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On Non-Supersoluble and Non-Polycyclic Normal Subgroups.

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ABSTRACT - This paper presents an investigation of groups G with the property that every non-polycyclic normal subgroup H has a finite G-invariant insoluble image and continues a previous investigation of groups with a similar property in relation to non-supersolubility.

0. Introduction.

This work is concerned mainly with groups having a certain property in relation to their non-polycyclic normal subgroups. As such, it represents a continuation of the investigations carried out in papers [1] and [2], where properties υ and σ (respectively) were introduced. We recall that a group G was said to have property υ (respectively σ) if, given any non-nilpotent (respectively non-supersoluble) normal subgroup H of G, there exists a G-invariant subgroup K of finite index in H such that H/K is non-nilpotent (respectively non-supersoluble). By Theorem 2 of [2], any group with σ also has υ.

The firts section proper presents some further results on groups with property σ and, it is hoped, provides some motivation for the introduction of the property π which is defined in Section 2 and which is discussed throughout the remainder of the paper.

1. Non-supersoluble normal subgroups.

If the group G has σ, then we know from Proposition 1 and Theorem 1 of [2] that the union $H(G)$ of the series of iterated Hirsch-Plotkin

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radicals is polycyclic and that the intersection $\phi_f(G)$ of all maximal subgroups of finite index in G is contained in the Fitting radical $\mathrm{Fit}(G)$ (which is finitely generated). These results draw attention to normal polycyclic subgroups, which are always contained in $H(G)$. Defining $P(G)$ to be product of all normal polycyclic subgroups of the group G, we have the following.

LEMMA 1. *Let G be a group with property σ. Then $\phi_f(G) \leqslant P(G)$, $P(G)$ is polycyclic and $P(G/\phi_f(G)) = P(G)/\phi_f(G)$.*

PROOF. In view of the preceding remarks, all that needs to be shown is that, in a group G with σ, $P(G/\phi_f(G)) \leqslant P(G)/\phi_f(G)$ (the reverse inclusion being obvious). But, from Theorem 1 of [2], we know that $\overline{G} = G/\phi_f(G)$ has σ and so, again from our remarks, $P(\overline{G})$ is polycyclic. The result follows.

Now Theorem 1 of [1] tells us that a group G has ν if and only if $\phi_f(G) \leqslant \mathrm{Fit}(G)$, $\mathrm{Fit}(G)$ is nilpotent and $\mathrm{Fit}(G/\phi_f(G)) = \mathrm{Fit}(G)/\phi_f(G)$. Further, we have seen that all normal locally nilpotent subgroups of a group G with σ are polycyclic. Accordingly, it would perhaps be reasonable to hope that the converse of Lemma 1 holds, but it is easy to show that this is not the case, for if G is the group with presentation $\langle x, y \,|\, y^{-1}xy = x^2\rangle$, then G has no non-trivial normal polycylic subgroups and $\phi_f(G) = 1$, but G does not have σ since $\mathrm{Fit}(G)$ is not finitely generated. (This is the same example as that mentioned at the end of Section 3 of [2]). To find necessary and sufficient conditions for a group to satisfy σ, conditions which are in some way similar to those for ν-groups as given in Theorem 1 of [1], we alter our perspective a little and notice that, for groups G with υ, $\mathrm{Fit}(G)$ coincides with the Baer radical of G (and, indeed, with the Hirsch-Plotkin radical). For any group G, let us denote by $P_1(G)$ the subgroup generated by all subnormal polycyclic subgroups of G. Then the following result holds.

THEOREM 1 (c.f. Theorem 1 of [1] and Theorem 1 of [2]). *A group G has property σ if and only if*

(*a*) $\phi_f(G) \leqslant P_1(G)$ *and* $P_1(G)$ *is polycyclic.*

Further, if G has σ, then

(*b*) $P_1(G/\phi_f(G)) = P_1(G)/\phi_f(G)$.

PROOF. If A is a subnormal polycyclic subgroup of the group G then A is contained in the iterated Hirsch-Plotkin radical of G (since it is contained in the iterated Baer radical). If G has σ, then this radical is polycyclic. Arguing as in the proof of Lemma 1, we see that properties (*a*) and (*b*) hold. Conversely, assume that (*a*) holds. Then (*b*) also holds. Further, $\mathrm{Fit}(G)$ is finitely generated (since it is contained in the Baer radical). We shall show that $\phi_f(G)$ is nilpotent and that $\mathrm{Fit}(G)/\phi_f(G) = \mathrm{Fit}(G/\phi_f(G))$. It will follow, by Theorem 1 of [2], that G has σ.

Suppose L is a G-invariant subgroup of finite index in $\phi_f(G)$. Then $\phi_f(G)/L = \mathrm{Frat}(G/L)$ is finite and hence nilpotent, by a result of Gaschütz (see 5.2.15 (i) of [7]). Since $\phi_f(G)$ is polycyclic, it follows that all finite images of $\phi_f(G)$ are nilpotent and hence, by a theorem of Hirsch (see 5.4.18 of [7]), $\phi_f(G)$ is nilpotent. Now let $F/\phi_f(G)$ be a normal nilpotent subgroup of $G/\phi_f(G)$. We need to show that F is nilpotent. By (*a*) and (*b*), F is polycyclic and so, as above, if F is not nilpotent there is a G-invariant subgroup T of finite index in F such that $\overline{F} = F/T$ is non-nilpotent. Since $\overline{\phi_f(G)} = \phi_f(G)\,T/T$ is finite, it is contained in $\mathrm{Frat}(G/T)$. Also, $\overline{F}/\overline{\phi_f(G)}$ is nilpotent. We can now apply the Frattini argument as in the proof of 5.2.15(i) of [7] to conclude that F/T is nilpotent, a contradiction which completes the proof of the theorem.

A further radical and one which is related to those discussed above, is $LP(G)$, the (unique) maximal normal locally polycyclic subgroup of the group G. Clearly $P(G) \leqslant P_1(G) \leqslant LP(G)$, and we have already mentioned an example of a group G in which $P(G) = 1 = \phi_f(G)$ and $P_1(G) = \mathrm{Fit}(G)$ is not finitely generated. An example due to McLain and referred to in Section 4 of [2] provides us with a non-polycyclic, locally polycyclic group G with σ in which $\phi_f(G) = 1$. Thus $P_1(G)$ cannot be replaced by $LP(G)$ in the statement of Theorem 1. Another example worth mentioning (with regard to the relationship between the various radicals) is one due to Dark [3]. His group G is a (non-trivial) Baer group with no non-trivial normal abelian subgroup. In particular, $P_1(G) = G$ and $P(G) = 1$.

To illustrate further the significance of polycyclic subgroups with regard to σ, we present the following result, which follows easily from Theorem 1, and which should be compared with Corollary 5 of [1] and Theorem 4 of [2].

THEOREM 2. *A group G has σ if and only if*

(*a*) *whenever $H \lhd G$, $\phi_f(G) \leqslant H$ and $H/\phi_f(G)$ is polycyclic, then H is polycyclic and*

(*b*) *$P_1(G)$ is polycyclic.*

We note that condition (*b*) cannot be dispensed with, as is shown by the example mentioned following the proof of Lemma 1 (or by any group G satisfying $\phi_f(G) = 1$ but not σ).

2. Non-polycyclic normal subgroups.

A group G shall be said to have property π if, given any non-polycyclic normal subgroup H of G, there exists a G-invariant subgroup K of finite index in H such that H/K is non-polycyclic (i.e. insoluble). Among other things, our intent is to establish, for the property π, results similar to those obtained for υ and σ. In some cases, the proofs are not substantially different from those of the corresponding results for υ and σ and are therefore sketched briefly or omitted altogether. We begin with a couple of rather straightforward results.

LEMMA 2 (c.f. Lemma 1 of [1] and Lemma 1 of [2]). *A group G has π if and only if, whenever H is a non-polycyclic normal subgroup of G, there is a normal subgroup N of finite index in G such that HN/N is insoluble.*

LEMMA 3. *If G has property π then G has σ.*

We note that McLain's (locally finite) group, mentioned earlier, has the property σ but not the property π, since a locally soluble group with π must be polycyclic.

PROOF OF LEMMA 3. If G has π and H is a normal, non-supersoluble subgroup of G, then either H is non-polycyclic, in which case there is clearly a G-invariant subgroup K of finite index in H with H/K not supersoluble, or else H is polycyclic, in which case there is, by a theorem of Baer (see [8], p. 162), a normal subgroup K of finite index in H such that H/K is non-supersoluble. Since H is finitely generated, we may take K to be normal in G, and the proof of Lemma 3 is complete.

Using these lemmas, the following theorem may be proved without difficulty.

THEOREM 3. *A group G has π if and only if $\phi_f(G)$ is polycyclic and $G/\phi_f(G)$ has π. Further, if G has π, then $P(G)$ (indeed $LP(G)$) is polycyclic and $P(G/\phi_f(G)) \leq P(G)/\phi_f(G)$.*

As a consequence of Theorem 3, we note that a group G has π if and only if G has σ and $G/\phi_f(G)$ and π.

Our sufficient conditions for a group G to have π include the hypothesis that $G/\phi_f(G)$ have π. This is, perhaps, a little unsatisfactory. Accordingly, it would be interesting to know of a (reasonable) sufficient condition for a group G satisfying $\phi_f(G) = 1$ to have π. Also, there is perhaps a theorem with regard to the property π which is comparable to Theorem 1 of [1] and Theorem 1 of [2] (or Theorem 1 above) for υ and σ respectively. One conjecture might be that a group G has π if and only if $\phi_f(G) \leq LP(G)$, $LP(G)$ is polycyclic and $LP(G/\phi_f(G)) = LP(G)/\phi_f(G)$. However, in the final section we give an example which shows that this is not the case. (The given hypotheses are, nonetheless, seen to imply that G has σ and if, in addition, G is locally finite then G has π — this follows from Theorem 8 below).

3. Closure properties and finiteness conditions.

We begin this section with some results concerning π which recall similar results from [1] and [2]. (Part (iii) of course has no counterpart for the property υ).

THEOREM 4. (i) *Let G be a group with property π and let H be a non-polycyclic ascendant subgroup of G. Then there is a normal subgroup L of finite index in G such that HL/L is insoluble* (*see Corollary* 2 *of* [1]).

(ii) *Let H be an ascendant subgroup of the group G. If G has π then so does H* (*see Lemma* 2 *of* [1] *and Lemma* 3 *of* [2]).

(iii) *Let G be a group with a subgroup H of finite index. Then G has property π if and only if H does* (*see Theorem* 10 *of* [2]).

We now present a theorem which indicates one way of distinguishing the class of groups with π within the class of groups satisfying σ. To do this we require the following definition.

A group G shall be said to satisfy the property π^* if, for each normal subgroup N whose finite G-quotients are all soluble, there is a bound (depending in general on N) for the derived lengths of these quotients.

We are now ready to prove:

THEOREM 5. *Let G be a group with σ. Then G has π if and only if G satisfies π^*.*

PROOF. It is easily seen that π implies π^*. Suppose then that G satisfies π^*. Then $\phi_f(G)$ is polycyclic and $G/\phi_f(G)$ has σ ([2], Theorem 1). It is straighforward to verify that $G/\phi_f(G)$ has π^* and, in order to show that G has π, we can (in view of the remark following Theorem 3 above) assume that $\phi_f(G) = 1$. Let N be a normal subgroup of G all of whose finite G-quotients are soluble and hence of derived length at most d, say. Let M be a maximal subgroup of finite index in G. It suffices to prove that $N^{(d)} \leq M$, for then $N^{(d)} \leq \phi_f(G) = 1$ and so N is soluble and hence, by Proposition 1 of [2], polycyclic. Clearly we may suppose $N \nleq M$, so that $G = MN$. Let $T = M \cap N$. Then $|N:T|$ is finite and the conjugates of T in G are precisely the conjugates in N. Write $T_0 = \mathrm{Core}_G T$. Then $|N: T_0|$ is finite and so $N^{(d)} \leq T_0 \leq M$, as required.

Using results of Mal'cev and Zassenhaus (4.2 and 3.7 of [8]), the above theorem and Theorem 7 of [2], we obtain the following.

THEOREM 6. *Let G be a subgroup of $GL(n, R)$, where R is a finitely generated domain. Then the following statements are equivalent:*

(a) G has π.

(b) G has σ.

(c) The unipotent radical of G is finitely generated.

COROLLARY 1. Let G be a subgroup of $GL(n, \mathbb{Z})$. Then G has π.

Returning now to closure properties, we shall establish a result which depends partly on the above corollary.

THEOREM 7. *Let G be a group with normal polycyclic- by-finite subgroup S. Then G has π if and only if G/S has π.*

PROOF. Suppose G/S has property π and let N be a normal subgroup of G with all of its finite G-quotients soluble. Then NS/S is polycyclic and so N is polycyclic-by-finite-by-polycyclic and hence polycyclic-by-finite and thus polycyclic. So G has π. Conversely, suppose that G has π and let $C = C_G(S)$. Assume that S is finite. Then, by Theorem 4, both C and CS have π. Since $C \cap S \leq Z(C)$, it follows easily that $C/C \cap S$ has π and thus, by Theorem 4 again, that G/S has π. By induction on the Hirsch length, we may thus suppose that S is finitely generated and free abelian. Then G/C *is* $\mathbb{Z}$-linear and thus has π, by Corollary 1.

Now let N be a normal subgroup of G such that $S \leq N$ and all finite G-quotients of N/S are soluble. Since $S \leq C$, we have NC/C polycyclic. If $N_1 = N \cap C$ is not polycyclic then there is a G-invariant subgroup T of finite index in N_1 with N_1/T insoluble. Let $D = C_N(N_1/T)$, which is normal in G and of finite index in N. Since $S \leq D$ we have N/D soluble and so N_1/T is centre-by-soluble and thus soluble. This contradiction completes the proof of the theorem.

We note the following consequence, whose proof requires (besides Theorem 7) Theorem 6 and Lemma 3 above and Proposition 1 of [2].

COROLLARY 2. Let S be a soluble-by-finite normal subgroup of the group G. Then G has π if and only if G/S and S both have π.

It is easy to see that any polycyclic-by-finite group G has π, and we know from Theorem 6 of [2] that a group G with finite rank has σ if and only if G is polycyclic-by-finite. Thus a group G of finite rank has π if and only if it has σ (if and only if it is polycyclic-by-finite).

Next, we give a couple of results on locally finite groups.

LEMMA 4. *Let G be a locally finite group which is also residually finite. Then G has π if and only if every locally soluble normal subgroup of G is finite.*

PROOF. Suppose G has π and let N be a locally soluble, normal subgroup of G. Since every finite image of N is soluble, N is polycyclic and therefore finite. Conversely, assume G satisfies the given hypothesis on locally soluble normal subgroups and let N be a normal subgroup of G all of whose finite G-quotients are soluble. Then N is residually soluble and hence locally soluble. This gives N finite and thus polycyclic. Hence G has π.

THEOREM 8. *A locally finite group G has property π if and only if the following conditions hold:*

(*a*) *Every locally soluble, normal subgroup of G is finite.*

(*b*) *G has σ.*

PROOF. If the locally finite group G has σ, then the finite residual R of G is finite (and nilpotent), by Theorem 8 of [2]. Suppose G has π and let N be a locally soluble, normal subgroup of G. Then G/R has π and so, by Lemma 4, NR/R is finite. Hence, by Lemma 3, N is finite. Now assume G satisfies (*a*) and (*b*) and let N/R be a locally soluble, normal subgroup of G. Then N is locally soluble and therefore finite. Thus N/R is finite and it follows from Lemma 4 that G/R has π. Let H be a normal subgroup of G all of whose finite G-quotients are soluble. Then HR/R is polycyclic and therefore H is polycyclic. Thus G has π.

We conclude this section with some consequences of Theorem 8, Theorem 6 of [1] and Theorem 4.32 of [6] on groups with finiteness conditions on conjugates. For the second theorem here, we recall that a group G is a CC-group if $G/C_G(\langle x\rangle^G)$ is a Černikov group, for all x in G. By Lemma 2.1 of [5], $R' \leq Z(G)$ and G/R is an FC-group, where R denotes the radicable part of G.

THEOREM 9. *Let G be an FC-group. Then G has π if and only if, whenever H is a locally soluble, normal subgroup of G, then H has a subgroup $K \leq Z(G)$ such that H/K is finite.*

THEOREM 10. *Let G be a CC-group. Then*

(i) *G has υ if and only if* $\mathrm{Fit}(G)$ *is nilpotent and* $\mathrm{Fit}(G/R) = (\mathrm{Fit}(G))/R$.

(ii) *G has σ if and only if* $\mathrm{Fit}(G)$ *is finitely generated and* $\mathrm{Fit}(G/R) = (\mathrm{Fit}(G))/R$.

(iii) *G has π if and only if G has σ and G/R has π.*

4. An example and an algorithm.

In this final section we present an algorithm which determines whether certain normal subgroups of a group with π are polycyclic, but first we give an example which disposes of the conjecture mentioned at the end of Section 2. In fact, we present a (non-trivial) locally soluble group G with trivial locally polycyclic radical in which $\phi_f(G) = 1$. Clearly this group cannot satisfy π. The group is easily described.

For each integer $i \geq 0$, let H_i be an infinite cyclic group. Define G_1 to be H_0 and, inductively, $G_{i+1} = H_i \,\mathrm{wr}\, G_i$ for each $i \geq 1$, where the wreath product in each case is the standard one. Then let G be the ascending union of the G_i, $i = 1, 2, \ldots$. Clearly G is locally soluble and torsion-free.

CLAIM. $LP(G) = 1 = \phi_f(G)$.

In order to establish this claim, we require two lemmas, the first of which is almost obvious. Incidentally, we remark that the group G is not a «wreath power» of infinite cyclic groups in the sense of P. Hall (see 6.2 of [6]).

LEMMA 5. *With G defined as above, let i be an arbitrary positive integer and let N be the normal closure in G of $\langle H_i, H_{i+1}, \ldots\rangle$. Then $G = G_i N$ and $G_i \cap N = 1$.*

LEMMA 6. *Suppose $G = \langle a\rangle \,\mathrm{wr}\, H$, the standard wreath product* *of the infinite cyclic group* $\langle a\rangle$ *with a group* H *satisfying* $\phi_f(H) = 1$. *Then* $\phi_f(G) = 1$.

Before sketching the proofs of these lemmas, let us see how they suffice to establish our claim.

Firstly, suppose K is a non-trivial, locally polycyclic, normal subgroup of our group G and let g be a non-trivial element of K. Then $g \in G_i$, for some i, and $\langle g\rangle^{H_i}$ is locally polycyclic. Suppose $H_i = \langle h\rangle$. Then $[h, g] \neq 1$ and, since g has infinite order, $\langle [h, g]\rangle^{\langle g\rangle}$ is not finitely generated. Thus $\langle [g, h], g\rangle$ is not polycyclic, a contradiction.

Now let x be any non-trivial element of G. Again $x \in G_i$, for some i. By Lemma 6 (and induction), $\phi_f(G_i) = 1$. By Lemma 5, therefore, we can find a maximal subgroup M of G which does not contain x. Thus $\phi_f(G) = 1$.

Proof of Lemma 5. Clearly we have $G = G_i N$. Now, for each $j \geq i$, let $L_j = \langle H_i, \ldots, H_j\rangle$, a subgroup of G_{j+1}. Suppose $x \in N \cap G_i$. Then $x \in L_k^{G_i} \cap G_i$, for some $k \geq i$. If $k > i$, then $L_k^{G_i} \leq L_{k-1}^{G_i} H_k^{G_{k+1}} = L_{k-1}^{G_i} H_k^{G_k}$ and so $L_k^{G_i} \cap G_i \leq L_{k-1}^{G_i} (H_k^{G_k} \cap G_k) \cap G_i = L_{k-1}^{G_i} \cap G_i$. Repeating this argument (if necessary) we obtain (eventually) $x \in H_i^{G_i} \cap G_i = 1$. The result follows.

Proof of Lemma 6. The proof here imitates (so far as possible) a couple of the proofs from [4]. Let G be as stated, and suppose first that H is finite. Let x be a non-trivial element of the base group $A = \langle a\rangle^H$ and choose a prime p not dividing the order of H such that $x \notin A^p$. Using Maschke's Theorem, one then easily obtains a G-invariant subgroup D of A such that $x \notin D$ and A/D is a simple H-module. Then $x \notin HD$, a maximal subgroup of finite index in G. It follows that $\phi_f(G) = 1$. In the general case, again assume x is a non-trivial element of $A = \langle a\rangle^H$. As in the proof of Lemmas 3.2 and 3.3 of [4], but using the stronger condition $\phi_f(H) = 1$, we can obtain a normal subgroup K of finite index in H such that $\mathrm{Frat}(H/K) = 1$ and a homomorphism θ from G onto $\langle a\rangle \,\mathrm{wr}\, H/K$ such that $x\theta \neq 1$. This allows us to reduce to the case where H is finite and the result follows.

Corollary 1 of each of papers [1] and [2] describes an algorithm for detecting nilpotency or supersolubility of certain normal subgroups. We conclude with a similar algorithm for the property π.

Proposition. *Let* G *be a finitely presented group with the property* π. *Then there is an algorithm which, when a finite subset* $\{x_1, \ldots, x_m\}$

of G is given, together with the information that $N = \langle x_1, \ldots, x_m \rangle$ *is normal in G, decides whether N is polycyclic.*

PROOF. As in the proof of Corollary 1 of [1], we introduce two recursive procedures, one of which must terminate, by Lemma 2. The result will follow immediately.

The first procedure is the obvious analogue of that described in the above-mentioned corollary.

Now let $(\{w_{1,1}, \ldots, w_{1,m_1}\}, \{w_{2,1}, \ldots, w_{2,m_2}\}, \ldots, \{w_{n,1}, \ldots, w_{n,m_n}\})$ be an n-tuple of subsets of words $w_{i,j}$ in the generators $x_1, \ldots, x_m$. Further, for each $i = 1, \ldots, n$, let H_i be the subgroup (of N) generated by the elements $w_{1,1}, \ldots, w_{1,m_1}, \ldots, w_{i,1}, \ldots, w_{i,m_i}$. The second procedure attempts to show that the subgroups H_i form a polycyclic series for N. Again referring to the proof of Corollary 1 of [1], it is not difficult to see how to proceed. The details are omitted here.

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