Rendiconti
del
Seminario Matematico
della
Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 89 (1993), p. 171-175
<http://www.numdam.org/item?id=RSMUP_1993__89__171_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 89 (1993)

On the Fitting Length of $H_n(G)$.

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For a finite group G and $n \in N$ the generalized Hughes subgroup $H_n(G)$ of G is defined as $H_n(G) = \langle x \in G \mid 1 \neq x^n \rangle$. Recently, there has been some research in the direction of finding a bound for the Fitting length of $H_n(G)$ in a solvable group G with a proper generalized Hughes subgroup in terms of n. In this paper we want to present a proof for the following

THEOREM 1. *Let G be a finite solvable group, $p_1, p_2, \ldots, p_m$ pairwise distinct primes and $n = p_1 \cdot p_2 \cdot \ldots \cdot p_m$. If $H_n(G) \neq G$, then the Fitting length of $H_n(G)$ is at most $m + 3$.*

This result is an immediate consequence of

THEOREM 2. *Let G be a finite solvable group, H a proper, normal subgroup of G such that the order of every element of $G \setminus H$ divides $n = p_1 \cdot p_2 \cdot \ldots \cdot p_m$, where $p_1, p_2, \ldots, p_m$ are pairwise distinct primes. Then the Fitting length $f(H)$ of H is at most $m + 3$.*

The proof of Theorem 2 will be given as usual by showing that a counterexample to the theorem does not exist. If G is a minimal counterexample to the theorem, then clearly $|G:H| = p$ is a prime, $G = H\langle \alpha \rangle$ for some element $\alpha \in G \setminus H$ of order p and every element of $G \setminus H$ has order dividing $n = p \cdot q_1 \cdot q_2 \cdot \ldots \cdot q_{m-1}$ for pairwise distinct primes $p, q_1, q_2, \ldots, q_{m-1}$.
Therefore Theorem 2 is a corollary of the following result

THEOREM 3. *Let H be a finite solvable group, α an automorphism of H of prime order p and let $G = H\langle \alpha \rangle$ be the natural semidirect*

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product of H with $\langle\alpha\rangle$. *Suppose that* α *acts on* H *in such a way that the order of any element of* $G\setminus H$ *divides* $N = p\cdot q_1\cdot q_2\cdot\ldots\cdot q_m$ *where* $q_1, \ldots, q_m$ *are (not necessarily distinct) primes different from* p. *If* $4 \nmid N$, *then the Fitting length of* H *is at most* $m+4$. *Furthermore, if* $H = [H, \alpha]$, *then the Fitting length of* H *is at most* $m+2$.

Unfortunately, we were not able to see whether the bound given in the above theorem is the best possible bound although one can construct an example to show that the best bound in the case $H = [H, \alpha]$ must be greater than or equal to $m+1$.

For the proof of the theorem, we need a technical lemma, which is essentially well known.

LEMMA 1. *Let the cyclic group* Z *of prime order* p *act on the finite solvable group* $1 \neq H$ *in such a way that the orders of elements of the natural semidirect product* $G = HZ$ *lying outside* H *are not divisible by* p^2. *If* $f = f(H)$ *is the Fitting length of* H, *then there exist subgroups* $C_1, C_2, \ldots, C_f$ *of* H *and subgroups* $D_i \lhd C_i$, $i = 1, 2, \ldots, f$ *and an element* $x \in G\setminus H$ *of order* p *such that the following conditions are satisfied:*

(i) C_i *is a* p_i*-subgroup for some prime* p_i, $i = 1, 2, \ldots, f$ *and* $p_i \neq p_{i+1}$ *for* $i = 1, 2, \ldots, f-1$.

(ii) C_i *and* D_i *are* $C_{i+1}C_{i+2}\ldots C_f\langle x\rangle$*-invariant for any* $i = 1, 2, \ldots, f$.

(iii) $\overline{C}_i = C_i/D_i$ *is a special group on the Frattini factor group of which* $C_{i+1}C_{i+2}\ldots C_f\langle x\rangle$ *acts irreducibly for any* $i = 1, 2, \ldots, f$. C_{i+1} *acts trivially on* $\Phi(\overline{C}_i)$, $i = 1, 2, \ldots, f$.

(iv) $[C_i, C_{i+1}] = C_i$ *for* $i = 1, 2, \ldots, f-1$. *The same equation holds also for* $i = f$, *if* $[H/F_{f-1}(H), x^{(s)}] \neq 1$ *for any* $s \in N$; *otherwise* $[C_f, x] \leq D_f$ *and* C_f/D_f *is of prime order. (The notation* $[G, x^{(s)}]$ *for a group* G *and an element* x *is defined inductively as* $[G, x^{(s)}] = [[G, x^{(s-1)}], x]$ *for any* $s \in N$).

(v) $C_{C_{i+1}}(\overline{C}_i/\Phi(\overline{C}_i)) = C_{C_{i+1}}(\overline{C}_i)$ *is contained in* $\Phi(C_{i+1} \bmod D_{i+1})$, $i = 1, 2, \ldots, f-1$.

(vi) *For any* $i = 2, \ldots, f$ *and any* $1 \leq j < i$, $[C_j, C_i]$ *is not contained in* $\Phi(C_j \bmod D_j)$.

PROOF. A slight modification of Lemma 2.7 in [3] gives that H has nilpotent subgroups $H_1, \ldots, H_f$ and that there exists $x \in G\setminus H$ of order p such that H_i is $H_{i+1}\ldots H_f\langle x\rangle$-invariant, $F_i(H) = F_{i-1}(H)H_i$ and H_i is a π_i-group, where $\pi_i = \pi(F_i(H)/F_{i-1}(H))$ for any $i = 1, 2, \ldots, f$. Observe that for any prime q and any $i = 1, 2, \ldots, f$ a Sylow q-subgroup of H_i is $H_{i+1} \ldots H_f\langle x\rangle$-invariant.

If $[H/F_{f-1}(H), x^{(s)}] = 1$ for some $s \in N$, then there exists a subgroup $\bar{Y}$ of $H/F_{f-1}(H)$ of prime order which is centralized by x. Let $p_f = |\bar{Y}|$. If $[H/F_{f-1}(H), x^{(s)}] \neq 1$ for all $s \in N$, then the same result holds for some Sylow subgroup of $H/F_{f-1}(H)$. Let p_f be the corresponding prime. By a Hall-Higman reduction, there exists an $\langle x\rangle$-invariant subgroup $\bar{Y}$ of $O_{p_f}(H/F_{f-1}(H))$ of minimal order on which $\langle x\rangle$ acts nontrivially. In this case $p_f \neq p, [\bar{Y}, x] = \bar{Y}$, $\bar{Y}$ is a special group, $[\Phi(\bar{Y}), x] = 1$ and $\langle x\rangle$ acts irreducibly on $\bar{Y}/\Phi(\bar{Y})$.

In both cases, there exists an $\langle x\rangle$-invariant subgroup C_f of $O_{p_f}(H_f)$ of minimal order such that $C_f F_{f-1}(H)/F_{f-1}(H) = \bar{Y}$. Let $C_f \cap \cap F_{f-1}(H) = D_f$. Suppose now, we have already chosen $C_{i+1}, C_{i+2}, \ldots, C_f$ such that C_j is a p_j-subgroup of $F_j(H)$ contained in H_j such that C_j is $C_{j+1}C_{j+2}\ldots C_f\langle x\rangle$-invariant for any $j = i+1, \ldots, f$. $C_j/D_j = \bar{C}_j$ is a nontrivial special group on the Frattini factor group of which $C_{j+1}\ldots C_f\langle x\rangle$ acts irreducibly for any $j = i+1, \ldots, f$, where $D_j = C_j \cap F_{j-1}(H)$, C_{j+1} acts trivially on $\Phi(\bar{C}_j)$ and $[C_j, C_{j+1}] = C_j$ for $j = i+1, \ldots, f-1$. C_{i+1}/D_{i+1} acts faithfully on the Frattini factor group of $O_{p'_{i+1}}(F_i(H)/F_{i-1}(H))$. So there exists a prime $p_i \neq p_{i+1}$ such that C_{i+1} acts nontrivially on $O_{p_i}(F_i(H)/F_{i-1}(H))$ and hence on $O_{p_i}(H_i/H_i \cap \cap F_{i-1}(H))$. Let now C_i be a $C_{i+1}C_{i+2}\ldots C_f\langle x\rangle$-invariant subgroup of $O_{p_i}(H_i)$ of minimal order such that C_{i+1} acts nontrivially on $C_iF_{i-1}(H)/F_{i-1}(H)$ but trivially on any $C_{i+1}C_{i+2}\ldots C_f\langle x\rangle$-invariant subgroup of it. Then $[C_i, C_{i+1}] = C_i$ and C_i/D_i is a special group on the Frattini factor group of which $C_{i+1}\ldots C_f\langle x\rangle$ acts irreducibly and the Frattini subgroup of which is centralized by C_{i+1} where $D_i = C_i \cap \cap F_{i-1}(H)$. Clearly, $[D_{i+1}, C_i] \leqslant D_i$ and $C_{C_{i+1}}(\bar{C}_i)$ is contained in $\Phi(C_{i+1} \bmod D_{i+1})$ as $1 \neq C_{i+1}/\Phi(C_{i+1} \bmod D_{i+1})$ is irreducible. So, recursively C_i's can be constructed such that (i)-(v) are satisfied.

If $[C_j, C_i] \leqslant \Phi(C_j \bmod D_j)$ for some i, j with $2 \leqslant i \leqslant f$ and $1 \leqslant j \leqslant i$, then three subgroup lemma yields that $[C_{j+1}, C_i, C_j] \leqslant \Phi(C_j \bmod D_j)$, i.e. $[C_i, C_{j+1}] \leqslant \Phi(C_{j+1} \bmod D_{j+1})$. Repeating this argument, one gets $[C_i, C_k] \leqslant \Phi(C_k \bmod D_k)$ for any $j \leqslant k < i$ and hence $C_{i-1} = [C_i, C_{i-1}] \leqslant \leqslant \Phi(C_{i-1} \bmod D_{i-1})$ which is not the case. This completes the proof.

PROOF OF THEOREM 3. Let $f = f(H)$. By lemma, there exist subgroups $C_1, \ldots, C_f$ of H and subgroups $D_i \lhd C_i$ for $i = 1, \ldots, f$ and an element $x \in G \setminus H$ of order p satisfying (i)-(vi). Put $K = C_1 \ldots C_f$. Now $K\langle x\rangle$ satisfies the hypothesis of the theorem. Note that if $[H, \alpha] = H$, then we have $[C_f, x] = C_f$ and so we may assume that $[K, x] = K$.

Suppose that there exist k and l in $\{1, \dots, f\}$ with $k < l$ so that C_k and C_l are both p-groups. Put $L = C_k C_{k+1} C_l$. Obviously, $f(L) = 3$. By lemma, there exist $\langle x \rangle$-invariant subgroups E_1, E_2, E_3 of L and subgroups $F_i \lhd E_i$ for $i = 1, 2, 3$ satisfying (i)-(vi), where E_1 and E_3 are p-groups. $1 \neq C_{E_1/\Phi(E_1)}(F_3)$ is $E_2 E_3 \langle x \rangle$-invariant and hence $[E_1, F_3] \leq \Phi(E_1)$ where also we have $C_{E_3}(E_1/\Phi(E_1)) \leq F_3$. Thus $F_3 = C_{E_3}(E_1/\Phi(E_1))$. Put $\overline{E} = E_1 E_2 E_3 / \Phi(E_1) F_2 F_3$. Observe that $f(\overline{E}) = 3$ and $[\overline{E}_3, x] = 1$. If $[\overline{E}_2, x] = 1$, then $[\overline{E}_1, x] = 1$ whence $[\overline{E}, x] = 1$. Then a Sylow p-subgroup of $\overline{E}$ has exponent p and ([5], IX. 4.3) gives that p-length of $\overline{E}$ is one which is not the case. Thus $[\overline{E}_2, x] = \overline{E}_2$. As in the proof of Proposition 1, the exceptional action of x on the elementary abelian p-group $\overline{E}_1$ gives that $\overline{E}_2$ is a nonabelian 2-group. Using ([2], 5.3.16), we get an element $\overline{y} \in \overline{E}_3 \langle x \rangle \setminus \overline{E}_3$ such that $\overline{y}$ centralizes a nontrivial element in the Frattini factor group of $\overline{E}_2$ on which $\overline{E}_3 \langle x \rangle$ acts irreducibly. Thus $\overline{y}$ must centralize $\overline{E}_2$. It follows that $\overline{E}_2$ is of exponent 2 and hence abelian. This contradiction shows that there is at most one p-group among the C_i's say C_k. Thus $U = \prod_{i \neq k} C_i$ is a p'-group and $f - 2 \leq f(U)$. By ([5], IX.4.3) q-length of $C_U(x)$ is at most the multiplicity of q in N for any prime q. Thus $f(C_U(x)) \leq m$ and hence $f(U) \leq m + 2$ by ([8], 3.2). It follows that $f \leq m + 4$.

Furthermore, assume that $[K, x] = K$. Take C_j for $j > k$. Let V be an irreducible composition factor of $GF(p)[C_j, x]\langle x \rangle$-submodule of the Frattini factor group of C_k/D_k on which $[C_j, x]$ acts nontrivially and let $C = \ker([C_j, x]\langle x \rangle \text{ on } V)$. If $x \in C$, then $[C_j, x] \leq C \cap [C_j, x] < [C_j, x]$. So $C < [C_j, x]$. Now applying ([7], 2.8) to $[C_j, x]/C$ on V, we get $[C_j, x]/C$ is a nonabelian special group by ([4], III.13.6) as x acts exceptionally on V. The irreducibility of V and ([5], IX.3.2) yields that C_j is a 2-group. Thus $f = k + 1$. If there exists $s < k$ such that C_s is a 2-group, put $M = [C_s, C_k] C_k C_{k+1}$. We have $[M, x] = M$ and M is a $\{2, p\}$-group. It follows that $f(M) \leq 2$ by [1] which is not the case. Thus $Y = \prod_{i=1}^{k-1} C_i$ is a $\{2, p\}'$-group and so $\exp(C_Y(x))$ divides a product of $m - 1$ primes. ([6], Satz 3) implies that $f([Y, x]) \leq m$. If $[Y, x] \leq F_{k-2}(Y)$, then $[C_{k-1}, x] \leq F_{k-2}(Y) \cap C_{k-1} = D_{k-1}$ which is not the case. Consequently, $f(K) = f(Y) + 2 = f([Y, x]) + 2 \leq m + 2$.

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Manoscritto pervenuto in redazione il 13 marzo 1992.