Rendiconti del Seminario Matematico della Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 89 (1993), p. 29-36

<http://www.numdam.org/item?id=RSMUP_1993__89__29_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 89 (1993)

On Vector Bundles whose General Sections Have All Projectively Equivalent Zero-Loci.

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The starting question of this paper (as in [3] and [4]) is the following one: what can be said when all general «sections» (linear section or hyperplane sections or zero-loci of sections of a given vector bundle or ...) are «equivalent»? Here «equivalent» could have many different means: same moduli or ...; in this paper (as in [3] and [4]) the «sections» can be identified with subvarieties of a given variety **P** (usually a projective space or a Schubert cycle) and «equivalent» means «in the same orbit for the action of Aut(**P**) (or of another suitable group $PGL(x)$)». Since a group acts naturally on the objects under study, it is natural to use theorems on existence of quotients by actions plus the trivial fact that a prevariety in the sense of Serre is a T_1 topological space (here we work always over an algebraically closed field **K**). On the subject we want to stress an important theorem of Seshadri ([13]): the existence of a so-called Seshadri covering for group actions with finite stabilizers (see also [6] and papers quoted there for the precise statement and for related applications). However in this paper almost always we need much less (and probably always). In this note we use essentially two tools. The first tool is the (elementary) general set up written in [4], § 1; for the second tool (i.e. monodromy arguments) see the first part of § 1. In § 2 we consider the main example considered in this paper: rank 2 vector bundles spanned (by 4 sections) on an integral surface, proving the following result (the reader will find just after its statement a brief reminder of the notions involved in the statement).

THEOREM 0.1. *Let X be an integral projective surface (over an algebraically closed base field* **K**) *and E a rank 2 vector bundle on X. Assume the existence of a vector space $W \subseteq H^0(X, E)$ with* $\dim(W) = 4$, *W spanning E and such that the morphism $\boldsymbol{i}$ from X to the Grassmannian $G(2, 4) \subset \mathbf{P}^5$ induced by (E, W) is an embedding. Set $c := c_2(E)$.*

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Assume that $\mathbf{i}(X)$ is ordinary and that there is $s \in W$ with zero-locus $(s)_0$ zero dimensional and union of $c-2$ smooth points of X (with reduced structure) and a length 2 scheme supported by another smooth point of X. Assume that all the general $(s)_0$ are reduced and projectively equivalent. Then $c \leq 28$.

Here remember that a section s of the tautological quotient bundle U on the Grassmannian $G(2,4)$ (the quadratic hypersurface of $\mathbf{P}^5$) with 0-locus $(s)_0$ of codimension 2 vanishes on a plane, Π. Consider X as embedded in $G(2,4)$ by the morphism induced by the 4 given sections of E and the universal property of the Grassmannians (hence E will be the restriction of U to X); in general $\Pi \cap X$ will be a finite set by a Bertini type theorem ([9]); the assumption of 0.1 means that, varying s (i.e. Π) we get projectively equivalent (under $Aut(\mathbf{P}^2)$) finite subsets of a plane. To motivate the restriction «$\dim(W)=4$», remember that a lemma of Serre (essentially [1], Th. 2 at page 426) gives that if E is spanned by its global sections, there is a subspace $W \subseteq H^0(X,E)$ with $\dim(W) \leq (\mathrm{rank}(E) + \dim(X))$ and spanning E. An integral subvariety V of a projective space $\mathbf{P}$ is called ordinary if it is reflexive (see [10] for this notion) and it has a hypersurface in the dual projective space $\mathbf{P}^*$ as dual variety V^*; every variety is reflexive if $\mathrm{char}(\mathbf{K})=0$. Note also that if $c_2(E)$ is very low, say $c_2(E) \leq 3$ and almost always if $c_2(E)=4$, then all general $(s)_0$ must be projectively equivalent (since any 3 distinct points on a line are projectively equivalent and the same is true for 4 points in a plane if no 3 of them are collinear); pairs (X,E) with very low $c_2(E)$ can often be classified (see the references in [2]), and hence may be considered «few» and «known».

Using only the general set up, it is very easy to show that, under assumptions weaker than the ones of 0.1, the triple (X,E,W) has very strange properties. To prove exactly 0.1 we find useful to use some monodromy argument (in the style of [7], (or see [8], chapter 3; in positive characteristic, see [5] and [12])) for sections of vector bundles. These monodromy arguments are the second main tool of this paper; we think that they an independent interest. We collect more than we need in § 1. The very short proof of 0.1 written here in § 2 uses only in a very minor way the general set up. We stress again that 0.1 is only a sample result; similar results (neither weaker nor stronger than 0.1) could be proved using much more ink and no new idea (only the general set up), and without even mention the word «monodromy».

Then in § 3 we consider the case of a plane curve C, but assuming only that all the general tangent lines to C intersect C in projectively equivalent subsets (see 3.1 and 3.2). Usually, (and always if $\mathrm{char}(\mathbf{K})=0$ by the properties of reflexive curves ([10]) and the general set up) this assumption is much weaker than the assumption that all general lines of $\mathbf{P}^2$ intersect C in projectively equivalent subsets. At the end of the section we show that «in general» (see 3.3 and 3.4 for a precise statement) the set $\{C \cap L;$ line in $\mathbf{P}^2\}$ has dimension two.

I want to thank Ciro Ciliberto for many stimulating conversations.

The author was partially supported by MURST and GNSAGA of CNR (Italy).

1. – The general (elementary) set up of [4], § 1, applies in many situations not explicitely stated in [4]. In particular it applies verbatim to sections of a vector bundle (the case stated in [4] being the case in which the vector bundle has rank 1). We refer to reader to [4], § 1, for that. We only note that [4], Remark 1.1, implies that, with the notations of 0.1, if all general $(s)_0$ are projectively equivalent and $m \in H^0(X, E)$ has $\dim((m)_0) = 0$ but $(m)_0$ is not projectively equivalent to a general $(s)_0$, then the stabilizer of $(m)_0$ is not finite.

In this section we fix the following notations. Let X be an integral complete veriety (over $\mathbf{K}$); set $n := \dim(X)$; for any sheaf $\boldsymbol{F}$ on X, write $H^i(\boldsymbol{F})$ (or $h^i(\boldsymbol{F})$) instead of $H^i(X, \boldsymbol{F})$ (or $h^i(X, \boldsymbol{F})$) Let E be a rank n vector bundle on X with E spanned by its global sections; let $W \subseteq H^0(E)$ be a vector space spanning E; set $w := \dim(W)$. Let h_W (or h (or h_E if $W = H^0(E)$)) be the morphism from X to the Grassmanian $G := G(n, w)$ of n-dimensional linear quotient spaces of $\mathbf{K}^w$ induced by W. Denote by h'_W (or h') the composition of h_W and the Plucker embedding of G into a projective space, $\mathbf{P}$. Let $\Gamma' \subset \mathbf{P}(W^*) \times X = \{([s], x): s(x) = 0\}$ be the incidence correspondence; Γ' is a $\mathbf{P}^{w-n-1}$ bundle over X (hence it is integral); set $\Gamma \subseteq \Gamma' := \{([s], x): \dim((s)_0) = 0\}$ (hence with projection $\Gamma \to \mathbf{P}(W^*)$ quasi-finite; the Galois group, M, of the projection $\Gamma \to \mathbf{P}(W^*)$ will be called the monodromy group of the incidence correspondence (or the monodromy group, for short).

DEFINITION 1.1. (*a*) W is called *birational* if h is birational.

(*b*) W is called *ordinary* if it is birational and $h'(X)$ is an ordinary variety of $\mathbf{P}$ in the sense of [10] (i.e. it is reflexive and with a hypersurface as dual variety $h'(X)^*$).

(*c*) W is called *strongly reflexive* if it is ordinary, $(s)_0$ is reduced for a general $s \in W$ and there is $x \in X_{\text{reg}}$ and $t \in W$ such that $(t)_0$ is 0-dimensional, supported on X_{reg} and reduced except for a length 2 subscheme supported on x.

(*d*) W is called *monodromic* if it is birational and the monodromy group M of the incidence correspondence contains the alternating group on $c_n(E)$ objects.

REMARK/DEFINITION 1.2. In [2] we defined an integer $c(E)$; here we recall the definition in the more general case of $W \subseteq H^0(E)$ ((with possibly $W \neq H^0(E)$); the integer $c(E, W)$ is by definition the maximal integer t such that for a general $S \subset X$ with $\text{card}(S) = t$, there is $s \in W$ with $S \subseteq (s)_0$ and $\dim((s)_0) = 0$; we have $c(E, W) \leqslant c_n(E)$; $c(E, W) > 0$ if W spans E (by Bertini theorem ([19])). Note that the monodromy group of W acts at least $c(E, W)$-transitively (since X is integral).

LEMMA 1.3. *If W is strongly reflexive then the monodromy group M is the full symmetric group on $c_n(E)$ objects. In particular W is monodromic.*

PROOF. The path is clear, after [7]. It is sufficient to show that M is doubly transitive and contains a double transposition. Take $y := ([t], x)$ as in the Definition 1.1(c). To find the double transposition it is sufficient to check that Γ is irreducible at y; indeed it is smooth because $x \in \in X_{\text{reg}}$ and $\Gamma' \to X$ is a $\mathbf{P}^{w-n-1}$-bundle. Now we will check that M is double transitive. Fix $s \in W$ with $\dim((s)_0) = 0$ and $(s)_0$ reduced; call s' the corresponding section of the tautological quotient bundle, F, on G. Fix $\{a, b, c\} \subset (s)_0$ with $\text{card}(\{a, b, c\}) = 3$. It is easy to check that the Schubert cycles σ corresponding to the 0-loci of sections of F and containing a can move b into c (see geometrically at least the case $n = 2$, $w = 4$ needed in § 2); note that the case «$n = 2$ and $w = 4$» is one the most difficult ones, since almost always we have $c(E, W) = 1$ if $\dim(W) \leqslant 2\dim(X)$. ■

DEFINITION 1.4. Assume the existence of an integers t such that there is an open subset U of $S^t(X)$ such that for every $A \in U$ there is $s \in W$ with $A \subseteq (s)_0$ and such that the stabilizer (in the projective group considered!) of A is finite; by definition if such a t exists, then $t \leqslant c(E, W)$ and every integer t' with $t \leqslant t' \leqslant c(E, W)$ has the same property; if $t(n, w)$ is the minimal such integer we will denote by $\gamma(n, w)$ the cardinality of the corresponding stabilizer (for general A).

DEFINITION 1.5. W is said to have the property ($) if the non degeneracy condition described in the Definition 1.4 holds.

We have $t(2, 4) = 4$ and $\gamma(2, 4) = 4! = 24$ (with respect to the obvious choice of the projective group: $\text{Aut}(\mathbf{P}^2)$).

PROPOSITION 1.6.' *Assume W monodromic, satisfying the non degeneracy condition* ($) *and that all general* 0*-loci of W are projectively equivalent. Then* $c_n(E) \leq t(n, w) + (t(n, w)!)$.

PROOF. Assume that the inequality fails. Fix a general $s \in W$ and set $A := (s)_0$; fix $B \subset A$ with $\mathrm{card}(B) = t(n, w)$ and 2 points a, b of $A \setminus B$. By the monodromy assumption $\Gamma^{(c-2)}$ is irreducible. Hence there is an algebraic affine integral curve τ in $\Gamma^{(c)}$ connecting a subset B' containing $B \cup \{a\}$ and a subset B'' containing $B \cup \{b\}$; up to a covering of τ we may order the points in B (hence in $B \cup \{a\}$ and $B \cup \{b\}$ with a and b as last elements). Consider a Seshadri covering $Sh \to \Gamma^{(\mathrm{card}(B)+1)}$; taking an irreducible component of the fiber product of $\tau \to \Gamma^{(\mathrm{card}(B)+1)}$ with $SH \to \Gamma^{(\mathrm{card}(B)+1)}$ we obtain that along τ the corresponding subsets with $\mathrm{card}(B) + 1$ elements have the same image in the quotient of Sh (in the case in which the Schubert cell is a projective space with its automorphism group (e.g. the case $n = 2$, $w = 4$ considered in § 2) this means the constance along τ of the cross ratio). But since τ permutes B and exchange a and b, we obtain card $(A \setminus B) \leq t(n, w)!$, as wanted. ■

We think that the definitions given in this section are useful. However these definition are not the only reasonable candidates. To decide between competing nearby definitions, one need more experimental work in higher dimensions and ranks than in the case $n = 2$, $w = 4$ considered in the next section. Furthermore, here we do not use $\gamma(n, w)$ and the bound on $c_n(E)$ in 1.6 seems to be very bad.

2. – Using 1.6 now we may give in a few lines a proof of 0.1.

PROOF OF 0.1. By 1.6 and the fact that $t(2, 4) = 4$, it is sufficient to check the non degeneracy condition ($). Assume it fails. Then the 0-locus of a general section is formed by $c := c_2(E)$ points, all except at most one on a line. Hence by the monodromy assumption all such points must be collinear. By assumption they are projectively equivalent as points in a plane. Hence, varying the section, we find points which are projectively equivalent as points on a line. Hence we reduce to another group action. By the reflexivity assumption, we find some section with as 0-locus a double point and $c - 2$ simple points; by semicontinuity again this configuration must be contained in a line. Since $c > 3$, this unreduced configuration has finite stabilizer. Hence the contradiction comes from the often quoted [4], Remark 1.1. ■

We leave to the interested reader the task to extend the case «$n = 2$, $w = 4$» considered here to the case «$n = 2$, $w > 4$».

3. – Let C be a plane curve; let $C^* \subset \mathbf{P}^{2^*}$ be its dual curve. A point $P \in C$ will be called «very ordinary double point» if C has multiplicity 2 at P, with as tangent cone two distict lines each of them intersecting with multiplicity 2 at P the corresponding branch of the formal completion of C at P (i.e. each of them with intersection multiplicity 3 with C at P) and if each of these tangent lines intersect C_{sing} only at P.

PROPOSITION 3.1. *Let C be a plane integral curve of degree $d \geq 6$ with k very ordinary double points as only singularities. Assume that all the general tangent lines to C have projectively equivalent (for the action of* $\mathrm{Aut}(\mathbf{P}^1)$) *intersections with C. Then*

$$2k(d-2) \leq 3d^2 - 6d .$$

PROOF. Note that by the often quoted [4], Remark 1.1 for every $P \in C_{\text{sing}}$ each of the tangent lines to C at P intersects C at most at another point; by our definition of very ordinary double point and the assumption on C, indeed these lines must intersect C at a smooth point and with intersection multiplicity $d-3$. Since $d \geq 6$, we see that C has at least $2k$ flexes (in the sense of [11]) each of them with weight at least $d-5$ (see [11], Definition at page 54). Since (1) is trivial if $k = 0$, we may assume $k > 0$. By the definition of (very) ordinary double point and the fact that $k > 0$, we may assume that a general tangent line to C intersects C with multiplicity 2 at the point of tangency. By [11], § 3 (in particular the last formula in the statement of Th. 9 at page 54 of [11]) the sum of the weights of the flexes of C is $3(2g-2) + 3d$, where $g = -2k + 1 + (d-1)(d-2)/2$ is the geometric genus of C. Hence we have (1). ■

REMARK 3.2. It is straightforward to apply verbatim the Proof of 3.1 to curves with other singularities.

Now we show that «in general» for a plane curve C we have $\dim(\{C \cap L : L \text{ line in } \mathbf{P}^2\} = 2)$.

PROPOSITION 3.3. *Let $C \subset \mathbf{P}^2$ be an integral non degenerate curve such that for every line $L \subset \mathbf{P}^2$ $(C \cap L)_{\text{red}}$ has cardinality at least 3 (i.e. finite stabilizer as subset of L under the group* $\mathrm{Aut}(L)$). *Then there is no positive dimension family of lines $T \subset \mathbf{P}^{2^*}$ such that all $\{C \cap L : L \in$*

$\in T\}$ are projectively equivalent and T is neither contained in the dual curve C^* nor formed by lines in a pencil through a singular point of C. In particular, varying the line M in $\mathbf{P}^{2^*}$ the projective equivalence classes of the sets $\{C \cap M\}$ vary in a two dimensional family.

PROOF. Assume by contradiction the existence of such T. Since by assumption C is not a line, $\dim(C^*) = 1$. Hence there is a line $R \in C^*$ in the closure of T. Since for a general $L \in T$, $C \cap L$ is reduced and $R \cap C$ is not reduced but with finite stabilizer, we have a contradiction. ■

REMARK 3.4. The assumption that $\text{card}((L \cap C)_{\text{red}}) \geq 3$ for every line L is satisfied in particular if C is a general curve of a given degree $d \geq 5$ and in many other cases (general plane curve with a small number of nodes, ...). To prove this type of results for C general in a given family, $\boldsymbol{H}$, of plane curves, it is often very useful to know that on the boundary of $\boldsymbol{H}$ there are suitable reducible curves (e.g. suitable union of conics and cubics).

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Manoscritto pervenuto in redazione l'1 settembre 1991.