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Multiple homoclinic orbits for a class of conservative systems

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Multiple Homoclinic Orbits for a Class of Conservative Systems.

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1. Introduction.

This paper deals with the existence of multiple homoclinic orbits for second order autonomous systems

$$\ddot{q} + V'(q) = 0 \tag{1}$$

where $q \in \boldsymbol{R}^N$ and $V: \boldsymbol{R}^N \to \boldsymbol{R}$ is smooth and such that $V'(0) = 0$. A homoclinic orbit is a solution $q \in C^2(\boldsymbol{R}, \boldsymbol{R}^N)$ of (1) which satisfies the asymptotic conditions

$$q(t) \to 0 \quad \text{as} \quad t \to \pm\infty .$$

The typical situation where homoclinics arise is the case in which the origin $(p, q) = (0, 0) \in \boldsymbol{R}^{2N}$ is a hyperbolic point for the hamiltonian

$$H(p, q) = \frac{1}{2}|p|^2 + V(q), \qquad (p, q) \in \boldsymbol{R}^{2N}$$

namely when $q = 0$ is a strict local maximum for the potential V.

The existence of homoclinic motions has been deeply investigated. For example, we refer to [2], [7], [15], [16] for results concerning second order systems and [11], [18] and [19] for first order systems.

When the potential V depends on time in a periodic fashion, multiple homoclinic orbits arise. Actually, the existence of many homoclinics is a very classical problem and the first multiplicity results go back to

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Poincaré [14] and Melnikov [12]. By means of perturbation techniques they proved that in the one dimensional case ($N = 1$) and when V depends in a periodic fashion on t, forced systems like

$$\ddot{q} + V_q(t, q) = 0 \tag{2}$$

possess finitely many homoclinics. The extension of this kind of results to any finite number of degrees of freedom (namely, when $N \geqslant 1$) is a quite non trivial matter. Recently, there has been a great progress in such a problem by using critical point theory. Actually equations (1) or (2) are variational in nature and homoclinics can be found as critical points of

$$F(u) = \int_{-\infty}^{+\infty} \left\{ \frac{1}{2} |\dot{u}|^2 - V(t, u) \right\} dt, \tag{3}$$

on the Sobolev space $E = H^{1,2}(\boldsymbol{R}, \boldsymbol{R}^N)$. The groundwork for employing variational tools has been given in [8] and expecially in the remarkable paper [17], dealing with first order convex forced hamiltonian systems

$$\dot{p} = -H_q(t, p, q) \qquad \dot{q} = H_p(t, p, q).$$

Subsequently, the convexity assumption has been dropped in [9], in the case of second order systems like (2). Both in [17] and in [9] the existence of infinitely many homoclinics is established. It is worth remarking that such a multiplicity result is not a consequence of any symmetry or invariance property of the functional f. Rather, it is obtained by using in a striking way the periodic time dependence of the hamiltonian.

On the contrary, in the autonomous case, no multiplicity result is known, and in fact one does not expect, in general, to find infinitely many homoclinics, nor multiple solutions in such a case.

The purpose of the present paper is to show that, for a class of second order autonomous systems, multiple homoclinic orbits actually occur.

Roughly, we deal with hamiltonians like

$$H(p, q) = \frac{1}{2} |p|^2 - \frac{1}{2} |p|^2 + W(q),$$

where $W(q)$ satisfies

$$a|q|^\alpha \leqslant W(q) \leqslant b|q|^\alpha \qquad (\alpha > 2). \tag{4}$$

Our main result states that (1) possesses at least two homoclinics provided $b < 2^{\alpha-2/2} \cdot a$. Other multiplicity results are also discussed.

Our approach is still variational, the main tool being a Lusternik-Schnirelman type lemma (Lemma 5 below) which allows us to take advantage of the «pinching» condition (4). Such a lemma is in a certain sense a comparison result, somewhat related with [4, Theorem 1.2], and we think it can be of possible interest in itself. It permits to estimate the number of critical points of a functional f on a manifold M, in relationship with those of a related functional g, whenever the sublevels of f and g are boxed in a suitable manner.

The main results of the present paper have been outlined in the preliminary note [3].

2. Main results.

Let us consider a potential V of the form

$$V(q) = -\frac{1}{2}|q|^2 + W(q) \tag{5}$$

where $q \in \boldsymbol{R}^N$, $|q|$ denotes the Euclidean norm in $\boldsymbol{R}^N$ and W satisfies:

(W_0) $W \in C^2(\boldsymbol{R}^N, \boldsymbol{R})$;

(W_1) $\exists \theta \in]0, 1/2[$ such that $0 < W(q) \leqslant \theta W'(q) \cdot q$, $\forall q \in \boldsymbol{R}^N$;

(W_2) $W''(0) = 0$ and $W'(q) \cdot q < W''(q) q \cdot q$, $\forall q \in \boldsymbol{R}^N$, $q \neq 0$;

(W_3) $\exists \varphi_1, \varphi_2 \subset C^2(\boldsymbol{R}^N, \boldsymbol{R})$, homogeneus of degree α, $\alpha \geqslant \frac{1}{\theta} > 2$, such that

$$\varphi_1(q) \leqslant W(q) \leqslant \varphi_2(q), \qquad \forall q \in \boldsymbol{R}^N.$$

REMARK. From the preceding hypotheses it follows that $q = 0$ is a strict local maximum for V. Moreover the set $\{q \in \boldsymbol{R}^N : V(q) \leqslant 0\}$ is compact and $q = 0$ belongs to the interior. ∎

We set

$$a = \min_{|x|=1} \varphi_1(x), \qquad b = \max_{|x|=1} \varphi_2(x). \tag{6}$$

Our main result is the following one

THEOREM 1. *Let V of the form* (5) *with W satisfying* (W_0)-(W_3), *and let*

$$\frac{b}{a} < 2^{\alpha - 2/2} \tag{7}$$

Then (1) *possesses at least two homoclinic orbits.*

REMARK. A condition like (7) recalls a similar one used in a celebrated multiplicity result by Ekeland-Lasry [10], see in particular the proof given in [5]. Actually, even if the approach in the present paper is somewhat similar to that one of [5], the discussion in the sequel will make it clear that the two results are different in nature. ■

When W is even, the preceding theorem can be improved. For $k \leq N$, let Π_k denote the class of k-dimensional linear subspaces of $\boldsymbol{R}^N$ and let us set

$$a_k = \max_{\pi \in \Pi_k} \min_{|x|=1, x \in \pi} \varphi_1(x). \tag{8}$$

THEOREM 2. *Let V be of the form* (5) *with W satisfying* (W_0)-(W_3) *and suppose, in addition, that W is even. If, for some $k \leq N$, there results*

$$\frac{b}{a_k} < 2^{\alpha - 2/2} \tag{9}$$

then (1) *possesses at least k homoclinic orbits. In particular, if* (7) *holds, then* (1) *has at least N homoclinic orbits.*

As a consequence of the preceding theorems, one can deduce multiplicity results, perturbative in nature. For example, one has

THEOREM 3. *Let V be of the form*

$$V(q) = -\frac{1}{2}|q|^2 + \frac{1}{\alpha}|q|^\alpha + \varepsilon R(q), \tag{10}$$

with $\alpha > 2$ and R satisfying (W_0) *and*

(W_4) $R(q) \leq 0, \qquad R''(q)q \cdot q = o(|q|^\alpha)$ *as* $q \to 0$.

Then there exists $\varepsilon_0 > 0$ such that for all $0 \leq \varepsilon \leq \varepsilon_0$ one has:

(*a*) (1) *possesses at least two homoclinic orbits;*

(*b*) *if, in addition, R is even, then* (1) *has at least N pairs of homoclinic orbits.*

3. The variational setting.

Let E denote the Sobolev space $H^{1,2}(\boldsymbol{R}, \boldsymbol{R}^N)$ with scalar product

$$(u \mid v) = \int_{-\infty}^{+\infty} [\dot{u}(t) \cdot \dot{v}(t) + u(t) \cdot v(t)]\, dt$$

and norm

$$\|u\|^2 = (u \mid u) = \int_{-\infty}^{+\infty} [\,|\dot{u}(t)|^2 + |u(t)|^2]\, dt .$$

Let V be of the form (5). With the preceding notation, the functional F defined in (3) becomes

$$F(u) = \frac{1}{2}\|u\|^2 - \int_{-\infty}^{+\infty} W(u) . \tag{11}$$

In order to prove the preceding theorems it is convenient to use a different variational principle, substituting the search of critical point of F on E with a constrained problem. This device is closely related to that one used, for example, in [5]. See also [6] and [1], Proposition 1.4.

Let us consider the unit sphere $S = \{u \in E\colon \|u\| = 1\}$; for $u \in S$ and $\lambda \in \boldsymbol{R}^+$, one finds

$$F(\lambda u) = \frac{1}{2}\lambda^2 - \int_{-\infty}^{+\infty} W(\lambda u) , \tag{12}$$

$$\frac{d}{d\lambda} F(\lambda u) = \lambda - \int_{-\infty}^{+\infty} W'(\lambda u) \cdot u , \tag{13}$$

$$\frac{d^2}{d\lambda^2} F(\lambda u) = 1 - \int_{-\infty}^{+\infty} W''(\lambda u)\, u \cdot u . \tag{14}$$

Let λ_0 be such that $(d/d\lambda)F(\lambda_0 u) = 0$. Then (13) yields

$$\lambda_0 = \int_{-\infty}^{+\infty} W'(\lambda_0 u)\cdot u,$$

and from (14) it follows

$$(15)\qquad \frac{d^2}{d\lambda^2}F(\lambda u)|_{\lambda=\lambda_0} = 1 = \int_{-\infty}^{+\infty} W''(\lambda_0 u)u\cdot u = \int_{-\infty}^{+\infty} \frac{W'(\lambda_0 u)\cdot u}{\lambda_0} -$$

$$- \int_{-\infty}^{+\infty} W''(\lambda_0 u)u\cdot u - \int_{-\infty}^{+\infty} \frac{W'(\lambda_0 u)\cdot \lambda_0 u - W''(\lambda_0 u)\lambda_0 u\cdot \lambda_0 u}{\lambda_0^2}$$

Using (W_2) we infer from (15) that

$$(16)\qquad \frac{d^2}{d\lambda^2}F(\lambda u)|_{\lambda=\lambda_0} < 0.$$

Moreover (W_1) (or else (W_3)) readily implies that

$$(17)\qquad \lim_{\lambda\to 0+} \frac{d}{d\lambda}F(\lambda u) = 0+, \qquad \lim_{\lambda=+\infty} \frac{d}{d\lambda}F(\lambda u) = -\infty.$$

Then (16) and (17) imply that for all $u \in S$ there exists a unique $\lambda(u)$ such that

$$\frac{d}{d\lambda}F(\lambda u)_{|\lambda=\lambda(u)} = 0.$$

The preceding discussion allows us to define a new function $f: S \to \boldsymbol{R}$ by setting

$$(18)\qquad f(u) = \max_{\lambda\geq 0} F(\lambda u) = F(\lambda(u)u).$$

Let $\nabla f|_S$ denote the contrained gradient of f on S.

LEMMA 4. *Suppose that W satisfies (W_1)-(W_2). Then*

(a) $f \in C^1(S, \boldsymbol{R})$;

(b) $\inf_S f \geq m > 0$;

(c) $\nabla f|_S(u) = 0$ *if and only if* $F'(\lambda(u)u) = 0$.

PROOF. From the preceding discussion it follows that $\lambda(u)$ is the unique solution of

$$(F'(\lambda u)|u) = 0. \tag{19}$$

Then the regularity of F, and (16) imply that $\lambda \mapsto \lambda(u)$ is C^1 and hence (*a*) follows.

From (W_1)-(W_2) we infer that $W(q) = o(|q|^2)$ as $q \to 0$ and thus there is $\delta > 0$ such that

$$|W(q)| \leqslant \frac{1}{4}|q|^2 \qquad \forall |q| \leqslant \delta. \tag{20}$$

Since for all $v \in E$ one has $\|v\|_{L^\infty} \leqslant c_1 \|v\|$ then for any $v \in E$ such that $\|v\| \leqslant \delta_1 \equiv \delta / c_1$ (hereafter c_1, c_2, etc., denote constants) there results $|v(t)| \leqslant \delta$. Hence (20) yields

$$F(v) \geqslant \frac{1}{2}\|v\|^2 - \frac{1}{4}\int_{-\infty}^{+\infty} |v|^2 \geqslant \frac{1}{4}\|v\|^2, \qquad \forall \|v\| \leqslant \delta_1 .$$

This, jointly with (W_1) and (18), implies that for all $u \in S$ there results

$$\frac{1}{2}(\lambda(u))^2 \geqslant \frac{1}{2}(\lambda(u))^2\|u\|^2 - \int_{-\infty}^{+\infty} W(\lambda(u)u) =$$

$$= F(\lambda(u)u) \geqslant F(\delta_1 u) \geqslant \frac{1}{4}\|\delta_1 u\|^2 = \frac{\delta_1^2}{4}.$$

Therefore one infers there exists $\rho > 0$ such that

$$\lambda(u) \geqslant \rho > 0 \qquad \forall u \in S, \tag{21}$$

and (*b*) follows.

Since $f(u) = F(\lambda(u)u)$ and $\lambda(u)$ satisfies (19), then one finds that

$$f'(u) = \lambda(u) F'(\lambda(u)u).$$

Thus, if $u \in S$ is a critical point of $f|_S$ then there results

$$\lambda(u) F'(\lambda(u)u) = \gamma u, \tag{22}$$

for some Lagrange multiplier $\gamma \in \boldsymbol{R}$. Taking the scalar product with u and using again (19) it follows that

$$\gamma = \lambda(u)\,(F'(\lambda(u)u)|u) = 0.$$

Therefore (22) yields $\lambda(u)F'(\lambda(u)u) = 0$ and finally (21) implies $F'(\lambda(u)u) = 0$. This proves (*c*). ∎

4. A Lusternik-Schnirelman lemma.

This section is devoted to state in a general form a theoretical lemma which will furnish a basic tool for proving our multiplicity results. We will use the Lusternik-Schnirelman (L-S, for short) theory of critical points. See for example [1] for a short review.

Let us consider a C^1 Riemannian manifold M and a functional $f \in C^1(M, \boldsymbol{R})$. For $c \in \boldsymbol{R}$ we set $f^c = \{u \in M: f(u) \leq c\}$. Let $K(f)$ denote the set of critical points of $f|_M$, $K(f) = \{u \in M: \nabla f|_M(u) = 0\}$. We say that $f|_M$ satisfies the condition (C) on f^c whenever for any ε-neighbourhood U of $K(f)$ there exists $\delta > 0$ such that

$$\|\nabla f|_M(u)\| \geq \delta \qquad \forall u \in f^c \setminus U.$$

REMARK. In using variational tools, it is usually employed the so called (PS) condition introduced by Palais and Smale [13] rather than condition (C). Verifying (PS) on f^c amounts to requiring that any sequence $u_n \in f^c$ such that $f(u_n)$ is bounded (below) and $\nabla f|_M(u_n) \to 0$, possesses a converging subsequence. Actually, condition (C) suffices to prove the main results in L-S critical point theory. ∎

Another main tool of the L-S theory is the L-S category. Let T be a topological space and let $X \subset T$. The L-S category of X with respect to T, $\operatorname{cat}(X, T)$, is the least integer k such that $X \subset \bigcup_{1 \leq i \leq k} X_i$, X_i being closed subset of T, contractible in T. Among the properties of the L-S category, let us recall here for future references the following monotonicity properties:

$$X_1 \subset X_2 \subset T \Rightarrow \operatorname{cat}(X_1, T) \leq \operatorname{cat}(X_2, T), \tag{23}$$

$$X \subset T_1 \subset T_2 \Rightarrow \operatorname{cat}(X, T_2) \leq \operatorname{cat}(X, T_1). \tag{24}$$

In the sequel we will make use of the following result in the L-S theory:

LEMMA 5. *Let $f \in C^1(M, \boldsymbol{R})$ be bounded below on M and satisfy* (C) *on f^c. Then $f|_M$ has at least* $\operatorname{cat}(f^c, f^c)$ *critical points in f^c.*

The main result we will prove in this section is the following one.

LEMMA 6. *Suppose that exist* c, $m \in \mathbf{R}$ *and subsets* $X \subset Y \subset M$ *such that*:

1) $X \subset f^c \subset Y$;

2) $\exists \eta \in C(Y, X)$ *such that* $\eta(x) = x$ *for all* $x \in X$;

3) $\inf_M f \geq m$ *and* $f|_M$ *satisfies* (C) *in* f^c.

Then $f|_M$ *has al least* $\operatorname{cat}(X, X)$ *critical points in* f^c.

PROOF. First of all, assumption 3) allows us to apply the L-S theory to f on f^c yielding the existence of (at least) $\operatorname{cat}(f^c, f^c)$ critical points for $f|_M$ on f^c.

Next, using assumption 1) jointly with (23) and (24), one infers

$$\operatorname{cat}(f^c, f^c) \geq \operatorname{cat}(X, Y). \tag{25}$$

We claim that, if 2) holds, then one has

$$\operatorname{cat}(X, Y) = \operatorname{cat}(X, X) \tag{26}$$

Indeed, since $X \subset Y$, (23) yields that $\operatorname{cat}(X, Y) \leq \operatorname{cat}(X, X)$. Conversely, let $\operatorname{cat}(X, Y) = k$. Then $X \subset \bigcup_{1 \leq i \leq k} X_i$, with X_i closed and contractible in Y. Let $h_i \in C([0, 1] \times X_i, Y)$ denote the homotopy such that, for all $x \in X_i$:

$$h_i(0, x) = x$$

$$h_i(1, x) = \xi_i \in Y$$

Let us set $H_i = \eta \circ h_i$. Then $H_i \in C([0, 1] \times X_i, X)$ and there results

$$H_i(0, x) = \eta(h_i(0, x)) = \eta(x) = x,$$

$$H_i(1, x) = \eta(h_i(1, x)) = \eta(\xi_i) = \tilde{\xi}_i \in X.$$

This means that X_i is contractible in X and therefore $\operatorname{cat}(X, X) \leq k = \operatorname{cat}(X, Y)$, proving (26). As a consequence, (25) becomes

$$\operatorname{cat}(f^c, f^c) \geq \operatorname{cat}(X, X),$$

and an application of Lemma 5 completes the proof. ■

5. Condition (C).

It is well known that the usual (PS) condition does not hold for F on E because F is invariant under the (non-compact) action $\tau \to u_\tau \equiv u(\cdot + \tau)$. To overcome this difficulty, the following result has been proved in [8] and [9]. Let $K_0(F)$ denote the set $\{u \in E\colon u \neq 0, F'(u) = 0\}$.

LEMMA 7. *Suppose that* $l = \inf_{u \in K_0(F)} F > 0$ *and let* $v_n \in E$ *be a sequence such that*

$$F(v_n) \to \beta \in [l, 2l[\,,$$

$$F'(v_n) \to 0.$$

Then there exist $\tau_n \in \boldsymbol{R}$ *and* $z_0 \in E$ *such that, setting* $z_n(t) = v_n(t + \tau_n)$ *there results* (*up to a subsequence*)

$$\begin{cases} \text{i)}\ z_n \to z_0\,, \\ \text{ii)}\ F'(z_0) = 0\,, \\ \text{iii)}\ l \leqslant F(z_0) \leqslant \beta < 2l\,. \end{cases} \tag{27}$$

REMARK. Actually, the proof in [8] and [9] is given in the case of non-autonomous potentials, but it can be carried over without changes in the autonomous case, too. Moreover, let us explicitly point out that, in the present setting. Lemma 4 (*b*)-(*c*) readily implies that $l = m(= \inf_S f)$. ■

In analogy with Lemma 7, a condition similar to (27) holds true for $f|_S$. Precisely one has

LEMMA 8. *Let* $u_n \in S$ *be a sequence such that*

$$f(u_n) \to \beta \in [m, 2m[\,,$$

$$\nabla f|_S(u_n) \to 0\,.$$

Then there exist $\tau_n \in \boldsymbol{R}$ *and* $u_0 \in E$ *such that, setting* $w_n(t) = u_n(t + \tau_n)$ *there results* (*up to a subsequence*)

$$\begin{cases} \text{i)}\ w_n \to u_0\,, \\ \text{ii)}\ \nabla f|_S(u_0) = 0\,, \\ \text{iii)}\ m \leqslant f(u_0) \leqslant \beta < 2m\,. \end{cases} \tag{28}$$

PROOF. There results:

$$F(\lambda(u_n)\, u_n) = f(u_n) \to \beta \in [m, 2m[\tag{29}$$

$$f'(u_n) = \gamma_n u_n + \sigma_n\,, \tag{30}$$

where $\sigma_n \to 0$. From (30) it follows that

$$\lambda(u_n) F'(\lambda(u_n) u_n) = \gamma_n u_n + \sigma_n .$$

Multiplying by u_n and taking into account (19),we get $0 = \gamma_n + (\sigma_n \,|\, u_n)$ and hence

$$\lambda(u_n) F'(\lambda(u_n) u_n) = -(\sigma_n \,|\, u_n) u_n + \sigma_n .$$

Since $\lambda(u_n) \geq \rho > 0$ (see (21)) then one infers

$$F'(\lambda(u_n) u_n) = \frac{1}{\lambda(u_n)} [\sigma_n - (\sigma_n \,|\, u_n)] .$$

There results

$$\left\| \frac{1}{\lambda(u_n)} [\sigma_n - (\sigma_n \,|\, u_n)] \right\| \leq \frac{1}{\lambda(u_n)} \cdot 2 \|\sigma_n\|$$

and hence, using again (21), we deduce that

$$F'(\lambda(u_n) u_n) \to 0 .$$

This and (29) say that Lemma 7 applies to the sequence $v_n = \lambda(u_n) u_n$ with $l = m$, see the Remark after Lemma 7. Therefore v_n satisfies (27) as well as u_n verifies (28), proving the Lemma. ■

As an immediate consequence of Lemma 8 we infer

LEMMA 9. *For all* $c < 2m$ $f|_S$ *satisfies condition* (C) *on* f^c.

6. Proofs.

The proofs of Theorems 1, 2 and 3 will rely on an application of Lemma 6 to f, with a suitable choice of X and Y. First, some preliminaries are in order.

For $\kappa = a, b$ let us consider functionals $G_\kappa : E \to \boldsymbol{R}$ defined by setting

$$G_\kappa(u) = \frac{1}{2} \|u\|^2 - \kappa \int_{-\infty}^{+\infty} |u(t)|^\alpha dt .$$

Moreover, the same arguments developed in Section 3 permit to define (smooth) functionals $g_\kappa : S \to \boldsymbol{R}$,

$$g_\kappa(u) = \max_{\lambda \geq 0} G_\kappa(\lambda u) .$$

From assumption (W_3) it follows that

$$a|q|^{\alpha} \leqslant W(q) \leqslant b|q|^{\alpha},$$

and hence

$$G_b(u) \leqslant F(u) \leqslant G_a(u) \qquad \forall u \in E.$$

This, in turn, implies

$$g_b(u) \leqslant f(u) \leqslant g_a(u) \qquad \forall u \in S. \tag{31}$$

LEMMA 10. *If* $b < a \cdot 2^{\alpha-2/2}$, *then* $g_b^c \subset f^{2c-\varepsilon} \subset g_b^{2c-\varepsilon}$, *for all* c *and all* $\varepsilon > 0$, *small.*

PROOF. From the left-hand side inequality in (31) we immediately infer that $f^{2c-\varepsilon} \subset g_b^{2c-\varepsilon}$. Furthermore, for all $u \in S$ there results

$$g_a(u) = \max_{\lambda \geqslant 0} \left[\frac{1}{2}\lambda^2 - a\lambda^{\alpha} \int_{-\infty}^{+\infty} |u|^{\alpha} \right].$$

Setting $A(u) = \alpha \int_{-\infty}^{+\infty} |u|^{\alpha}$, a direct calculation yields

$$g_a(u) = \left(\frac{1}{2} - \frac{1}{\alpha} \right) a^{2/2-\alpha} A(u)^{2/2-\alpha}. \tag{32}$$

In the same way one finds

$$g_b(u) = \left(\frac{1}{2} - \frac{1}{\alpha} \right) b^{2/2-\alpha} A(u)^{2/2-\alpha}. \tag{33}$$

Using (7), from (32) and (33) it follows

$$g_a(u) < 2g_b(u) \qquad \forall u \in S.$$

This and the right-hand side inequality in (31) yield

$$f(u) < 2g_b(u) \qquad \forall u \in S,$$

completing the proof of the lemma. ■

PROOF OF THEOREM 1. The proof will be carried our in several steps.

(i) If $u \in S$ is a possible critical point of g_b on S, then the arguments of Section 3 yield that $G_b'(\lambda(u)\,u) = 0$ and thus $q(t) = \lambda(u)\,u(t)$ is a homoclinic orbit of

$$\ddot{q} + q - b\alpha |q|^{\alpha-2} q = 0 .$$

From the conservation of the angular momentum, it follows that $g(t) = \xi r(t)$, where $\xi \in S^{N-1}$ and $r = r(t)$ satisfies the scalar equation

$$\text{(34)} \qquad \begin{cases} \text{i)}\ \ddot{r} + r - b\alpha r^{\alpha-1} = 0 , \\ \text{ii)}\ r(t) \to 0, \qquad \dot{r}(t) \to 0, \qquad \text{as } t \to \pm\infty . \end{cases}$$

Since (34) has, up to time translations, a unique solution it follows that g_b has a unique critical level on S.

(ii) Let $\mu = \inf_S g_b$. Clearly Lemma 9 applies to g_b with $m = \mu$ and therefore the L-S theory can be used to infer that, actually, $\mu = \min_S g_b$. Let $u^* \in S$ be such that $g_b(u^*) = \mu$. Then, according to point (i) above, μ is the only critical level of g_b and $q^*(t) \equiv \lambda(u^*) u^*(t)$ has the form $q^*(t) = \xi^* r(t)$, with $\xi^* \in S^{N-1}$ and r satisfying (34).

(iii) We will apply Lemma 6 to $M = S$ and f, by taking $c = 2\mu - \varepsilon$, with $\varepsilon > 0$ small, and

$$X = \{u \in S\colon g_b(u) = \mu\} \qquad (= g_b^{\mu}),$$

$$Y = g_b^{2\mu-\varepsilon} .$$

First of all, let us notice that $\operatorname{cat}(X, X) = 2$. To see this, we recall that $X = \{u = \xi r/\|r\|\colon \xi \in S^{N-1},\ r \text{ is a solution of (34)}\}$; moreover, the solutions of (34) have the form $r(t) = r_0(t + \tau)$, for a fixed r_0 and any $\tau \in \boldsymbol{R}$. Thus $X = S^{N-1} \times \boldsymbol{R}$ and the claim follows.

Next, from Lemma 10 (with $c = \mu$) it follows that assumption 1) of Lemma 6 is verified. Moreover, from point (ii) we deduce that X is a deformation retract of Y. Indeed, g_b has no other critical points than those in X, and Y can be deformed (actually, retracted) on X because, by Lemma 9, condition (C) holds true for $g_b|_S$ on $g_b^{2\mu-\varepsilon}$. In particular, such a retraction gives rise to the map $\eta \in C(Y, X)$ satisfying assumption 2) of Lemma 6.

From (31) it follows that $m = \min_S f \geq \mu$. Hence $2\mu - \varepsilon < 2m$ and therefore, as a consequence of Lemma 9, $f|_S$ verifies condition (C) on $f^{2m-\varepsilon}$.

(iv) The discussion of point (iii) allows us to apply Lemma 6 yielding $\operatorname{cat}(X, X) = 2$ critical points for $f|_S$. More precisely, according to the L-S theory, f_S has two critical levels $c_1 = m \leq c_2$ and if $c_1 = c_2 (= m)$ then $f|_S$ has at level m infinitely many critical points K_m, such that

$\operatorname{cat}(K_M, f^{2\mu-\varepsilon}) = 2$. It remains to show that, in any case, (1) possesses at least 2 geometrically distinct homoclinic orbits. To see this, let u_i be such that $f(u_i) = c_i$ and set $\mathcal{G}(u_i) = \{u \in E: u(t) = u_i(\pm t + \tau), \tau \in \boldsymbol{R}\}$. If there exists only one geometrically distinct homoclinic orbit, we would have $\mathcal{G}(u_1) = \mathcal{G}(u_2)$ and therefore $c_1 = c_2 = m$. But it is immediate to see that $\operatorname{cat}(\mathcal{G}(u), f^c) = 1$ for all $u \in f^c$ (one can assume, without loss of generality, that f^c is connected, otherwise the same would be true in each component). This completes the proof. ■

Before proving Theorem 2, let us recall that when W is even, then F and f are even functionals, and we can use the $\boldsymbol{Z}_2$-invariant L-S theory. Let $\gamma(A)$ denote the $\boldsymbol{Z}_2$-genus of the $\boldsymbol{Z}_2$-symmetric closed set A, with $0 \notin A$; see, for example [1, Section 2]. We anticipate that, due to the specific features of the $\boldsymbol{Z}_2$-genus, we do not need to use Lemma 6 any more.

For $k \leq N$, let π^* denote the k-dimensional linear subspace of $\boldsymbol{R}^N$ such that for the a_k defined in (8) there results

$$a_k = \min_{|x|=1, x \in \pi^*} \varphi_1(x).$$

Define $\Sigma_k = \{x \in R^N: |x| = 1, x \in \pi^*\}$, and

$$X_k = \left\{ u = \frac{\xi}{\|r\|} \cdot r: \xi \in \Sigma_k, \quad r \text{ is a solution of } (34) \right\}.$$

Since $X_k \simeq S^{k-1} \times \boldsymbol{R}$, then $\gamma(X_k) = k$, see Lemma 2.11-(vi) of [1]. We claim

LEMMA 11. *If* $b < a_k \cdot 2^{\alpha - 2/2}$ *then* $X_k \subset f^{2\mu-\varepsilon}$.

PROOF. For all $s \in \boldsymbol{R}$ and all $x \in \Sigma_k$ there results

$$W(sx) \geq \varphi_1(x)\,|s|^\alpha \geq a_k\,|s|^\alpha .$$

Therefore, for any $u = \xi/\|r\| \cdot r \in X_k$ one has

$$W\left(\frac{\xi}{\|r\|} \cdot r(t)\right) \geq a_k \cdot \frac{|r(t)|^\alpha}{\|r\|^\alpha},$$

and hence

$$f(u) \leq g_{a_k}(u) \equiv \max_{\lambda \geq 0} \left[\frac{1}{2}\lambda^2 - a_k \lambda^\alpha \int_{-\infty}^{+\infty} \frac{|r(t)|^\alpha}{\|r\|^\alpha} dt \right] \tag{35}$$

The same calculation made in Lemma 10 yields

$$g_k(u) = \left(\frac{1}{2} - \frac{1}{\alpha}\right) a_k^{2/2-\alpha} \tilde{A}^{2/2-\alpha}, \tag{36}$$

where $\tilde{A} = \alpha\|r\|^{-\alpha} \int_{-\infty}^{+\infty} |r(t)|^{\alpha} dt$. On the other side, as in Lemma 10, one also has

$$g_b(u) = \left(\frac{1}{2} - \frac{1}{\alpha}\right) b_k^{2/2-\alpha} \tilde{A}^{2/2-\alpha},$$

and hence, using (35), (36) and assumption (9)

$$f(u) < 2 \cdot g_b(u), \qquad \forall u \in X_k . \tag{37}$$

Finally, since $u = \xi/\|r\| \cdot r$ and r solves (34), there results $g_b(u) = \mu$ and (37) implies $f(u) < 2\mu$ for all $u \in X_k$. ∎

We are now in position to prove Theorem 2.

PROOF OF THEOREM 2. From Lemma 11 and the monotonicity property of the Z_2-genus (cfr. Lemma 2.11 of [1]) we infer $\gamma(f^{2\mu-\varepsilon}) \geqslant \gamma(X_k) = k$. Moreover, according to Lemma 9, $f|_S$ satisfies (C) on $f^{2\mu-\varepsilon}$ because $2\mu - \varepsilon < 2m$. Then $f|_S$ possesses at least k (pairs of) critical points, which give rise, as in the proof of Theorem 1, to k geometrically distinct homoclinic orbits. ∎

REMARK. As seen in the preceding proof, we did not need to find any set Y as claimed in Lemma 6. The reason is because, unlike the genus, we do not have, in general, that $\operatorname{cat}(X, X) \leqslant \operatorname{cat}(Y, Y)$ provided $X \subset Y$. ∎

In order to prove Theorem 3, les us begin modifying the potential R. Let $r_0 > 0$ be such that $0 < (1/\alpha) r^{\alpha} < (1/2) r^2$ implies $0 < r < r_0$ and let $\chi \in C^{\infty}(\boldsymbol{R}^+, \boldsymbol{R})$ be a nonincreasing function such that

$$\begin{cases} \text{i) } \chi(r) \equiv 1, & \forall r \leqslant r_0 , \\ \text{ii) } \chi(r) \equiv 0, & \forall r \geqslant r_0 + 1 . \end{cases}$$

Define $\tilde{R}_{(q)}$ by setting

$$\tilde{R}(q) = \chi(|q|) R(q),$$

and set

$$W_\varepsilon(q) = \frac{1}{\alpha}|q|^\alpha + \varepsilon\tilde{R}(q).$$

LEMMA 12. *Let $\varepsilon > 0$ and let q be a homoclinic orbit of*

$$\ddot{q} - q + W_\varepsilon'(q) = 0. \tag{38}$$

Then q is a homoclinic orbit of (1) *with V given by* (10).

PROOF. From the conservation of the energy we infer

$$\frac{1}{2}|\dot{q}(t)|^2 - \frac{1}{2}|q(t)|^2 + W_\varepsilon(q(t)) \equiv 0.$$

Hence it follows

$$\frac{1}{2}|q(t)|^2 \geqslant W_\varepsilon(q(t)) = \frac{1}{\alpha}|q(t)|^\alpha + \varepsilon\tilde{R}(q(t)).$$

Since $\varepsilon \geqslant 0$ and $\tilde{R}(q) \geqslant 0$, then we find that $1/2|q(t)|^2 \geqslant 1/\alpha|q(t)|^\alpha$. Therefore $|q(t)| \leqslant r_0$ and from the definition of χ it follows that $R = R$. Hence $q(t)$ solves (1), with $V(q) = -1/2|q|^2 + 1/\alpha|q|^\alpha + \varepsilon R(q)$. ■

PROOF OF THEOREM 3. First of all, let us check that for all $\varepsilon > 0$ and small enough, W_ε verifies assumptions (W_1) and (W_2). Let us take any $\theta \in]1/\alpha, 1/2[$. Then there results

$$\theta W_\varepsilon'(q)\cdot q - W_\varepsilon(q) = \left(\theta - \frac{1}{\alpha}\right)|q|^\alpha + \varepsilon(\theta\tilde{R}'(q)\cdot q - \tilde{R}(q)).$$

From (W_4) it follows that $R'(q)\cdot q$ and $R(q)$ are $o(|q|^\alpha)$ as $q \to 0$. Since $\tilde{R}(q) = R(q)$ for $|q|$ small, then there exists $\delta > 0$ such that

$$\theta W_\varepsilon'(q)\cdot q - W_\varepsilon(q) \geqslant 0, \qquad \forall |q| \leqslant \delta. \tag{39}$$

On the other hand, since $\tilde{R}(q) \equiv 0$ for all $|q| \geqslant r_0 + 1$, then there exist $\varepsilon_1 > 0$ such that

$$\varepsilon(\tilde{R}(q) - \tilde{R}'(q)\cdot q) \leqslant \left(\theta - \frac{1}{\alpha}\right)|q|^\alpha, \qquad \forall |q| > \delta,$$

whenever $0 \leqslant \varepsilon \leqslant \varepsilon_1$. This and (39) imply that W_ε verifies (W_1) provided $0 \leqslant \varepsilon \leqslant \varepsilon_1$.

In a quite similar way, one shows that there exists $\varepsilon_2 > 0$ such that W_ε verifies (W_2) provided $0 \leqslant \varepsilon \leqslant \varepsilon_2$.

Finally, let b be any real number such that $1 < b < 2^{\alpha-2/2}$. We claim that, for ε possibly smaller, (W_3) holds true with $\varphi_1(q) = (1/\alpha)\,|q|^\alpha$ and $\varphi_2(q) = (b/\alpha)\,|q|^\alpha$.

To see this, we argue as before. We first remark that, being $R(q) = o(|q|^\alpha)$ as $q \to 0$, then $\exists\,\delta' > 0$ such that

$$\varepsilon\tilde{R}(q) = \varepsilon R(q) \leqslant \frac{b-1}{\alpha}\,|q|^\alpha\,, \qquad \forall |q| \leqslant \delta'\,.$$

Moreover, since $\tilde{R}(q) \equiv 0$ for $|q| \geqslant r_0 + 1$, then there exists $\varepsilon_3 > 0$ such that

$$\varepsilon\tilde{R}(q) \leqslant \frac{b-1}{\alpha}\,|q|^\alpha\,, \qquad \forall |q| \geqslant \delta'\,, \qquad \forall \varepsilon \in [0, \varepsilon_3]\,.$$

Therefore $W_\varepsilon(q) \geqslant \varphi_2(q) = (b/\alpha)\,|q|^\alpha$, provided $0 < \varepsilon \leqslant \varepsilon_3$. Since, plainly, $W_\varepsilon(q) \geqslant (1/\alpha)\,|q|^\alpha$, $\forall \varepsilon > 0$, $\forall q \in \boldsymbol{R}^N$, the claim follows.

The preceding discussion allows us to apply Theorem 1 (respectively, Theorem 2) whenever $0 \leqslant \varepsilon \leqslant \varepsilon_0 \equiv \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and V has the form (10) (respectively, V has the form (10) and is even), yielding 2 (respectively, N) homoclinic orbits for equation (38). According to Lemma 12, these orbits are in fact solutions of (1), and this completes the proof of Theorem 3. ■

REFERENCES

[1] A. Ambrosetti, *Critical points and nonlinear variational problems*, Mem. Soc. Math. France, **120**, No. 49 (1992).

[2] A. Ambrosetti - M. L. Bertotti, *Homoclinics for a second order conservative systems*, in *Partial Differential Equations and Related Subjects*, M. Miranda Ed. Longman (1992), pp. 21-37.

[3] A. Ambrosetti - V. Coti Zelati, *Multiplicté des orbites homoclines pour des systémes conservatifs*, Compte Rendus Acad. Sci. Paris, **314** (1992), pp. 601-604.

[4] A. Ambrosetti - V. Coti Zelati - I. Ekeland, *Symmetry breaking in Hamiltonian systems*, J. Diff. Equat., **67** (1987), pp. 165-184.

[5] A. Ambrosetti - G. Mancini, *On a theorem by Ekeland and Lasry concerning the number of periodic Hamiltonian trajectories*, J. Diff. Equat., **43** (1982), pp. 249-256.

[6] A. Bahri - H. Berestycki, *A perturbation method in critical point theory and applications*, Trans. Amer. Math. Soc., **267** (1981), pp. 1-32.

[7] S. V. Bolotin, *The existence of homoclinic motions*, Vestnik Moscow Univ. Ser. I, Math. Mekh., **6** (1983), pp. 98-103; Moscow Univ. Math. Bull., 38-6 (1983), pp. 117-123.

[8] V. Coti Zelati - I. Ekeland - E. Seré, *A variational approach to homoclinic orbits in Hamiltonian systems*, Math. Ann., **288** (1990), pp. 133-160.

[9] V. Coti Zelati - P. H. Rabinowitz, *Homoclinic orbits for a second order Hamiltonian systems possessing superquadratic potentials*, Jour. Am. Math. Soc., 4 (1991), pp. 693-727.

[10] I. Ekeland - J. M. Lasry, *On the number of closed trajectories for a Hamiltonian flow on a convex energy surface*, Ann. Math., **112** (1980), pp. 283-319.

[11] H. Hofer - K. Wysocki, *First order elliptic systems and the existence of homoclinic orbits in Hamiltonian systems*, Math. Ann., **288** (1990), pp. 483-503.

[12] V. K. Melnikov, *On the stability of the center for periodic perturbations*, Trans. Moscow Math. Soc., **12** (1963), p. 1-57.

[13] R. Palais - S. Smale, *A generalized Morse theory*, Bull. Amer. Math. Soc., **70** (1964), p. 165-171.

[14] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Gauthier-Villars, Paris (1897-1899).

[15] P. H. Rabinowitz, *Homoclinic orbits for a class of Hamiltonian systems*, Proceed. Royal Soc. Edinburgh, **114-A** (1990), pp. 33-38.

[16] P. H. Rabinowitz - K. Tanaka, *Some results on connecting orbits for a class of Hamiltonian systems*, Math. Zeit., to appear.

[17] E. Séré, *Existence of infinitely many homoclinic orbits in Hamiltonian systems*, Math. Z., **209** (1992), pp. 27-42.

[18] E. Séré, *Homoclinic orbits on compact hypersurfaces in $\mathbf{R}^{2N}$ of restricted contact type*, preprint CEREMADE, 1992.

[19] K. Tanaka, *Homoclinic orbits in a first order superquadratic Hamiltonian system: convergence of subharmonic orbits*, to appear.

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