q > \alpha > 1, \\ d(x) \in L^{q,n-\alpha} & q > 1. \end{cases} \tag{2.2}$$

We recall that for $p \in]1, +\infty[$, $\lambda \in]0, n[$, $L^{p,\lambda} = L^{p,\lambda}(\Omega)$ denotes the classical Morrey space of the functions $f \in L^p(\Omega)$ satisfying

$$\sup_{\substack{x \in \Omega \\ \rho > 0}} \frac{1}{\rho^{\lambda}} \int_{B(x,\rho) \cap \Omega} |f(y)|^p \, dy = \|f\|_{p,\lambda}^p < +\infty \; (^1).$$

(1) Here $B(x, \rho)$ is the ball centered at x with radius ρ. Whenever x is not relevant we will write $B(\rho)$.

We will say that $u \in H^{1,\alpha}_{\text{loc}}(\Omega)$ is a local solution of (1.2) in Ω if

$$\int_\Omega \{A(x, u(x), u_x(x))\, \varphi_x(x) + B(x, u(x), u_x(x))\, \varphi(x)\}\, dx = 0 \qquad \forall \varphi \in C_0^\infty(\Omega) \tag{2.3}$$

In the following we will use C. Fefferman's inequality (see [2]):

THEOREM 1. *Let* $1 < p < n$, $1 < r \leqslant n/p$, $f \in L^{r,n-pr}$. *Then*

$$\int_{\mathbb{R}^n} |\varphi(x)|^p f(x)\, dx \leqslant c\|f\|_{r,\,n-pr} \int_{\mathbb{R}^n} |\nabla\varphi|^p\, dx \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^n)$$

Here c *depends on* n *and* p *only.*

Using Theorem 1 we can prove the following lemmas:

LEMMA 1. *Let* $q > 1$ *and* $d \in L^{q,\,n-\alpha}$. *Then there exist* $\varepsilon > 0$ *and* $r_1 > 1$ *such that*

$$\int_\Omega d(x)\,[\varphi(x)]^\alpha\, dx \leqslant \tag{2.4}$$

$$\leqslant K_1 \|d^{\alpha/(\alpha-\varepsilon)}\|^{(\alpha-\varepsilon)/\alpha}_{r_1, n-\alpha r_1} \left(\int_\Omega \varphi^\alpha\, dx\right)^{\varepsilon/\alpha} \left(\int_\Omega \varphi_x^\alpha\, dx\right)^{(\alpha-\varepsilon)/\alpha} \qquad \forall \varphi \in C_0^\infty(\Omega)$$

where $K_1 = K_1(\alpha, r_1, \operatorname{diam}\Omega)$.

PROOF. We start by showing that we can find $\varepsilon > 0$ and $r_1 > 1$ such that $d^{\alpha/(\alpha-\varepsilon)} \in L^{r_1,\, n-\alpha r_1}$.

Indeed define $r_1 = q(\alpha - \varepsilon)/\alpha$ with $\varepsilon > 0$ such that $r_1 > 1$.

Than we have

$$\int_{B(\rho)} (d^{\alpha/(\alpha-\varepsilon)})^{r_1}\, dx = \int_{B(\rho)} d^q\, dx \leqslant C\rho^{n-\alpha} = C\rho^{n-\alpha r_1 + \alpha r_1 - \alpha} \leqslant$$

$$\leqslant C(\operatorname{diam}\Omega)^{\alpha(r_1-1)} \rho^{n-\alpha r_1} = C\rho^{n-\alpha r_1}.$$

Using Theorem 1 and Hölder's inequality we obtain:

$$\int_\Omega d(x)\,[\varphi(x)]^\alpha\, dx \leqslant \left(\int_\Omega [\varphi(x)]^\alpha\, dx\right)^{\varepsilon/\alpha} \left(\int_\Omega [d(x)]^{\alpha/(\alpha-\varepsilon)} [\varphi(x)]^\alpha\, dx\right)^{(\alpha-\varepsilon)/\alpha} \leqslant$$

$$\leqslant K_1 \|d^{\alpha/(\alpha-\varepsilon)}\|^{(\alpha-\varepsilon)/\alpha}_{r_1,\, n-\alpha r_1} \left(\int_\Omega [\varphi(x)]^\alpha\, dx\right)^{\varepsilon/\alpha} \left(\int_\Omega [\varphi_x(x)]^\alpha\, dx\right)^{(\alpha-\varepsilon)/\alpha}.$$

LEMMA 2. *Let* $q > \alpha > 1$ *and* $c \in L^{q,\,n-1}$. *Then there exist* $\varepsilon > 0$ *and*

$r_2 > 1$ *such that*

$$\text{(2.5)} \qquad \int_\Omega c(x)\,\varphi(x)\,|\psi(x)|^{\alpha-1}\,dx \leq K_2\,\|c^{\alpha/(1-\varepsilon)}\|_{r_2,n-\alpha r_2}^{(1-\varepsilon)/\alpha}\left(\int_\Omega [\varphi(x)]^\alpha\,dx\right)^{\varepsilon/\alpha}\cdot$$

$$\cdot\left(\int_\Omega [\varphi_x(x)]^\alpha\,dx\right)^{(1-\varepsilon)/\alpha}\left(\int_\Omega |\psi(x)|^\alpha\,dx\right)^{(\alpha-1)/\alpha} \qquad \forall\varphi,\ \psi \in C_0^\infty(\Omega)$$

with $K_2 = K_2(\alpha, r_2, \operatorname{diam}\Omega)$.

PROOF. Following the pattern of Lemma 1 we prove that there exist $\varepsilon > 0$ and $r_2 > 1$ such that $c^{\alpha/(1-\varepsilon)} \in L^{r_2, n-\alpha r_2}$. Indeed taking $r_2 = q(1-\varepsilon)/\alpha$ with $\varepsilon > 0$ such that $r_2 > 1$, we have:

$$\int_{B(\rho)} \{[c(x)]^{\alpha/(1-\varepsilon)}\}^{r_2}\,dx = \int_{B(\rho)} [c(x)]^q\,dx \leq C\rho^{n-1} \leq C\rho^{n-\alpha r_2}.$$

By Theorem 1 and Hölder's inequality we have:

$$\int_\Omega c(x)\,\varphi(x)\,|\psi(x)|^{\alpha-1}\,dx \leq \left(\int_\Omega [\varphi(x)]^\alpha\,dx\right)^{\varepsilon/\alpha}\left(\int_\Omega [c(x)]^{\alpha/(1-\varepsilon)}[\varphi(x)]^\alpha\,dx\right)^{(1-\varepsilon)/\alpha}\cdot$$

$$\cdot\left(\int_\Omega |\psi(x)|^\alpha\,dx\right)^{(\alpha-1)/\alpha} \leq K_2\,\|c^{\alpha/(1-\varepsilon)}\|_{r_2,n-\alpha r_2}^{(1-\varepsilon)/\alpha}\left(\int_\Omega [\varphi(x)]^\alpha\,dx\right)^{\varepsilon/\alpha}\cdot$$

$$\cdot\left(\int_\Omega [\varphi_x(x)]^\alpha\,dx\right)^{(1-\varepsilon)/\alpha}\left(\int_\Omega |\psi(x)|^\alpha\,dx\right)^{(\alpha-1)/\alpha}.$$

LEMMA 3. *Let* $q > \alpha/(\alpha-1) > 1$ *and* $b \in L^{q,n-\alpha}$. *Then there exist* $\varepsilon > 0$ *and* $r_3 > 1$ *such that*

$$\text{(2.6)} \qquad \int_\Omega b(x)\,\psi(x)\,[\varphi(x)]^{\alpha-1}dx \leq K_3\|b^{\alpha/(\alpha-1)}\|_{r_3,n-\alpha r_3}^{(\alpha-1)/\alpha}\left(\int_\Omega [\psi(x)]^\alpha dx\right)^{1/\alpha}\cdot$$

$$\cdot\left(\int_\Omega [\varphi_x(x)]^\alpha\,dx\right)^{(\alpha-1)/\alpha} \qquad \forall\varphi,\ \psi \in C_0^\infty(\Omega)$$

where $K_3 = K_3(\alpha, r_3, \operatorname{diam}\Omega)$.

PROOF. The proof is similar to that of Lemmas 1 and 2.

REMARK. We wish now to compare our hypotheses with those in [8].

In [8] is assumed

$$d \in L^{n/(\alpha-\varepsilon)}; \qquad c \in L^{n/(1-\varepsilon)}; \qquad b \in L^{n/(\alpha-1)}.$$

We have:

$$\begin{aligned} L^{n/(\alpha-\varepsilon)} &\subsetneqq L^{q,\,n-\alpha} && 1 < q < n/(\alpha-\varepsilon), \\ L^{n/(1-\varepsilon)} &\subsetneqq L^{q,\,n-1} && \alpha < q < n/(1-\varepsilon), \\ L^{n/(\alpha-1)} &\subsetneqq L^{q,\,n-\alpha} && \alpha/(\alpha-1) < q < n/(\alpha-1). \end{aligned}$$

Let us prove the first one. For $f \in L^{n/(\alpha-\varepsilon)}$ clearly

$$\int_{B(\rho)} [f(x)]^q\,dx \leq \left(\int_{B(\rho)} [f(x)]^{n/(\alpha-\varepsilon)}\,dx\right)^{q(\alpha-\varepsilon)/n} |B(\rho)|^{1-q(\alpha-\varepsilon)/n} \leq$$

$$\leq C(\operatorname{diam}\Omega)^{\alpha-q\alpha+q\varepsilon}\rho^{n-\alpha} = C\rho^{n-\alpha}\ (^2).$$

The inclusion is proper because (see [7]) there are functions from space $L^{q,\,n-\alpha}$ which are not integrable with any exponent greater than q.

Also we wish to observe that for (2.4) to hold is necessary that $d \in L^{1,\,n-\alpha+\varepsilon}$. This because, taking in (2.4) $\varphi \in C_0^\infty(B(2\rho))$, $\varphi = 1$ in $B(\rho)$, $0 \leq \varphi \leq 1$ in $B(2\rho)$, $|\nabla\varphi| \leq K/\rho$, we have:

$$\int_{B(\rho)} d(x)\,dx \leq K_1(\rho^n)^{\varepsilon/\alpha}(\rho^{n-\alpha})^{(\alpha-\varepsilon)/\alpha} = K_1\rho^{n-\alpha+\varepsilon}.$$

Clearly (by Hölder's inequality) $L^{q,\,n-\alpha} \subseteq L^{1,\,n-\alpha+(\alpha-\alpha/q)}$ and then our assumption is very close to be necessary for (2.4) (taking q very close to one).

Finally let us point that for (2.5) to hold true we need to take $c(x)$ in the space L^α_{loc}. This because, for given $\varphi \in C_0^\infty(\Omega)$ from (2.5) we have:

$$\int_\Omega (c\varphi)\,|\psi|^{\alpha-1}\,dx \leq K(\varphi)\left(\int_\Omega (\psi^{\alpha-1})^{\alpha/(\alpha-1)}\,dx\right)^{(\alpha-1)/\alpha}$$

(2) Here $|B(\rho)|$ is the measure of $B(\rho)$.

or, letting $\eta = \psi^{\alpha-1}$, we have:

$$\int_\Omega c\varphi\eta\, dx \leq K\|\eta\|_{L^{\alpha/(\alpha-1)}} \qquad \forall \eta \in L^{\alpha/(\alpha-1)}$$

and then $c\varphi \in L^\alpha$ $\forall \varphi \in C_0^\infty(\Omega)$ which clearly implies $c \in L^\alpha_{\text{loc}}(\Omega)$.

Similar considerations for the coefficient b show that in order to have (2.6) $b \in L^{\alpha/(\alpha-1)}_{\text{loc}}(\Omega)$ is necessary.

3. Local boundedness of solutions.

In this section we will show that weak solutions of equation (1.2) are locally bounded.

THEOREM 2. *Let u a weak solution of eq.* (1.2) *defined in some ball $B(2\rho) \subset\subset \Omega$. We assume that conditions* (2.1) *and* (2.2) *hold. Then*

$$\|u\|_{L^\infty(B(\rho))} \leq C\rho^{-n/\alpha}\|u\|_{L^\alpha(B(2\rho))}$$

and

$$\|u_x\|_{L^\infty(B(\rho))} \leq C\rho^{-1}\|u\|_{L^\alpha(B(2\rho))}$$

where $C = C(\alpha, n, \varepsilon, a, \|b\|, \rho^\varepsilon\|c\|, \rho^\varepsilon\|d\|)$, $\varepsilon, r_1, r_2, r_3$ are those given by Lemmas 1, 2, 3 *and $\|b\|$, $\|c\|$, $\|d\|$ stand for $\|b^{\alpha/(\alpha-1)}\|^{(\alpha-1)/\alpha}_{r_3, n-\alpha r_3}$, $\|c^{\alpha/(1-\varepsilon)}\|^{(1-\varepsilon)/\alpha}_{r_2, n-\alpha r_2}$, $\|d^{\alpha/(\alpha-\varepsilon)}\|^{(\alpha-\varepsilon)/\alpha}_{r_1, n-\alpha r_1}$ respectively.*

PROOF. We consider first the case $\rho = 1$. The general case $\rho \neq 1$ follows by application of the linear transformation $y = \rho x$.

For fixed numbers $q \geq 1$ and $t > 0$ we consider the functions

$$F(u) = \begin{cases} |u|^q & \text{if } |u| \leq t, \\ qt^{q-1}|u| - (q-1)t^q & \text{if } |u| \geq t, \end{cases}$$

and

$$G(u) = \operatorname{sign} u\{F(|u|)F'(|u|)^{\alpha-1} - q^{\alpha-1}\}, \qquad -\infty < u < +\infty.$$

As a test function in (2.3) we take:

$$\Psi(x) = \eta^\alpha(x)G(u)$$

where $\eta \in C_0^1(B(2))$.

Substituting $\Psi(x)$ in (2.3) and using the assumptions (2.1) we obtain, as in [8]:

$$\int |\eta v_x|^\alpha dx \leq \alpha a \int |\eta_x v|\, |\eta v_x|^{\alpha-1} dx + \alpha q^{\alpha-1} \int b |\eta_x v|\, (\eta v)^{\alpha-1)} dx + \tag{3.1}$$

$$+ \int c(\eta v)\, |\eta v_x|^{\alpha-1} dx + (1+\beta)\, q^{\alpha-1} \int d(\eta v)^\alpha dx$$

where $v = v(x) = F(u)$ and q and β are related by $\alpha q = \alpha + \beta - 1$.

Using Hölder's inequality together with Lemmas 1, 2 and 3 we can estimate the terms on the right-hand side of (3.1) as follows

$$\int |\eta_x v|\, |\eta v_x|^{\alpha-1} dx \leq \|\eta_x v\|_{L^\alpha} \|\eta v_x\|_{L^\alpha}^{\alpha-1},$$

$$\int b |\eta_x v|\, (\eta v)^{\alpha-1} dx \leq K_3 \|b\| \Big(\int |\eta_x v|^\alpha dx \Big)^{1/\alpha} \Big(\int |(\eta v)_x|^\alpha dx \Big)^{(\alpha-1)/\alpha} \leq$$

$$\leq K_3 \|b\| \left\{ \|\eta_x v\|_{L^\alpha}^\alpha + \|\eta_x v\|_{L^\alpha} \|\eta v_x\|_{L^\alpha}^{\alpha-1} \right\},$$

$$\int c(\eta v)\, |\eta v_x|^{\alpha-1} dx \leq K_2 \|c\| \Big(\int (\eta v)^\alpha dx \Big)^{\varepsilon/\alpha} \Big(\int |(\eta v)_x|^\alpha dx \Big)^{(1-\varepsilon)/\alpha} \cdot$$

$$\cdot \Big(\int |(\eta v_x)|^\alpha dx \Big)^{(\alpha-1)/\alpha} \leq K_2 \|c\| \|\eta v\|_{L^\alpha}^\varepsilon \left\{ \|\eta_x v\|_{L^\alpha}^{1-\varepsilon} \|\eta v_x\|_{L^\alpha}^{\alpha-1} + \|\eta v_x\|_{L^\alpha}^{\alpha-\varepsilon} \right\},$$

$$\int d(\eta v)^\alpha dx \leq K_1 \|d\| \Big(\int (\eta v)^\alpha dx \Big)^{\varepsilon/\alpha} \Big(\int |(\eta v)_x|^\alpha dx \Big)^{(\alpha-\varepsilon)/\alpha} \leq$$

$$\leq k_1 \|d\| \left\{ \|\eta_x v\|_{L^\alpha}^{\alpha-\varepsilon} + \|\eta v_x\|_{L^\alpha}^{\alpha-\varepsilon} \right\}.$$

Using these inequalities in (3.1) we obtain:

$$\|\eta v_x\|_{L^\alpha}^\alpha \leq C\{ \|\eta_x v\|_{L^\alpha} \|\eta v_x\|_{L^\alpha}^{\alpha-1} + q^{\alpha-1} (\|\eta_x v\|_{L^\alpha}^\alpha + \|\eta_x v\|_{L^\alpha} \|\eta v_x\|_{L^\alpha}^{\alpha-1}) +$$

$$+ \|\eta v\|_{L^\alpha}^\varepsilon (\|\eta_x v\|_{L^\alpha}^{1-\varepsilon} \|\eta v_x\|_{L^\alpha}^{\alpha-1} + \|\eta v_x\|_{L^\alpha}^{\alpha-\varepsilon}) + (1+\beta)\, q^{\alpha-1} (\|\eta_x v\|_{L^\alpha}^{\alpha-\varepsilon} + \|\eta v_x\|_{L^\alpha}^{\alpha-\varepsilon})\}$$

where C depends on the quantities listed in the statement of the theorem.

Proceeding as in [8] we obtain

$$\|\eta v_x\|_{L^\alpha} \leq C q^{\alpha/\varepsilon} (\|\eta v\|_{L^\alpha} + \|\eta_x v\|_{L^\alpha}) \tag{3.2}$$

and by Sobolev inequality we have

$$\|\eta v_x\|_{L^{\alpha^*}} \leq C q^{\alpha/\varepsilon} (\|\eta v\|_{L^\alpha} + \|\eta_x v\|_{L^\alpha}). \tag{3.3}$$

Let now h, h' satisfy $h' < h \leq 2$ and choose $\eta(x)$ so that $\eta = 1$ in

$B(h')$, $0 \leq \eta \leq 1$ in $B(h)$, $\eta(x) = 0$ outside $B(h)$ and $|\eta_x| \leq 2(h - h')^{-1}$.

Substituting this function in (3.2) and (3.3) we have:

(3.4) $$\|v_x\|_{L^{\alpha}(B(h'))} \leq C q^{\alpha/\varepsilon}(h - h')^{-1}\|v\|_{L^{\alpha}(B(h))},$$

(3.5) $$\|v\|_{L^{\alpha^*}(B(h'))} \leq C q^{\alpha/\varepsilon}(h - h')^{-1}\|v\|_{L^{\alpha}(B(h))}.$$

Letting $t \to \infty$, since $v \to |u|^q$, (3.5) gives

(3.6) $$\|u^q\|_{L^{\alpha^*}(B(h'))} \leq C q^{\alpha/\varepsilon}(h - h')^{-1}\|u^q\|_{L^{\alpha}(B(h))}.$$

Finally, setting in (3.6)

$$p = \alpha q = \alpha + \beta - 1, \qquad \chi = \alpha^*/\alpha = n/(n - \alpha)$$

we obtain

(3.7) $$\|u\|_{L^{\chi p}(B(h'))} \leq [C(p/\alpha)^{\alpha/\varepsilon}(h - h')^{-1}]^{\alpha/p}\|u\|_{L^{p}(B(h))}.$$

This is the inequality which we will iterate.
We define for $\nu = 0, 1, 2, \ldots$

$$p_\nu = \chi^\nu \alpha, \qquad h_\nu = 1 + 2^{-\nu}, \qquad h_\nu' = h_{\nu+1}.$$

Hence from (3.7), we have:

$$\|u\|_{L^{p_{\nu+1}}(B(h_{\nu+1}))} \leq C^{1/\chi^\nu} K^{\nu/\chi^\nu}\|u\|_{L^{p_\nu}(B(h_\nu))} \leq C^{\Sigma 1/\chi^\nu} K^{\Sigma \nu/\chi^\nu}\|u\|_{L^{\alpha}(B(2))} \leq C\|u\|_{L^{\alpha}(B(2))}$$

where $K = 2\chi^{\alpha/\varepsilon}$.

Letting $\nu \to \infty$ gives

$$\|u\|_{L^{\infty}(B(1))} \leq C\|u\|_{L^{\alpha}(B(2))}.$$

The second part of the theorem follows by setting $q = 1$, $h' = 1$, $h = 2$ in (3.2).

4. Harnack's inequality.

THEOREM 3. *Let $u \geq 0$ a weak solution of eq.* (1.2) *is some open ball $B(3\rho) \subset \Omega$. We suppose that conditions* (2.1) *and* (2.2) *holds.*
Then

$$\max_{B(\rho)} u \leq \min_{B(\rho)} u$$

where C is a constant depending on the same arguments as in Theorem 2.

PROOF. The proof is the same as in [8]' following very closely the classical Moser's procedure. The main modification is in the first step which was given by us in detail in Theorem 2.

Let us only point that in the «central step», in which $\log u$ is estimated, we have once more to rely on our Lemmas 1, 2, 3 in order to bound the relevant integrals instead of Hölder's inequality as in [8].

With these remarks the conclusion of the theorem can be easily established by the reader.

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