RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

ZVI ARAD
GIDEON EHRLICH
OTTO H. KEGEL

An application of Ramsey's theory to partition in groups. - II

Rendiconti del Seminario Matematico della Università di Padova, tome 89 (1993), p. 57-81
<http://www.numdam.org/item?id=RSMUP_1993__89__57_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 89 (1993)

An Application of Ramsey's Theory to Partition in Groups. - II.

ZVI ARAD - GIDEON EHRLICH - OTTO H. KEGEL (*)

Introduction.

In 1916 I. Schur [Sc] proved the following theorem, one of the earliest results of Ramsey type:

THEOREM. *In every finite coloring of the positive integers* $\mathbb{Z}^+$ *there exists a monox solution to the equation* $x + y = z$.

In [AEKL] we applied Ramsey theory in order to generalize Schur's theorem to arbitrary groups, finite and infinite, and at the same time to weaken Schur's assumptions.

Define a group G (or partial semigroup G) to have an n-partition; in short, G is in the class $\boldsymbol{nP}$, if there exists a partition of the set G into subsets $\{1\}, A_1, \ldots, A_n$, $n \geq 2$, (A_i may be empty) such that if $x, y \in A_i$, $x \neq y$, $1 \leq i \leq n$, then $xy \notin A_i$.

We proved in [AEKL] that infinite groups are not in $\boldsymbol{nP}$, for any positive integer $n \geq 2$. Also finite groups of order greater than $R(2, 8, (1/2)(n^2+2))$ are not in $\boldsymbol{nP}$. In particular, we proved that for $n = 2$ groups of order greater than 9 are not in $\boldsymbol{2P}$ and that for $n = 3$ groups of order greater than 18 are not in $\boldsymbol{3P}$.

(*) Indirizzo degli AA.: Z. ARAD and G. EHRLICH: Department of Mathematics and Computer Science, Bar-Ilan University, Ramat-Gan 52900, Israel; O. H. KEGEL: Mathematisches Institut, Albert-Ludwigs-Universität, D-7800 Freiburg i. Br., Germany.

This research was supported by a Grant from the G.I.F., the German-Israeli Foundation for Scientific Research and Development.

The Ramsey numbers $R(2, 8, (1/2)(n^2+2))$ are large and it is an open question how to compute them [Gr1].

For a more complete background and more detailed information see [AEKL].

The goal of this paper is to obtain the following main theorem:

MAIN THEOREM. *An abelian group G is* **4P** *if and only if G is isomorphic to one of the following*:

a) A cyclic group C_k of order k where either $k \leqslant 45$ or $k = 48$.

b) A non-cyclic group of order $\leqslant 40$.

c) A non-cyclic group of order 45, 48 or 49.

d) The non-cyclic group $C_3 \times C_3 \times C_3 \times C_2$.

e) The non-cyclic group $C_3 \times C_3 \times C_3 \times C_3$.

The list in the main theorem is shorter if we delete the assumption $x \neq y$ in the definition of the class **4P**.

If $f(t)$ denotes the largest natural number n such that $\{1, \ldots, n\} \subseteq \mathbb{N}$ can be split into t set, none of which contains a solution of $x + y = z$, then by [Gr2, p. 88], $f(1) = 1$, $f(2) = 4$, $f(3) = 13$, $f(4) = 44$. The evaluation of $f(5)$ appears to be a difficult computational problem [Gr2].

If $f^*(t)$ denotes the largest natural number n such that $\{1, \ldots, n\} \subseteq \mathbb{N}$ is in t**P**, then we proved in [AEKL] that $f^*(1) = 2$, $f^*(2) = 8$ and $f^*(3) = 23$.

In this paper we find that $f^*(4) = 66$ and $f^*(5) \geqslant 195$. Our conjecture is that $f^*(5) = 195$ and $f^*(n) = 3[f^*(n-1) - 1]$ for $n \geqslant 4$.

The number of partitions of $\{1, \ldots, 66\} \subseteq \mathbb{N}$ into 4 subsets is known as the Stirling number of the second kind $S(66, 4)$; it is larger than 4^{62}. This relation shows that the present super computers need more than a life-time to handle all 4 partitions of $\{1, ..., 66\}$. We constructed a fast algorithm and found all the possible partitions of $\{1, ..., 66\}$ that are in **4P**; there are 29931 of them. We found also that $f^*(4) = 66$.

The evaluation of $f^*(4) = 66$ enables us to prove that a cyclic group C_k of order k is in **4P** if and only if either $k \leqslant 45$ or $k = 48$. Consequently, the largest prime power dividing the order of a cyclic subgroup of a **4P**-group is 43. This fact leads us to the proof of our main theorem.

It bears mentioning that there is a significant difference between the cases $n = 3$ and $n = 4$. This fact forces us to construct a new more delicate algorithm to deal with the case $n = 4$. This new algorithm is useful also in dealing with related problems; therefore we describe it here in full detail.

PROOF OF THE MAIN THEOREM. We use freely the following two statement about **4*P***-groups:

(*) If a proper subset of H of G is not in ***nP***, then G is not in ***nP***.

(**) Proper subsets of ***nP***-groups are in ***mP*** for some $m \leqslant n$.

In order to prove constructively that a given subset of a group G is in ***nP*** we constructed the following algorithm.

THE ALGORITHM. In order to constructively decide whether or not a group G of n elements has an r-partition for a given r, we apply the backtrack method.

By a sub-partition of the group G we mean a partition of a subset of the group elements to at most r mutually exclusive subsets, so that there is no solution to the equation $x * y = z$, $x \neq y$ within any of the subsets.

By systematically generating some collection of sub-partitions we will either get a partition—a sub-partition which is a partition of the whole group—, or no partition and thus know that no partition exists.

Given a random function the following is a definition of such a collection:

Our collection is the set of nodes of a tree T. The root of T is the empty sub-partition. Given a node SP, select randomly an x not yet in SP and join ut in turn to each subset of SP if possible, in order to get at most r children of SP.

Pay attention to the following facts:

a) All partitions of G are leaves of T no matter which random function was used.

b) It may happen that SP is a leaf—has no children—and SP is not a partition of G. Almost all leaves are of this type.

c) If $|\mathrm{SP}| < r$ then $\{x\}$ may be added to it as a singleton.

We can look for a partition by searching the above tree. We will generate each new node when we visit it. The first come-in-mind method is the «width first search» method. We start with visiting the 0 level of the tree—the empty sub-partition. After visiting all sub-partitions of some level k (sub-partitions of k elements) we will visit the next level—all children of the previous ones (sub-partitions of $k+1$ elements). This method requires saving all elements of the previous level and, hence, consumes space that may grow exponentially with the depth of the tree.

Another method is the «depth first search» method. We search the tree by examining the root and then all the sub-trees rooted on it using the same method. This method involves saving the parent of each node, except for the root, and thus the amount of space it consumes seemed to be about proportional to the depth of the tree.

Starting with the root, we search subtree rooted on a node by:

1) generating the node,

2) processing it (in our case checking it for being a solution),

3) searching in the same way all subtrees rooted on the node's childs.

This implies that if pk was the last element to enter a sub-partition SP then after checking all nodes in a subtree rooted on SP we transfer pk to some other subset of SP in order to get a new sub-partition SP$'$. After pk was located in all subsets it was able to join, and all the appropriate subtrees were searched we omit pk and try to transfer the remaining last joined element to a new subset and so on. The process is terminated after omitting the first joined element.

A general algorithm of the above backtrack algorithm is described in [RND].

The backtrack time-complexity is mainly affected by the number of nodes in the tree and by checking whether or not the new element may join a subset. In order to condense the tree we look for possible prunings. Namely, we ignore a sub-tree rooted on some sub-partition whenever we know that it does not contain a partition. We number the subsets of a partition from 1 to at most r and save a Boolean matrix, CAN, which for each element not in the current sub-partition and for each subset of the current sub-partition, indicates whether or not the element may join the subset. When an element x joins a subset A, we update that matrix by setting all solutions z of: $zx = y$, $zy = x$, and $xy = z$; y in A (we pre-prepare a matrix that gives the 3 appropriate z's for each x, y) and z not yet in the sub-partition, to «false». This by itself does not save time, but for each z not in the sub-partition we save the number of subsets it may join—its «degree». When the element x will be removed from the subset A (in order to be added to some other subset of its parent, or in order to switch one of its ancestor nodes for that node's brother), we will update the degrees and the Boolean values changed because of adding x to A. No re-computations are done since we save the needed information we got on adding x.

PRUNING. Pruning a sub-tree rooted on the current sub- partition SP means ignoring that sub-tree. That is, we move the last joined element x to another subset in order to get a brother of SP or we return to SP's parent. The following property A is a sufficient reason for pruning.

A: on joining x to the sub-partition SP, some z not in SP gets the degree 0.

We will ensure that A will never even be about to occur since we will use a stronger pruning rule.

Instead of selecting randomly the new joined x we will select just one of the degree 1 elements if there are such elements not in our sub-partition.

We keep the following property B as an invariant of our algorithm.

B: There are no 2-degree 1 elements outside the sub-partition such that each of them can, and hence must, join the same subset but one cannot join it if the other one does.

We claim: B is true when a new element x is about to join SP in order to generate SP$'$.

If B is false then we prune the sub-tree rooted on SP$'$, since SP$'$ cannot be completed to a partition of G. Under these circumstances, A cannot occur since if z is a degree-1 and x was selected to join SP, then according to our method, x is a degree-1, B is true and hence A is false.

Therefore, when joining x and updating the degree of some z we check if it should become 1. If it should, and it must join some other subset a', we check if B is about to be violated as to that a'. If it will be violated, then we prune the tree rooted on SP$'$.

Saving intermediate states: Since the Algorithm consumes a lot of time, for some cases days and for others much more, we save intermediate states of the algorithm each time we visit a node in some fixed level, for instance, 8 or 12.

The program was written in standard Pascal, and runs on a variety of computers: SUNs, Vaxes, and the RISC machines IBM RS/6000 and Decstation 5200, all under the UNIX opperating system. Each of the, up to 10, computers was assigned the lowest priority (there were some other users around).

Estimating running time:

The following two cases were checked on an IBM PC AT 18 mhz, Turbo Pascal 6.0.

3-Partitioning of C15: 83 partitions were found in 0.17 seconds.

4-Partitioning of C48: 2301 partitions were found in 37.5 hours.
7984 nodes at level 10 were visited.

This means that 7984 sub-partitions of 10 elements were generated during the whole process. This number is roughly the number of level 10 nodes we found for every other case of a 4-partitioning.

By measuring the time some level 10 nodes are generated a very rough estimation for the total needed time is known.

Subtrees rooted on nodes of the same level may be of very different size.

In the above C48 case some such subtrees were searched in less than a second, others in half an hour.

If $G = C_1 \times C_2 \times C_3 \times \ldots$ then the smaller the size of the groups C_i, the longer is the time needed for checking it. Checking the case $C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2$ seemed to require several years on the RISC machines which are about 100 times faster than the IBM PC AT.

In case the running time seemed to be long we let several computers work in parallel. Each of them starts from some node and stops on arriving to some other node. E.g. the input file describes a sub-partition of 6 elements and the program halts when the 4th element to join the sub-partition, P[4], moves to another subset in order to generate a new sub-partition of 4 elements. That new sub-partition defines the node another computer already got as his input.

Program Outline.

REMARK. Comments clarifying the program are given within curly brackets { }.

Constants

Some parameters of the program:

Should it work on groups or integers 1..n.

How many subsets should be in the partition.

Is $x * x = y$ allowed when x, y belongs to the same subset.

Is one partition enough, or do we want all partitions.

Data-structures (variables):

A file, GROUP-FILE, defining the group and the state of the program.

N: integer; {group size}

UNFIT: array [0..n, 0..n, 1..3] of integer;

{The first layer of UNFIT, UNFIT[*,*,1] represents the group table.

UNFIT $[i,j,t] = k$, $t = 2, 3 \rightarrow$ either $i * k = j$ or $j * k = i$

$k = 0$ means: ignore this entry.}

r-Partition of $k \leq n$ elements out of all n group elements.

We keep it in 2 forms:

1: (k, P, SETOF)

P: a permutation of the integers 1..n.

SETOF: a characteristic vector, defines for each $P[i]$, $i \leq k$ to which of the r sets of the partition it belongs.

2: r stacks.

CAN: array [0..n, 1..r] of boolean;

{shows for each element not in the partition and each set whether the element may join the set}.

top1: Integer; {defines the set of all degree-1 elements not yet in the partition.

$P[k+1]$...P[top1] are all degree-1 elements not yet in the sub-partition.

elements of P are swapped in order to keep this arrangment of P}.

For each element $P[k]$ enters the sub partition we keep a set of all pairs (wrong,t) such that CAN[wrong,t] is turned to false because $P[k]$ entered the sub-partition.

We keep those sets on a long enough array. Each set occupies one new segment of the array.

procedure elmntgen;

{Generating a sequence of all group elements, each element is represented by an array of nofcoord integers $0 \leq \text{grpelmnt}[k, i] < \text{order}[i]$.}

procedure bltunfit;

{Building the table UNFIT}

```
procedure showpartition;
      {output the r-partition of the group}
procedure savedata;
      {Some base data enbales us to recover the program state, in case of a
      computer shut-down. Besides the description of the group and some
      parameters, it saves the current sub-partition.}
procedure getdata:
      {Gets what was saved by savedata and use if for initiating varia-
      bles}
procedure check;
      {Checks if the current state is legal:.
         p is a pemeutation,
         ip is its inverse,
         no 3 elements x, y z located in the same subset solve x * y = z,
         If each of the defree-1 elements x, y must join the same subset t,
         then x and y can join it together
begin {main program}
         getdata;
         elmntgen;
         bltunfit;
         check;
{OUTER-LOOP}
   {INNER-LOOP} {pk is an element about to enter the subset t}.
                {Search the subtree rooted on the new sub-parti-
                tion}.
   add pk to subset t: update the two partitions representations.
   update array SETOF: SETOF[pk] ← t;
   update STACK[t]: top[t] ← top[t] + 1; STACK[t, top[t]] ← pk;
   if k = backing-level then savedata;
   if k = n then showpartition (and HALT if you like).
   {Preventing wrong elements from joining the set t.}
      for stackti ← all elements already in set t
         begin
            for wrong ← UNFIT[pk, stackti,i] i = 1,2,3
               begin
                  if CAN[wrong,t], wrong was able to join set t,
                        and wrong is not in the sub-partition then
                     begin
                           if now wrong can join just set tt then
                           if (another element that must join tt and still
                              another such element or one that is already
                              in tt prevents wrong from joining tt)
                              then pk can't join set t – goto BACKING
```

```
                        else
                        {wrong becomes a degree-1 element.}
                        Add wrong to the degree-1 elements segment
                        that follows the k location at P.
                        {Prevent wrong from joining set t.}
                        CAN[wrong,t] ← false;
                        add (wrong,t) to the set of denied elements.
                    end; {preventing wrong elements}
                k ← k + 1;
                Pk ← P[k];
                t ← a new set that pk can join {CAN[pk, t] = true}
                Repeat INNER-LOOP
      {END INNER-LOOP}
    {BACKING}
            k ← highest i ⩽ k s.t. P[i] has to be located in some new set t,
                {While decreasing k to find the new k, remove each P[k]
                from its old set (only the stack has to be updated).
                All elements that were denied joining some set by some
                element that now was removed from the sub-partition
                should be allowed again to join the appropriate set.}
            If k = 0 then the algorithm terminated – print a note and
            HALT.
            pk ← p[k];
            t ← a new set that pk can join {CAN[pk,t] = true}
            repeat OUTER-LOOP
      {END OUTER-LOOP}
end program.
```

The algorithm's Pascal code will be sent to the interested reader upon request.

Let us denote our algorithm by AL.

By AL the partial semigroup $\{1, ..., 66\} \subset \mathbb{N}$ has exactly 29931 **4*P***-partitions. All the **4*P***- partitions are extension of the following four subsets:

A_1: 1, 2, 4, 8, 11, 22, 25.

A_2: 3, 5, 6, 7, 19, 21, 23, 51, 52, 64, 65.

A_3: 9, 10, 12, 13, 14, 15, 17, 18, 20, 54, 55, 61, 62.

A_4: 24, 26, 27, 28, 29, 30, 33, 41, 42, 47, 49.

For example one of the **4*P***-partitions is:

A_1^*: 1, 2, 4, 8, 11, 22, 25, 37, 40, 43, 50, 53, 56, 63, 66.

A_2^*: 3, 5, 6, 7, 19, 21, 23, 34, 35, 36, 51, 52, 64, 65.

A_3^*: 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 54, 55, 57, 58, 59, 60, 61, 62.

A_4^*: 24, 26, 27, 28, 29, 30, 31, 32, 33, 38, 39, 41, 42, 44, 45, 46, 47, 48, 49.

The partial semigroup $T = \{1, 2, ..., 67\} \subseteq \mathbb{N}$ is not in **4*P*** as AL shows. Consequently, $f^*(4) = 66$. In each of the 29931 **4*P***-partitions, there exists a subset all of whose elements are larger than 23.

LEMMA 1. *If G is a* **4*P***-*group then G is finite.*

PROOF. As mentioned in the introduction, this Lemma was proved in [AEKL].

LEMMA 2. *If a group G is isomorphic to one of the groups in the list of the main theorem a)-e), then G is a* **4*P***-*group.*

PROOF. Examples of **4*P***-partitions of groups from the list of the main theorem *a*)-*e*) can be found in Appendix A. We found these partitions by AL. We leave to the reader to find examples of **4*P***-partitions of abelian non-cyclic groups of order $\leqslant 28$.

LEMMA 3. *Cyclic groups G are in* **4*P*** *if and only if* $G \simeq C_k$, *where either* $k \leqslant 45$ *or* $k = 48$.

PROOF. By Lemma 1 and 2 if G is isomorphic to C_k with either $k \leqslant 45$ or $k = 48$ then G is in **4*P***.

Using AL we found that C_k for $k = 46, 47$ and $49 \leqslant k < 67$ are not in **4*P***. If $k \geqslant 67$ then C_k is not in **4*P*** since $f^*(4) = 66$. Thus Lemma 3 holds.

Let us denote the abelian group $C_{n_1} \times \dots \times C_{n_k}$ by $(n_1, \dots, n_k)$. By AL and Lemma 12 we found that the groups in the list of Lemma 4 are not in **4*P***.

LEMMA 4. *The following abelian non-cyclic groups are not in* **4*P***.

a) $C_p \times C_p$, *where p is a prime such that* $11 \leqslant p \leqslant 43$.

b) The groups of this table:

Order	Groups
$44 = 2^2 \cdot 11$	(2, 2, 11)
$50 = 2 \cdot 5^2$	(2, 5, 5)
$52 = 2^2 \cdot 13$	(2, 2, 13)
$54 = 2 \cdot 3^3$	$(2, 3, 3^2)$
$56 = 2^3 \cdot 7$	(2, 2, 2, 7) (2, 4, 7)
$60 = 2^2 \cdot 3 \cdot 5$	(2, 2, 3, 5)
$63 = 3^2 \cdot 7$	(3, 3, 7)
$64 = 2^6$	(2, 2, 16) (2, 32) (2, 4, 8) (8, 8) (2, 2, 2, 8) (2, 2, 2, 2, 4) (2, 2, 4, 4) (4, 4, 4) (4, 16)
$68 = 2^2 \cdot 17$	(2, 2, 17)
$72 = 2^3 \cdot 3^2$	(2, 2, 2, 3, 3) (2, 3, 3, 4) (3, 3, 8) (2, 2, 2, 9) (2, 4, 9)
$75 = 3 \cdot 5^2$	(3, 5, 5)
$76 = 2^2 \cdot 19$	(2, 2, 19)
$80 = 2^4 \cdot 5$	(2, 2, 4, 5) (2, 2, 2, 2, 5) (4, 4, 5) (2, 5, 8)
$81 = 3^4$	(3, 3, 9) (3, 27) (9, 9)
$84 = 2^2 \cdot 3 \cdot 7$	(2, 2, 3, 7)
$88 = 2^3 \cdot 11$	(2, 2, 2, 11) (2, 4, 11)
$96 = 2^5 \cdot 3$	(2, 2, 2, 2, 2, 3) (2, 2, 2, 3, 4) (2, 2, 3, 8) (2, 3, 16) (2, 3, 4, 4)
$98 = 2 \cdot 7^2$	(2, 7, 7)
$99 = 3^2 \cdot 11$	(3, 3, 11)
$100 = 2^2 \cdot 5^2$	*has a subgroup of order* 50 *either cyclic or non-cyclic*
$117 = 3^2 \cdot 13$	(3, 3, 13)
$125 = 5^3$	(5, 5, 5) $(5, 5^2)$
$135 = 3^3 \cdot 5$	(3, 3, 3, 5) (3, 5, 9)
$147 = 3 \cdot 7^2$	(3, 7, 7)
$162 = 2 \cdot 3^4$	(2, 3, 3, 3, 3)
$175 = 5^2 \cdot 7$	(5, 5, 7)
$245 = 5 \cdot 7^2$	(5, 7, 7)
$343 = 7^3$	(7, 7, 7)

LEMMA 5. *The set $\pi(G)$ of prime divisors of a* **4P**-*group is included in* $T = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43\}$.

PROOF. This is an immediate consequence of (**) and Lemma 3.

LEMMA 6. *If G is an abelian* **4P**-*group such that $p^2 \mid |G|$, p a prime, then* $p \in \{2, 3, 5, 7\}$.

PROOF. Lemmas 3 and 4 yield Lemma 6.

LEMMA 7. *If G is an abelian $\mathbf{4P}$-group then*

$$|\pi(G) - \{2,3,5\}| \leqslant 1.$$

PROOF. Assume that $|\pi(G) - \{2,3,5\}| \geqslant 2$ then G contains a cyclic $\boldsymbol{mP}$-subgroup for $m \leqslant n$ and of order $\geqslant 49$; this contradicts Lemma 3. Thus $|\pi(G) - \{2,3,5\}| \leqslant 1$.

LEMMA 8. *If G is an abelian non-cyclic $\mathbf{4P}$-group then either $|G| \; |2^\alpha 3^\beta 5^\gamma t^\delta$ or $|G| \, |2^\alpha 3^\beta 5^\gamma \cdot p$.*

PROOF. Lemmas 4 and 7 imply Lemma 8.

LEMMA 9. *If G is an abelian $\mathbf{4P}$-group and for the prime p, $p \, | \, |G|$, $p \geqslant 23$, then $G \simeq G_p$.*

PROOF. Lemmas 3 and 6 imply our Lemma.

LEMMA 10. *If G is an abelian $\mathbf{4P}$-group, with a partition to 4 subsets A, B, C, D, then each subset of the partition contains at most 16 involutions.*

PROOF. W.l.o.g. assume that A contains 17 involutions $\{x, y_1, \ldots, y_{16}\} \subseteq A$.
By assumption $\{xy_1, \ldots, y_{16}\} \cap A = \emptyset$. W.l.o.g. assume that $|\{xy_1, \ldots, y_{16}\} \cap B| \geqslant 6$ and $\{xy_1, \ldots, xy_6\} \subseteq B$. By assumption $\{y_1 y_2, y_1 y_3, \ldots, y_1 y_6\} \cap (A \cup B) = \emptyset$ and w.l.o.g. we can assume that $\{y_1 y_2, y_1 y_3, y_1 y_4\} \subseteq C$. Furthermore $\{y_2 y_1, y_3 y_4\} \cap (A \cup B \cup C) = \emptyset$ and consequently $\{y_2 y_3, y_2 y_4\} \subseteq D$. Therefore $y_3 y_4 \notin (A \cup B \cup C \cup D)$, a contradiction.

LEMMA 11. *If the 2-group G is in an elementary abelian $\mathbf{4P}$-group then $|G| \leqslant 64$. Furthermore if $|G| = 64$ then w.l.o.g. $|A| = |B| = |C| = 16$ and $|D| = 15$.*

PROOF. Lemma 11 is an immediate corollary of Lemma 10.

LEMMA 12. *The abelian group $G \simeq C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_3$ is not a $\mathbf{4P}$-group.*

PROOF. The group G contains 31 involutions $x_1, \ldots, x_{31}$, 2 elements θ and θ^2 of order 3, 62 elements of order 6, $x_1\theta, \ldots, x_{31}\theta, x_1\theta^2, \ldots, x_{31}\theta^2$ and the unit 1.

Let A, B, C, D be a partition of G to 4 subsets. W.l.o.g. A contains at least 22 nonidentity elements of order 2 and 6 denoted by $x_1, \ldots, x_i$, $y_1\theta, \ldots, y_j\theta$, $z_1\theta^2, \ldots, z_k\theta^2$, where x_i, y_j, z_k are involutions, $i+j+k \geqslant 22$ and $0 \leqslant i, j, k \leqslant 22$.

As in Lemma 1 we can construct an element $a \in G$ and $a \notin A \cup B \cup C \cup D$, a contradiction.

LEMMA 13. *The following abelian groups are not in* **4P**.

a) $C_3 \times C_3 \times C_3 \times C_3 \times C_3$,

b) $C_2 \times C_2 \times C_3 \times C_3 \times C_3$,

c) $C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2$.

PROOF. *a)* If $G = C_3 \times C_3 \times C_3 \times C_3 \times C_3$ then w.l.o.g. A contains at least 61 elements $x_1, \ldots, x_{61}$. By assumption the 60 distinct elements $x_1x_2, \ldots, x_1x_{61} \notin A$. Thus $|\langle x_1, \ldots, x_{61}\rangle| \geqslant 121$ and consequently $\langle x_1, \ldots, x_{61}\rangle = G$. W.l.o.g. assume that $\langle x_1, \ldots, x_6\rangle = G$. Hence $G = \langle x_1\rangle \times \ldots \times \langle x_6\rangle$. Using AL and the fact that the generators of G $x_1, \ldots, x_6 \in A$ we found that $G = C_3 \times C_3 \times C_3 \times C_3 \times C_3$ is not in **4P**. Thus *a)* holds.

b) If $G = C_2 \times C_2 \times C_3 \times C_3 \times C_3$ then w.l.o.g. A contains at least 27 elements. If $|A| > 27$ then the same arguments as in *a)* imply that $G = \langle x_1\rangle \times \ldots \times \langle x_5\rangle$ where $x_1, \ldots, x_5 \in A$ and then AL yields that $C_2 \times C_2 \times C_3 \times C_3 \times C_3$ is not in **4P**. Therefore we may assume that $|A| = |B| = |C| = 27$ and $|D| = 26$.

If $A = \{x_1, \ldots, x_{27}\}$ and $\langle x_1, \ldots, x_{27}\rangle = G$ then the same arguments as in *a)* together with AL imply that G is not in **4P**. Therefore we may assume that $A = \{x_1, \ldots, x_{27}\}$ and $\langle x_1, \ldots, x_{27}\rangle \subset G$. Since the distinct elements $x_1x_2, \ldots, x_1x_{27} \notin A$ we have $|\langle x_1, \ldots, x_{27}\rangle| = 54$ and w.l.o.g. $\langle x_1, x_2, x_3, x_4\rangle = C_2 \times C_3 \times C_3 \times C_3$. Consequently, $\langle x_1\rangle \times \langle x_2\rangle \times \langle x_3\rangle \times \langle x_4\rangle = C_2 \times C_3 \times C_3 \times C_3$, $|A| = |B| = |C| = |D| + 1 = 27$ and $x_1, x_2, x_3, x_4 \in A$. These last three results and AL imply that G is not in **4P** and thus *b)* holds.

c) Assume that $G = C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2$ is in **4P**. Then by Lemma 11 $|A| = |B| = |C| = |D| + 1 = 16$. Denote $A = \{x_1, \ldots, x_{16}\}$ and $y \in B \cup C \cup D$. If $y \neq x_ix_j\, \forall i, j \leqslant 16$, $i \neq j$, then the 2-elementary abelian group G is a **4P**-group also for the partition $A \cup \{y\}$, $B - \{y\}$, $C - \{y\}$ and $D - \{y\}$. But $|A \cup \{y\}| = 17$ contradicts Lemma 10. Therefore for every $y \in B \cup C \cup D$ there exist i, j, $i \neq j$, $1 \leqslant i, j \leqslant 16$ such that $y = x_ix_j$. Thus $\langle x_1, \ldots, x_{16}\rangle = G$ and w.l.o.g. $\langle x_1\rangle \times \ldots \times \langle x_6\rangle = G$. Since $x_1, \ldots, x_6 \in A$ the 15 elements $x_ix_j \notin A$, where $1 \leqslant i, j \leqslant 6$, $i \neq j$. At least 5 of these elements are w.l.o.g. in B. Let $X = \{x_m x_n \mid m, n \in \{1, \dots, 6\}\} \subseteq B$ be the set of these 5 elements.

CASE 1. $X \subseteq \{x_m x_n \mid m, n \in \{1, 2, 3, 4\}\}$.

Clearly $X \subseteq \{x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4, x_3 x_4\} \cap B$ and consequently G is not in $\mathbf{4P}$.

CASE 2. $X \subseteq \{x_m x_n \mid m, n \in \{1, 2, 3, 4, 5\}\}$.

Assume that the index m shows up in 4 pairs. W.l.o.g. assume $m = 1$.

Then $\{x_1 x_2, x_1 x_3, x_1 x_4, x_1 x_5\} \subseteq X$ and since G is in $\mathbf{4P}$, we obtain $|X| = 4$, a contradiction.

Assume now that the index m shows up in pairs. W.l.o.g. assume that $\{x_1 x_2, x_1 x_3, x_1 x_4\} \subseteq X$ and $x_1 x_5 \notin X$.

Therefore $|X \cap \{x_2 x_5, x_3 x_5, x_4 x_5\}| = 2$. W.l.o.g. assume that $X = \{x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_5, x_3 x_5\} \subseteq B$.

Therefore we have:

(i) $x_1, \dots, x_6 \in A$;

(ii) $X = \{x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_5, x_3 x_5\} \subseteq B$;

(iii) $|A| = |B| = |C| = |D| + 1 = 16$.

AL and (i), (ii), (iii) imply that G is not a $\mathbf{4P}$ group, a contradiction.

Thus we may assume that both indexes m and n show up only in 2 pairs.

Therefore we have (i), (iii) and w.l.o.g. $X = \{x_1 x_2, x_1 x_3, x_2 x_4, x_4 x_5, x_3 x_5\}$. Then AL implies that G is not a $\mathbf{4P}$-group, a contradiction.

CASE 3. $X = \{x_m x_n \mid m, n \in \{1, 2, 3, 4, 5, 6\}\}$.

Assume that the index m shows up in 5 pairs. W.l.o.g. assume $m = 1$.

Therefore we have (i), (iii) and $X = \{x_1 x_2, x_1 x_3, x_1 x_4, x_1 x_5, x_1 x_6\}$. Then AL implies that G is not in $\mathbf{4P}$, a contradiction.

Assume now that the index m shows up in 4 pairs. W.l.o.g. assume $m = 1$.

Therefore we have (i), (iii) and $X = \{x_1 x_2, x_1 x_3, x_1 x_4, x_1 x_5, x_1 x_6\}$, where $i \in \{2, 3, 4, 5\}$. Hence AL implies that G is not in $\mathbf{4P}$, a contradiction.

Assume now that the index m shows up in 3 pairs. W.l.o.g. assume that $m = 1$.

W.l.o.g. we have 2 cases:

$(*)$ $X = \{x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_5, x_2 x_6\}$ (1 and 2 appear in 3 pairs)

$(**)$ $X = \{x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_5, x_3 x_6\}$ (only 1 appear in 3 pairs).

In both cases the combination of AL and (i), (iii) imply that G is not in **4*P***, a contradiction.

Therefore we may assume that both indexes m and n appear at most in 2 pairs.

W.l.o.g. we obtain that $X = \{x_1 x_2, x_1 x_3, x_2 x_4, x_3 x_5, x_4 x_6\}$. Thus AL, (i) and (ii) imply that G is not in **4*P***, a final contradiction.

Consequently, Lemma 13 holds.

PROOF OF THE MAIN THEOREM. Assume that G is an abelian **4*P***-group of the main theorem type.

If G is also of Lemma 8 type then Lemmas 2, 3, 4, 11 and 13 imply that $\delta \leqslant 2$, $\gamma \leqslant 2$, $\beta \leqslant 4$, $\alpha \leqslant 5$.

The detailed information of our lemmas force G to be isomorphic to a group from the list which appears in the main theorem a)-e).

REMARKS. Our set is $\{1, 2, ..., n\} \subsetneq \mathbb{N}$ with the addition $+$.

We found by computing the following facts:

1) $f^*(2) = 8$ and there exists only one solution for **2*P***.

A: 1 2 4 8

B: 3 5 6 7

2) $f^*(3) = 23$ and there exist only 3 solutions for **3*P***.

(I): *A*: 1 2 4 8 11 16 22

B: 3 5 6 7 19 21 23

C: 9 10 12 13 14 15 17 18 20

(II): *A*: 1 2 4 8 11 17 22

B: 3 5 6 7 19 21 23

C: 9 10 12 13 14 15 16 18 20

(III): *A*: 1 2 4 8 11 22

B: 3 5 6 7 19 21 23

C: 9 10 12 13 14 15 16 17 18 20

Each solution of $n = 3$ is an extension of a solution for $n = 2$. In particular, the first element of the third subset C is $9 = f^*(2)+1$.

3) $f^*(4) = 66$ and there exist exactly 29931 solutions for **4*P***.

Let $k = f^*(n)$. Define $R(i,j)$; $i,j \leqslant k$ iff for each n- partition of $1 \dots k$, i belongs to the same subset that j does.

DEFINITION. The root of ***nP*** is the collection of the equivalence classes of size > 1 of R ordered according to their minimal elements.

EXAMPLES.

(1) The root of **2*P*** is

A: 1 2 4 8

B: 3 5 6 7

(2) The root of **3*P*** is

A: 1 2 4 8 11 22

B: 3 5 6 7 19 21 23

C: 9 10 12 13 14 15 18 20

(3) The root of **4*P*** is

A: 1 2 4 8 11 22 25

B: 3 5 6 7 19 21 23 51 52 64 65

C: 9 10 12 13 14 15 17 18 20 54 55 61 62

D: 24 26 27 28 29 30 33 41 42 47 49

CONJECTURES. *a*) The root for ***nP*** is an extension of the root for $(\boldsymbol{n} - 1)\boldsymbol{P}$.

b) A solution of ***nP*** is an extension of a solution for $(\boldsymbol{n} - 1)\boldsymbol{P}$. In particular, the smallest number in the n-th subset of an ***nP*** solution is $f^*(n - 1) + 1$.

c) $f^*(n) = 3[f^*(n - 1) - 1]$ for $n \geqslant 4$ (or $f^*(n) = 21\frac{1}{2} \cdot 3^{n-3} + 1.5$ for $n \geqslant 3$).

The conjectures are true for **3*P*** and **4*P***. Calculations show that there exist **3*P*** solutions which cannot be extended to **4*P*** solutions.

In fact the mentioned **3*P*** solution (I) can be extended to 8238 **4*P*** solutions. The **3*P*** solution (II) cannot be extended to **4*P*** solutions. The **3*P*** solution (III) can be extended to 21693 **4*P*** solutions. The total number of **4*P*** solutions is 29931.

EVALUATION OF $f^*(5)$. If we start from an arbitrary **4*P*** solution and try to extend the solution to **5*P*** solution there is a chance that this **4*P*** solution cannot be extended to the **5*P*** solution.

Therefore we prefer to start from the root of **4*P***. We will assume that the above-mentioned conjecture *a*) is true.

The numbers:

16, 31, 32, 34, 35, 36, 37, 38, 39, 40, 43, 44, 45, 46, 48,
50, 53, 56, 57, 58, 59, 60, 63, 66

are not in the root.

If we put 16 in the subset A of the root then we know that there exist 8238 extensions to **5*P*** solutions.

If we put 16 in B we have 0 extensions. If we put 16 in C we have 21693 extensions to **5*P*** solutions. According to our conjecture *b*) the smallest number in D is 24; thus if we put 16 in D we have 0 extensions. Thus statistically the best possibility is to put 16 in C.

Computing brings us to the *ideal* **4*P*** solution:

A: root, 50, 63

B: root, 53, 66

C: root, 16, 56, 57, 58, 59, 60

D: root, 31, 32, 34, 35, 36, 37, 38, 39, 40, 43, 44, 45, 46, 48

FOR EXAMPLE. If we put 31 in (A, B, C, D), the possible extensions are $(3456, 0, 0, 18237)$, respectively. Therefore we decided to put 31 in D.

Extension of this ideal solution and using other ideas brought us to the partition which illustrates that $f^*(5) \geqslant 195$. Our conjecture *c*) is that $f^*(5) = 195$.

APPENDIX A

4*P*-groups of small orders.

Cyclic groups of order $\leqslant 15$ are $3P$-groups by [AEKL] and consequently they are $4P$-groups.

Cyclic groups C_k of order k, where k is either 48 or $16 \leqslant k \leqslant 45$ are $4P$-groups as shown in the following list:

$n = 16$

1 2 6 10 14
4 5 7 8
3 9 11 13 15
12

$n = 17$

1 2 6 11 14
4 5 8 15
3 7 12 16
9 10 13

$n = 18$

1 2 6 11 15
4 5 7 13 14
3 8 9 10
12 16 17

$n = 19$

1 2 6 9 13 17
4 5 7 8 14
3 10 11 15 16
12 18

$n = 20$

1 2 6 11 14 18
4 5 8 15
3 7 9 13 17 19
10 12 16

$n = 21$

1 2 6 11 14 19
4 5 8 15 16
3 7 9 13 18
10 12 17 20

$n = 22$

1 2 6 11 14
4 5 8 15 16 17
7 10 12 18 21
3 9 13 19 20

$n = 23$

1 2 6 9 14 17 21
4 5 8 15 18
3 7 11 12 16 20
10 13 19 22

$n = 24$

1 2 5 8 12 15 19 22
3 7 9 11 13 17 21 23
4 6 18 20
10 14 16

$n = 25$

1 2 5 8 15
9 10 11 12 13 14 16 17
3 7 18 23 24
4 6 19 20 21 22

$n = 26$

1 2 5 8 11 15 18 21 24
3 7 9 13 17 19 23 25
4 6 20 22
10 12 14 16

$n = 27$

1 2 5 8 11 15 22 25
4 6 7 9 17 18 20
3 10 12 19 24 26
13 14 16 21 23

$n = 28$

1 2 5 8 12 15 19 26
9 10 11 13 14 16 17 18
3 20 25 27
4 6 7 21 22 23 24

$n = 29$

1 2 5 8 11 15 18 21 24 27
4 6 7 9 12 17 20 22 25
3 10 14 19 26 28
13 16 23

$n = 30$

1 2 5 8 11 15 18 22 25 28
3 7 9 13 17 19 21 23 27 29
6 10 12 20
4 14 16 24 26

$n = 31$

1 2 5 8 12 15 19 22 26 29
4 6 9 14 16 17 27
3 11 13 18 20 28 30
7 10 21 23 24 25

$n = 32$

1 2 5 8 11 15 21 24 27 30
9 12 13 14 16 17 18 19 20
3 10 22 29 31
4 6 7 23 25 26 28

$n = 33$

1 2 5 8 11 15 18 22 25 28 31
6 9 13 16 17 20 24 27
3 7 14 19 30 32
4 10 12 21 23 26 29

$n = 34$

1 2 5 8 11 15 22 29 32
4 6 7 9 17 25 27 28 30
3 10 19 24 26 31 33
12 13 14 16 18 20 21 23

$n = 35$

1 2 5 8 11 15 24 27 30 33
6 9 14 16 17 18 19 21 29
3 13 20 22 32 34
4 7 10 12 23 25 26 28 31

$n = 36$

1 2 5 8 11 15 18 21 25 28 31 34
4 6 7 9 17 19 27 29 30 32
3 10 12 24 26 33 35
13 14 16 20 22 23

$n = 37$

1 2 5 8 11 15 22 25 29 32 35
4 7 9 12 17 20 28 30 33 36
3 10 18 19 26 27 34
6 13 14 16 21 23 24 31

$n = 38$

1 2 5 8 11 15 27 30 33 36
4 6 7 9 17 19 29 31 32 34
3 10 12 21 26 28 35 37
13 14 16 18 20 22 23 24 25

$n = 39$

1 2 5 8 11 15 18 21 24 28 31 34 37
4 6 7 9 17 19 20 22 30 32 33 35
3 10 12 14 16 23 25 27 29 36 38
13 26

$n = 40$

1 2 5 8 11 15 25 29 32 35 38
9 12 13 17 20 23 27 28 31
3 10 14 18 22 26 30 37 39
4 6 7 16 19 21 24 33 34 36

$n = 41$

1 2 6 11 15 18 23 30 35 39
3 5 12 14 16 25 27 29 36 38 40
8 10 13 17 19 22 24 28 31 33
4 7 9 20 21 26 32 34 37

$n = 42$

1 2 5 11 15 19 23 27 31 37 40
3 8 13 18 20 22 24 29 34 39 41
6 9 10 12 14 17 25 28 30 32 33
4 7 16 21 26 35 36 38

$n = 43$

1 2 9 12 15 20 23 28 31 34 41
4 5 6 7 14 16 17 26 27 29 37 38 39
8 10 19 21 22 24 33 35 36
3 11 13 18 25 30 32 40 42

$n = 44$

1 2 5 8 11 15 29 33 36 39 42
4 7 9 12 17 22 27 32 35 37 40 43
3 10 14 18 19 25 26 30 34 41
6 13 16 20 21 23 24 28 31 38

$n = 45$

1 2 9 12 15 20 25 30 33 36 43
3 5 7 16 18 27 29 38 40 42 44
8 10 17 19 21 22 23 24 26 28 35 37
4 6 11 13 14 31 32 34 39 41

$n = 48$

1 2 6 9 14 19 22 26 29 34 39 42 46
3 4 5 15 17 23 24 25 31 33 43 44 45
7 8 10 12 21 27 36 38 40 41
11 13 16 18 20 28 30 32 35 37 47

Abelian non-cyclic groups of order $\leqslant 28$ are **4*P***-groups. We leave to the reader as an exercise to find examples of **4*P***-partitions for these groups.

The following **4*P***-partitions illustrate why the last 19 groups from the list of the main theorem are **4*P***-groups.

Let us denote the generators of $Cm_1 \times \ldots \times Cm_k$ by $x, y, z, r, s, \ldots$ respectively. Then **4*P*** partitions of the following groups are illustrated as follows:

$C_2 \times C_2 \times C_2 \times C_2 \times C_3$
size of group = 48
$s\ s^2\ r\ z\ zrs\ ys^2\ yz\ x\ xzs\ xyr\ xyzrs$
$rs\ rs^2\ zs\ zs^2\ yzs\ yzs^2\ xz\ xzs^2\ xyzr\ xyzrs^2$
$zr\ zrs^2\ y\ ys\ yr\ yrs\ yrs^2\ xy\ xys\ xys^2\ xyz\ xyzs\ xyzs^2$
$yzr\ yzrs\ yzrs^2\ xs\ xs^2\ xr\ xrs\ xrs^2\ xzr\ xzrs\ xzrs^2\ xyrs\ xyrs^2$

$C_2 \times C_2 \times C_2 \times C_5$
size of group = 40
$r\ r^4\ z\ zr^3\ yzr\ yzr^4\ xzr\ xzr^4 xyzr\ xyzr^4$
$r^2\ r^3\ zr\ yzr^2\ yzr^3\ xr^2\ xr^3\ xyr^2\ xyr^3$
$zr^2\ zr^4\ y\ yr\ yr^2\ yr^3\ yr^4\ x\ xz$
$yz\ xr\ xr^4\ xzr^2\ xzr^3\ xy\ xyr\ xyr^4\ xyz\ xyzr^2\ xyzr^3$

$C_2 \times C_2 \times C_9$
size of group = 36
$z\ z^2\ z^7\ y\ yz^3\ yz^6\ x\ xz^3\ xz^6\ xyz^2\ xyz^7$
$z^3\ z^4\ z^5\ z^6\ yz\ yz^8\ xz\ xz^8\ xyz\ xyz^8$
$z^8\ yz^2\ yz^4\ yz^7\ xy\ xyz^3\ xyz^6$
$yz^5\ xz^2\ xz^4\ xz^5\ xz^7\ xyz^4\ xyz^5$

$C_3 \times C_3 \times C_3 \times C_2$
size of group = 54
$u\ z\ z^2\ y\ yzu\ y^2\ y^2z^2u\ x\ xz^2u\ xy^2u\ xy^2z^2\ x^2\ x^2zu\ x^2yu\ x^2yz$
$zu\ z^2u\ yu\ yz^2u\ y^2u\ y^2zu\ xz^2\ xy\ xy^2zu\ x^2z\ x^2yz^2u\ x^2y^2$
$yz\ y^2z^2\ xu\ xzu\ xyu\ xyz\ xyz^2\ xy^2zx^2u\ x^2z^2u\ x^2yz^2\ x^2y^2u$
$x^2y^2z\ x^2y^2z^2$
$yz^2\ y^2z\ xz\ xyzu\ xyz^2u\ xy^2\ xy^2z^2u\ x^2z^2\ x^2y\ x^2yzu\ x^2y^2zu\ x^2y^2z^2u$

$C_3 \times C_3 \times C_3 \times C_3$

size of group = 81

$u\ u^2\ z\ z^2\ y\ yz^2u^2\ y^2\ y^2zu\ x\ xz^2u^2\ xyu\ xyz\ xy^2zu^2\ xy^2z^2u\ x^2\ x^2zu\ x^2yzu^2\ x^2yz^2u\ x^2y^2u^2\ x^2y^2z^2$

$zu\ zu^2\ z^2u\ z^2u^2\ yu\ yu^2\ y^2u\ y^2u^2\ xu\ xzu\ xyz^2\ xyz^2u^2\ xy^2u\ xy^2zu\ x^2u^2\ x^2z^2u^2\ x^2yu^2\ x^2yz^2u^2\ x^2y^2z\ x^2y^2zu$

$yz\ yzu\ yz^2\ yz^2u\ y^2z\ y^2zu^2\ y^2z^2\ y^2z^2u^2\ xu^2\ xy\ xyu^2\ xyzu\ xyz^2u\ xy^2\ x^2u\ x^2y\ x^2y^2\ x^2y^2u\ x^2y^2zu^2\ x^2y^2z^2u^2$

$yzu^2\ y^2z^2u\ xz\ xzu^2\ xz^2\ xz^2u\ xyzu^2\ xy^2u^2\ xy^2z\ xy^2z^2\ xy^2z^2u^2\ x^2z\ x^2zu^2\ x^2z^2\ x^2z^2u\ x^2yu\ x^2yz\ x^2yzu\ x^2yz^2\ x^2y^2z^2u$

$C_3 \times C_3 \times C_2 \times C_2$

size of group = 36

$r\ z\ y\ yzr\ y^2\ x\ xyr\ xyz\ x^2\ x^2zr\ x^2y^2r\ x^2y^2z$

$zr\ yz\ y^2z\ y^2zr\ xz\ xzr\ xyzr\ xy^2z\ xy^2zr\ x^2z\ x^2yz\ x^2yzr\ x^2y^2zr$

$yr\ y^2r\ xy\ xy^2\ x^2y\ x^2y^2$

$xr\ xy^2r\ x^2r\ x^2yr$

$C_4 \times C_3 \times C_3$

size of group = 36

$z\ z^2\ y\ y^2\ x\ xy^2z\ x^2\ x^2yz\ x^2y^2z^2\ x^3yz^2$

$yz\ yz^2\ y^2z\ y^2z^2\ xy\ xyz\ xyz^2\ x^2z\ x^2z^2\ x^3y^2\ x^3y^2z\ x^3y^2z^2$

$xz\ xz^2\ x^2y\ x^2yz^2\ x^2y^2\ x^2y^2z\ x^3\ x^3z\ x^3z^2$

$xy^2\ xy^2z^2\ x^3y\ x^3yz$

$C_4 \times C_4 \times C_2$

size of group = 32

$z\ y\ y^2\ x\ xy^3z\ x^2y^2z\ x^2y^3\ x^3\ x^3yz$

$yz\ y^3\ y^3z\ xy^2\ xy^2z\ x^2y^2\ x^3y^2\ x^3y^2z$

$y^2z\ xy\ xyz\ x^2\ x^2z\ x^2yz\ x^2y^3z\ x^3y^3\ x^3y^3z$

$xz\ xy^3\ x^2y\ x^3z\ x^3y$

$C_8 \times C_4$

size of group = 32

$y\ y^2\ x\ x^2\ x^3y^3\ x^4\ x^5y\ x^6y^3$

$y^3\ xy^2\ x^2y^2\ x^3y^2\ x^4y\ x^4y^3\ x^5y^2\ x^6y^2\ x^7y^2$

$xy\ xy^3\ x^3\ x^4y^2\ x^5\ x^7\ x^7y\ x^7y^3$

$x^2y\ x^2y^3\ x^3y\ x^5y^3\ x^6\ x^6y$

$C_{16} \times C_2$

size of group = 32

$y\ x\ x^3y\ x^6y\ x^8\ x^{12}\ x^{13}y\ x^{14}$

$xy\ x^3\ x^4\ x^8y\ x^{10}\ yx^{13}\ x^{15}y$

$x^2\ x^2y\ x^5\ x^5y\ x^{11}\ x^{11}y\ x^{14}y\ x^{15}$

$x^4y\ x^6\ x^7\ x^7y\ x^9\ x^9y\ x^{10}\ x^{12}y$

$C_4 \times C_9$

size of group = 36

$y\ y^2\ y^5\ xy\ xy^4\ xy^7\ x^2\ x^2y^3\ x^2y^6\ x^3y^2\ x^3y^5\ x^3y^8$
$y^3\ y^6\ y^8\ x\ x^2y^4\ x^3y$
$y^4\ y^7\ xy^2\ xy^5\ xy^8\ x^3y^4\ x^3y^7$
$xy^3\ xy^6\ x^2y\ x^2y^2\ x^2y^5\ x^2y^7\ x^2y^8\ x^3\ x^3y^3\ x^3y^6$

$C_8 \times C_2 \times C_2$

size of group = 32

$x\ y\ x\ xyz\ x^2\ x^3z\ x^4yz\ x^5y\ x^6$
$yz\ xy\ x^3y\ x^3yz\ x^5yz\ x^7\ x^7y$
$xz\ x^2z\ x^2y\ x^2yz\ x^5z\ x^6z\ x^6y\ x^6yz\ x^7z$
$x^3\ x^4\ x^4y\ x^5\ x^7yz$

$C_4 \times C_2 \times C_2 \times C_2$

size of group = 32

$r\ z\ y\ yzr\ x\ xzr\ xyz\ x^2\ x^3yr$
$zr\ yz\ xyzr\ x^2z\ x^2zr\ x^2yz\ x^3\ x^3yzr$
$yr\ xr\ xz\ x^2r\ x^2yr\ x^2yzr\ x^3r\ x^3z$
$xy\ xyr\ x^2y\ x^3zr\ x^3y\ x^3yz$

$C_2 \times C_2 \times C_2 \times C_2 \times C_2$

size of group = 32

$s\ r\ zs\ zr\ y\ yrs\ xz\ xzrs\ xy\ xyrs$
$ys\ yr\ xs\ xr\ xys\ xyr\ xyz\ xyzs\ xyzr\ xyzrs$
$x\ zrs\ yz\ yzs\ yzr\ yzrs\ x\ zrs\ xzs\ xzr$
rs

$C_3 \times C_3 \times C_5$

size of group = 45

$z\ z^2\ y\ y^2\ x\ xyz\ xy^2z^2\ x^2\ x^2yz^3\ x^2y^2z^4$
$z^3\ z^4\ yz\ y^2z\ xz^2\ xy\ xy^2\ x^2z^3\ x^2y\ x^2y^2$
$yz^2\ yz^3\ y^2z^2\ y^2z^3\ xz^3\ xyz^2\ xyz^3\ xy^2z^3\ x^2z^2\ x^2yz^2\ x^2y^2z^2\ x^2y^2z^3$
$yz^4\ y^2z^4\ xz\ xz^4\ xyz^4\ xy^2z\ xy^2z^4\ x^2z\ x^2z^4\ x^2yz\ x^2yz^4\ x^2y^2z$

$C_4 \times C_2 \times C_2 \times C_3$

size of group = 48

$r\ r^2\ z\ y\ xr^2\ xz\ xyzr\ x^2\ x^2yzr^2\ x^3r\ x^3y\ x^3yzr^2$
$zr\ zr^2\ yz\ yzr\ yzr^2\ xr\ xzr\ x^2z\ x^2yz\ x^2yzr\ x^3r^2\ x^3zr^2$
$yr\ yr^2\ xzr^2\ xy\ xyr\ xyr^2\ xyzr^2\ x^2y\ x^3zr\ x^3yr\ x^3yr^2\ x^3yzr$
$x\ xyz\ x^2r\ x^2r^2\ x^2zr\ x^2zr^2\ x^2yr\ x^2yr^2\ x^3\ x^3z\ x^3yz$

$C_4 \times C_4 \times C_3$

size of group = 48

$z\ z^2\ y\ y^2\ x\ xyz\ xy^2z^2\ x^2\ x^2yz^2\ x^2y^3z\ x^3y^2z\ x^3y^3z^2$
$yz\ yz^2\ y^3\ y^3z\ y^3z^2\ xy\ xyz^2\ x^2z\ x^2z^2\ x^3y^3\ x^3y^3z$
$y^2z\ y^2z^2\ xy^2\ xy^2z\ xy^3\ xy^3z\ xy^3z^2\ x^3y\ x^3yz\ x^3yz^2\ x^3y^2\ x^3y^3z^2$
$xz\ xz^2\ x^2y\ x^2yz\ x^2y^2\ x^2y^2z\ x^2y^2z^2\ x^2y^3\ x^2y^3z^2\ x^3\ x^3z\ x^3z^2$

$C_7 \times C_7$

size of group = 49

$y\ y^6\ x\ xy^3\ xy^5\ x^2y^2\ x^2y^4\ x^5y^3\ x^5y^5\ x^6\ x^6y^2\ x^6y^4$
$y^2\ y^5\ xy\ xy^2\ x^2\ x^2y\ x^3y^4\ x^4y^3\ x^5\ x^5y^6\ x^6y^5\ x^6y^6$
$y^3\ y^4\ xy^4\ xy^6\ x^2y^6\ x^3y\ x^3y^6\ x^4y\ x^4y^6\ x^5y\ x^6y\ x^6y^3$
$x^2y^3\ x^2y^5\ x^3\ x^3y^2\ x^3y^3\ x^3y^5\ x^4\ x^4y^2\ x^4y^4\ x^4y^5\ x^5y^2\ x^5y^4$

$C_8 \times C_2 \times C_3$

size of group = 48

$z\ z^2\ y\ x\ xyz^2\ x^3z^2\ x^3y\ x^4\ x^5z\ x^5y\ x^7\ x^7yz$
$yz\ yz^2\ xz\ x^2z\ x^3\ x^4y\ x^4yz\ x^4yz^2\ x^5\ x^5z^2\ x^6z^2\ x^7z^2$
$xz^2\ xyz\ x^2\ x^2yz\ x^2yz^2\ x^3z\ x^3yz^2\ x^4z\ x^4z^2\ x^5yz\ x^6\ x^7z\ x^7yz^2$
$xy\ x^2z^2\ x^2y\ x^3yz\ x^5yz^2\ x^6z\ x^6y\ x^6yz\ x^6yz^2\ x^7y$

The partial semigroup $\{1, \dots, 195\} \subset \mathbb{N}$ has a $5P$-partition, and consequently, $f^*(5) \geqslant 195$ as illustrated here:

A: 1 2 4 8 11 22 25 50 63 68 136 149 154 159 168 177 182 189 192 195

B: 3 5 6 7 19 21 23 51 52 53 64 65 66 137 138 139 150 151 152 163 164 165 179 180 181 193 194

C: 9 10 12 13 14 15 16 17 18 20 54 55 56 57 58 59 60 61 62 140 141 142 143 144 145 146 147 148 183 184 185 186 187 188 190 191

D: 24 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 153 155 156 157 158 160 161 162 166 167 169 170 171 172 173 174 175 176 178

E: 67 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 109 110 111 112 113 114 115 116 117 118 119 120 121 122 123 124 125 126 127 128 129 130 131 132 133 134 135

REFERENCES

[AEKL] Z. Arad - G. Ehrlich - O. H. Kegel - J. Lennox, *An application of Ramsey's theory to partitions in groups, I*, Rend. Sem. Sem. Mat. Univ. Padova, 84 (1990), pp. 143-157.

[Gr1] R. L. Graham, *Rudiments of Ramsey theory*, Conference Board of the Mathematical Sciences, Regional Conference Series in Mathematics, 45 (1981).

[Gr2] R. L. Graham - B. L. Rothschild - J. H. Spencer, *Ramsey Theory*, Wiley-Interscience Series in Discrete Math. (1977), pp. 106-112.

[RND] E. Reingold - J. Nievergelt - N. Deo, *Combinatorial Algorithms*, Prentice Hall (1977), pp. 106-112.

[Sc] I. Schur, *Über die Kongruenz* $x^m + y^m$ *congruent* $2m$ $(\mathrm{mod}\, p)$, Iber Deutsche Math. Verein., 25 (1916), pp. 114-116.

Manoscritto pervenuto in redazione l'1 settembre 1991 e, in forma revisionata il 12 dicembre 1991.