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On a Question of Deaconescu About Automorphisms.

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1. Introduction.

At a recent meeting in Barnaul, Siberia, Marian Deaconescu posed the following problem.

1.1. *Do there exist infinite groups* G *such that* $\mathrm{Aut}\, H \cong \cong N_G(H)/C_G(H)$ *for all subgroups* H *of* G?

Let us call (finite or infinite) groups with this property MD-groups. We note at the outset that the infinite dihedral group D has the MD-property, and this is fairly easy to prove. Theorem 2.1 characterizes the infinite metabelian MD-groups; there are just $2^{\aleph_0}$ of them, and they all have very simple structure. A corollary is that D is the only finitely generated infinite metabelian MD-group.

It is quite probable that the groups covered in Theorem 2.1 are the only infinite MD-groups, because automorphism groups tend to be insoluble and big. For example, every free group F_m of infinite rank m has F_{2^m} in its automorphism group; thus, by the splitting property of free groups, any MD-group containing a subgroup whose automorphism group contain a non-abelian free subgroup would have to be of order bigger than all the cardinals obtained from $\aleph_0$ by related exponentiation ω times; that is, bigger than $\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \ldots,$.

On the other hand, there are very few finite MD-groups:

1.2. *The only non-trivial finite* MD-*groups are* Z_2 *and* S_3.

PROOF. Let G be a finite non-trivial MD-group. We claim first that every non-trivial Sylow subgroup is of prime order. If not, G has a p-subgroup P of order bigger than p, and the celebrated Gaschütz theo-

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rem [2] shows that Out P has an element of order p. Let $X/C_G(P)$ be a Sylow p-subgroup of $N_G(P)/C_G(P)$. Then $X/C_G(P)$ is of order more that $P/Z(P)$, so that X contains a p-subgroup strictly bigger than P. This process continues and we can construct p-subgroups of arbitrarily large order. This is impossible since G is finite.

So G is metacyclic, and we can choose a Carter subroup H of G, that is, a self-normalizing nilpotent subgroup. Then $\mathrm{Aut}\, H \cong$ $\cong N_G(H)/C_G(H) = H/Z(H) = 1$, so again by Gaschütz's theorem, H must be cyclic of order 2, with generator h, say. But then $C_G(h) = H$, and h has $|G|/2$ conjugates, so that $|G'| = |G|/2$. Let K be any subgroup of G'; then $C_G(K) \geqslant G'$ and so Aut K has order at most 2. This means that G' is of order 1, 2, 4 or 6, and a routine calculation completes the proof.

The only general property of MD-groups we shall require in the sequel is this:

1.3. *For every* MD-*group,* G/G' *is of exponent dividing* 2.

Proof. For every x in G, $N_G(\langle x \rangle)/C_G(\langle x \rangle) \cong \mathrm{Aut}\langle x \rangle$ so there must exist $g \in G$ such that $x^g = x^{-1}$. Thus $[x, g] = x^{-2}$, so every square is a commutator.

2. Infinite metabelian ND-groups.

Here is the characterization promised in the introduction.

Theorem 2.1. *An infinite metabelian group G is an* MD-*group if and only if G can be generated by a non-trivial torsion-free locally cyclic group A and an element x of order* 2 *such that*

(i) $a^x = a^{-1}$ *for all a in A,*

(ii) *the type of A is finite at every prime.*

Corollary 2.2. *There are precisely* $2^{\aleph_0}$ *infinite metabelian* MD-*groups.*

The corollary follows since $G' = A^2$ and there are $2^{\aleph_0}$ different structure for A; for any locally cyclic group A of the type under consideration, $A \cong A^2$.

Before embarking on the proof of Theorem 2.1, we establish the following simple lemma, which is probably known:

Lemma 2.3. *Every subgroup H of a torsion-free abelian group A*

has automorphism group of exponent at most 2 *if and only if* A *is locally cyclic with type that is finite at each prime. In fact,* $\operatorname{Aut} H$ *is of order at most* 2 *in all such cases.*

PROOF. Let A be a torsion-free abelian group such that $\operatorname{Aut} H$ is of exponent at most 2 for all subgroups H. Then A is certainly locally cyclic since $Z \times Z$ has non-abelian automorphism group. If A is p-divisible for some prime p, the map $a \to a^p$ is an automorphism of infinite order; so the type of A is finite at all primes. Thus A is isomorphic to the subgroup A^* of the additive group Q of rationals generated by reciprocals $p_n^{-k_n}$, where p_n is the n-th prime and k_n is a non-negative integer for each n (see Fuchs [1]). Let α be an automorphism of A^*, and write $1\alpha = rs^{-1}$, where r and s are relatively prime integers. We claim that $s = 1$. If not, then s is divisible by p_m for some m, so that $k_m > 0$; but, since A^* is torsion-free, this means that $p_m^{-k_m}\alpha = rs^{-1}p_m^{-k_m}$, whereas this number has denominator divisible by $p_m^{k_m+1}$, and so is not in A^*. Thus $1\alpha = r$, and a similar argument on α^{-1} shows that $r = \pm 1$, so that $\operatorname{Aut} Z = Z_2$, as required. The same holds for all subgroups H of A, of course.

The converse is clear, and the proof of the lemma is complete.

We turn now to the proof of Theorem 2.1. Let G be a group having a subgroup A and an element x with the properties mentioned in the statement, and let H be a non-trivial subgroup of G. If $H \leqslant A$, then $H \lhd G$ and $C_G(H) = A$, so that $N_G(A)/C_G(A) \cong Z_2$, and so $N_G(H)/C_G(H) \cong \operatorname{Aut} H$, by Lemma 2.3. If $H \nleqslant A$, then $H = \langle K, bx\rangle$ for some $K \leqslant A$ and some element b of A. If $K = 1$, all is well. If not, $C_G(K) = A$ and $C_G(bx) = \langle bx\rangle$, so $C_G(H) = 1$. A very quick calculation shows that $N_G(H) = \langle \sqrt{K}, bx\rangle$, where $\sqrt{K} = \{a\colon a \in A \text{ and } a^2 \in K\}$. However, we have $\sqrt{K} \cong K$ since K/K^2 is of order precisely 2 (this is true for any nontrivial subgroup of A), and the isomorphism is the squaring map. This means that $N_G(H) \cong H$ and we simply need to show that $\operatorname{Aut} H \cong H$ in this case. Without loss of generality, we may suppose that $b = 1$, so we are considering a group of the form $H = \langle K, x\rangle$, where K is a non-trivial subgroup of A.

Now K is characteristic in H, since every element of H–K is of order 2. Thus, for every automorphism α of H, $\alpha|_K$ is the identity or the inverting map, and $\alpha\colon x \to bx$ for some $b \in K$. Conversely, every map $\phi_{\varepsilon, l}$ of the form

$$k \to k^{\varepsilon} \quad \text{for } k \in K, \qquad x \to lx,$$

where $\varepsilon = \pm 1$ and $l \in L$, yields an automorphism of K.

Those with $\varepsilon = 1$ form a subgroup isomorphic with K; the automorphism $\xi = \phi_{-1,1}$ is of order 2, and

$$\xi^{-1}\phi_{1,l}\xi = \phi_{1,l^{-1}},$$

so that $\operatorname{Aut} H \cong H$, as required.

Conversely, let G be an infinite metabelian MD-group, and A a maximal abelian subgroup containing G'. For every subgroup H of A, $C_G(H) \geqslant A \geqslant G'$, so $N_G(H)/C_G(H)$ is of exponent at most 2, by 1.3. Thus the periodic part of A is of order at most 6 and so $A = S \times P$ for some finite group P and torsion-free group S. Since $C_G(A) = A$ by maximality we have $S \neq 1$ since G is infinite. If $P \neq 1$, then A contains a subgroup isomorphic to $Z \times Z_n$, $n \neq 1$, which has non-abelian automorphism group. So A is torsion-free and thus it is locally cyclic since $Z \times Z$ has non-abelian automorphism group. By 1.3 and 2.3, A must be a locally cyclic group, and its type is finite for each prime, so $\operatorname{Aut} A$ is of order 2. Thus $Z_2 \cong N_G(A)/C_G(A) = G/A$, so $G = \langle x, A\rangle$ for some x such that $x^2 \in A$. But x^2 is central and this means that $x^2 = 1$ else $N_G(\langle x^2\rangle)/C_G(\langle x^2\rangle)$ is of order 1, not 2 as it is for infinite cyclic groups. This completes the proof of the theorem.

Finally, the proof of our last corollary is obvious:

COROLLARY 2.4. *This only infinite finitely generated metabelian* MD-*group is D.*

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