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A contribution to the theory of finite supersoluble groups

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A Contribution to the Theory of Finite Supersoluble Groups.

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In memory of my father.

1. Introduction.

Throughout this paper the term group always means a finite group. It is well-known that a *supersoluble group* is a group whose chief factors are all cyclic. The class of supersoluble groups lies between nilpotent and soluble groups. In these last years a number of papers have investigated the influence of the embedding properties of some subgroups of a group on its supersolubility (cf. [1], [4] and [6]). Our aim is to continue these investigations analyzing the cover and avoidance property.

DEFINITIONS. Let G be a group, H/K a chief factor of G and M a subgroup H of G. We say that

i) M *covers* H/K if $H \leqslant KM$;

ii) M *avoids* H/K if $H \cap M \leqslant K$;

iii) M has the *cover and avoidance property* in G, M is a *CAP-subgroup* of G for short, if it either covers or avoids every chief factor of G

Normal subgroups are clearly *CAP*-subgroups. Copious examples of *CAP*-subgroups in the universe of soluble groups are well-known; amongst them the most remarkable are perhaps the Hall subgroups. By an obvious consequence of the definitions, in a supersoluble group every subgroup is a *CAP*-subgroup.

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In Section 3 some characterizations of p-supersoluble groups involving *CAP*-subgroups are presented. If p is a prime, a *p-supersoluble group* is a group whose p-chief factors are all cyclic. A p-solubility condition must be imposed. Some examples illustrate the discussion.

In Section 4 we deduce some characterizations of supersoluble groups involving *CAP*-subgroups; we prove here that a group G is supersoluble if and only if all subgroups of G are *CAP*-subgroups of G. As a matter of fact, what we really prove is that is enough to impose the cover and evoidance property only on certain subgroups to characterize the supersolubility.

Finally in Section 5 we expose an example to distinguish our contribution from some others.

2. Three preparatory known lemmas.

The following three lemmas are known; we include them here for the sake of completeness.

LEMMA 1 ([5], § 1, Lemma 1.4). *Let G be a group, N a normal subgroup of G and M a CAP-subgroup of G. Then MN is a CAP-subgroup of G.*

PROOF. Let H/K be a chief factor of G. If N covers H/K, so does NM. Suppose $H \cap N \leqslant K$. Then HN/KN is a chief factor of G, G-isomorphic to H/K; if M covers HN/KN then $H \leqslant HN \leqslant KNM$ and NM covers H/K; if M avoids HN/KN: $KN \cap M = HN \cap M$; then $HN \cap MN = (HN \cap M)N = (KN \cap M)N \leqslant KN$ and $MN \cap H \leqslant KN \cap H = K(N \cap H) = K$ and MN avoids H/K.

LEMMA 2 ([2], Proposition 2.3). *Let G be a group, N a normal subgroup of G such that $G = QN$ for some subgroup Q of G. Take a maximal subgroup M of G with $N \leqslant M$. Then $M \cap Q$ is a maximal subgroup of Q.*

PROOF. It is clear from the isomorphism between G/N and $Q/(Q \cap N)$ that $(M \cap Q)/(Q \cap N)$ is maximal in $Q/(Q \cap N)$ and therefore $Q \cap M$ is a maximal subgroup of Q.

LEMMA 3 ([1], Lemma 3.1). *Let G be a group, p a prime, H a subgroup of G and P a normal p-subgroup of $N_G(H)$. Then $F(HP) = F(H)P$.*

PROOF. Let $F = F(HP)$; then $F \cap H = F(H)$ and $F = F \cap HP = P(F \cap H) = PF(H)$.

3. Characterizations of p-supersoluble groups.

THEOREM A. *Let p be a prime, G be a p-soluble group and H a normal subgroup of G such that G/H is p-supersoluble. Suppose that all maximal subgroups of the Sylow p-subgroups of H are CAP-subgroups of G. Then G is p-supersoluble.*

PROOF. We prove the theorem by induction on the order of G.

a) If N is a minimal normal subgroup of G then G/N is p-supersoluble.

If $N \leq H$ we check that all hypotheses hold for G/N and H/N. Notice that if Q is a Sylow p-subgroup of H and M is a maximal subgroup of QN with $N \leq M$ then $M = N(Q \cap M)$. By Lemma 2, $Q \cap M$ is a maximal subgroup of Q. By hypothesis, $Q \cap M$ is a *CAP*-subgroup of G and by Lemma 1 so is M. Thus, M/N is a *CAP*-subgroup of G/N. By induction, G/N is p-supersoluble.

Otherwise $N \cap H = 1$. Take Q a Sylow p-subgroup of HN. If $(|Q|, |N|) = 1$ then there exists a Sylow p-subgroup Q^* of H such that $Q^* = Q^x$ for some $x \in N$; so, $QN = Q^*N$. If $(|Q|, |N|) = p \neq 1$ then $Q = Q^*N$ for some $Q^* \in \mathrm{Syl}_p(H)$. Therefore, in any case, $QN = Q^*N$ for some Sylow p-subgroup Q^* of H. Applying again Lemmas 1 and 2 it is easy to check the hypotheses hold for G/N and HN/N. By induction we have again that G/N is p-supersoluble.

b) We can suppose that G is a primitive group.

If G has two different minimal normal subgroups, say N_1 and N_2, then G/N_i is p-supersoluble for $i = 1, 2$, and $G = G/(N_1 \cap N_2)$ is p-supersoluble. So we can assume that G is monolithic.

Denote by N the unique minimal normal subgroup of G. If $N \leq \Phi(G)$ then $G/\Phi(G)$ is supersoluble and so is G. The remaining case is $\Phi(G) = 1$ and G is a primitive group.

c) Conclusion.

If G is not p-supersoluble then N is a p-group for some prime p, and p^2 divides $|N|$. Let T be a complement of N in G and let $P \in \mathrm{Syl}_p(H)$. Then $T \cap P$ is a complement to N in P. Let M be a maximal subgroup of P containing $T \cap P$. Then $|N : N \cap M| = |P : M| = p$ contrary to the hypothesis that M either covers or avoids N. Thus, N is cyclic and G is p-supersoluble.

LEMMA 4. *Let p be a prime, G be a p-soluble group and H a normal subgroup of G such that G/H is p-supersoluble. Assume $O_{p'}(G) = \Phi(G) = 1$. Suppose that all maximal subgroups of $O_p(H)$ are CAP-subgroups of G. Then G is supersoluble.*

PROOF. Since G is p-soluble and $O_{p'}(G) = 1$ we have $C_G(O_p(G)) \leqslant O_p(G)$. Now $\Phi(G) = 1$ implies that $F(G) = O_p(G) = \mathrm{Soc}(G)$ is an elementary abelian group, by Satz III.4.5 of [3]. Thus $C_G(F(G)) = F(G)$.

Now we claim that all minimal normal subgroups of G are cyclic.

Take N a minimal normal subgroup of G; if $N \cap H = 1$ then NH/H is p-chief factor of G/H and therefore is cyclic; since $N \cong NH/H$, N is cyclic. Otherwise $N \leqslant H$ and indeed $N \leqslant O_p(H)$. Since $\Phi(O_p(H)) = 1$ there exists a maximal subgroup of $O_p(H)$, say S, such that N is not contained in S: $O_p(H) = NS$. By hypothesis, S is a CAP-subgroup of G and then $N \cap S = 1$ and therefore we have $p = |O_p(H): S| = |N|$. (Notice that this argument holds even in $N = O_p(H)$; then $S = 1$). So, our claim is true: every minimal normal subgroup of G is cyclic.

Recall that $F(G) = \mathrm{Soc}(G) = N_1 \times \ldots \times N_r$, where each N_i is a minimal normal subgroup of G. For each minimal normal subgroup N_i of G the quotient group $G/C_G(N_i)$ is a subgroup of the group of automorphisms of a cyclic group and therefore is an abelian group and is indeed a supersoluble group. Therefore $G\Big/\left(\bigcap_{i=1}^{r} C_G(N_i)\right)$ is supersoluble. In fact, what we really have is that $G/F(G)$ is supersoluble inasmuch as $\bigcap_{i=1}^{r} C_G(N_i) = C_G(F(G)) = F(G)$. But all chief factors of G below $F(G)$ are cyclic and hence the whole of G is supersoluble.

THEOREM B. *Let p be a prime, G a p-soluble group and H a normal subgroup of G such that G/H is p-supersoluble. Suppose that all maximal subgroups of $O_{p'p}(H)$ containing $O_{p'}(H)$ are CAP-subgroups of G. Then G is p-supersoluble.*

PROOF. We prove the theorem by induction on $|G|$.

Denote $R = O_{p'}(G)$ and suppose $R \neq 1$. We check the hypotheses on G/R and HR/R. Denote $T = O_{p'}(H)$ and notice that HR/R is isomorphic to H/T. Given a subgroup $M/R \leqslant HR/R$, $M = R(H \cap M)$ and under the isomorphism the image of M/R is $(H \cap M)/T$. If M/R is a maximal subgroup of $O_{p'p}(HR/R)$ then $H \cap M$ is a maximal subgroup of $O_{p'p}(H)$ containing T and by hypothesis is a CAP-subgroup of G. Hence M is a CAP-subgroup of G and so is M/R in G/R. By induction

G/R is p-supersoluble and this implies obviously that G is p-supersoluble.

Therefore we assume henceforth that $R = 1$. So, $T = 1$ and $O_{p'p}(H) = O_p(H) = F(H)$.

Suppose $P = O_p(\Phi(G)) = \Phi(G) \neq 1$. By Satz III.3.5 of [3], $F(HP/P) = F(HP)/P$ and by Lemma 3, $F(HP) = F(H)P$; therefore, $O_p(H)P/P = F(H)P/P = F(HP/P)$; hence $O_p(H)P/P = O_p(HP/P)$. On the other hand if we denote $K/P = O_{p'}(HP/P)$ and S is a Hall p'-subgroup of K we have $K = SP$ and by the Frattini argument $G = KN_G(S) = PN_G(S) = N_G(S)$ and S is normal in G. Therefore $S = 1$ and $O_{p'}(HP/P) = 1$. This implies $O_{p'p}(HP/P) = O_p(HP/P) = O_p(H)P/P$. If M/P is a maximal subgroup of $O_p(H)P/P$ then $M \cap O_p(H)$ is a maximal subgroup of $O_p(H)$ and, by hypothesis, is a CAP-subgroup of G. Now usual arguments and the induction hypothesis give G/P is p-supersoluble and then so is G.

Hence, we can assume $O_{p'}(G) = \Phi(G) = 1$. Clearly $O_{p'}(H) = 1$ and $O_{p'p}(H) = O_p(H)$ and therefore we are in the hypothesis of Lemma 4 and we are done.

These two theorems give characterizations of p-supersolubility:

COROLLARY 1. *Let p be a prime and G a p-soluble group. Then the following are equivalent*:

i) *G is p-supersoluble*;

ii) *all p-subgroups of G are CAP-subgroups of G*;

iii) *all maximal subgroups of the Sylow p-subgroups of G are CAP-subgroups of G*;

iv) *all maximal subgroups of $O_{p'p}(G)$ containing $O_{p'}(G)$ are CAP-subgroups of G*;

v) *there exists a normal subgroup H of G such that G/H is p-supersoluble and all maximal subgroups of any Sylow p-subgroup of H are CAP-subgroups of G*;

vi) *there exists a normal subgroup H of G such that G/H is p-supersoluble and all maximal subgroups of $O_{p'p}(H)$ containing $O_{p'}(H)$ are CAP-subgroups of G.*

In Theorems A and B we have restricted ourselves to p-soluble groups. Theorem A does not hold in general.

EXAMPLE 1. Consider the group $G = \mathrm{Alt}(5)$. Clearly G is not 5-supersoluble and 1 is the maximal subgroup of any Sylow 5-subgroup of G.

EXAMPLE 2. Take $C = C_3$. C has an irreducible and faithful module V over $GF(2)$. Construct $A = VC \cong \mathrm{Alt}(4)$. A has an irreducible and faithful module W over $GF(3)$. Construct $B = WA$. If $D = C_2$ consider $G = D \times B$ and $H = D \times WV$.

G is soluble and $G/H \cong C_3$ is 2-supersoluble; $O_{2'}(H) = W$, $O_2(H) = = D \neq 1$ and 1 is the maximal subgroup of D: all maximal subgroups of $O_2(H) \neq 1$ are CAP-subgroups of G; however G is a non-2-supersoluble group.

A π-soluble group is *π-supersoluble* if its π-chief factors are all cyclic, i.e. if it is p-supersoluble for all primes $p \in \pi$. Obviously, results for π-supersolubility can be obtained just by taking the «intersection» of the corresponding results for p-supersolubility for all primes $p \in \pi$. One might ask whether the results of this section can be generalized by changing p by π to obtain results about π-supersolubility, where π is a set of prime numbers with $|\pi| > 1$. The answer is negative.

EXAMPLE 3. Take $\pi = \{2, 3\}$. Consider the soluble group $G = \mathrm{Sym}(4)$ and $H = \mathrm{Alt}(4)$. G/H is π-supersoluble; the maximal subgroups of the Hall π-subgroups of $H = O_\pi(H)$ are the Sylow subgroups of H and they are CAP-subgroups of G. $O_{\pi'}(G) = \Phi(G) = 1$. But G is not π-supersoluble.

4. Characterizations of supersoluble groups.

A particular case of π-supersolubility, when π is the set of all primes dividing the order of G, is the usual supersolubility. In this section we deduce some characterizations of supersolubility.

Theorem C is clearly inspired in Theorem A. However no hypothesis on the solubility is needed here. In fact the solubility is deduced from the other hypothesis.

THEOREM C. *Let G be a group and H a normal subgroup of G such that G/H is supersoluble. Suppose that all maximal subgroups of the Sylow subgroups of H are CAP-subgroups of G. Then G is supersoluble.*

PROOF. We prove first that, under these conditions, G is soluble. Suppose there exists a nonabelian chief factor of G, say N/K. If H avoids N/K, $H \cap N \leq K$, then NH/KH is a chief factor of the supersoluble group G/H and is G-isomorphic to H/K; this cannot happen and therefore $N \leq KH$. So, H/K is G-isomorphic to $(N \cap H)/(N \cap K)$ and we can suppose without loss of generality that the non-abelian chief factor N/K of G is below H.

Take P a Sylow subgroup of H and M a maximal subgroup of P. By hypothesis M is a CAP-subgroup of G. If M covered N/K, then the chief factor $N/K \cong (N \cap M)/(K \cap M)$ would be nilpotent; therefore N/K must be avoided by every maximal subgroup of every Sylow subgroup of H.

On the other hand $|N/K|$ is not square-free; so, there exists a prime q such that q^2 divides $|N/K|$; if $Q \in \mathrm{Syl}_q(H)$ then q^2 divides the index $|Q \cap N : Q \cap K|$. Suppose $Q \cap N$ is a strict subgroup of Q and consider a maximal subgroup M of Q with $Q \cap N \leqslant M$. M avoids N/K and therefore we have $Q \cap N \leqslant M \cap N = M \cap K \leqslant Q \cap K$, a contradiction. Then $Q = Q \cap N$ and $Q \leqslant N$ and for any maximal subgroup M of Q, we have that $M = M \cap N = M \cap K \leqslant Q \cap K < Q \cap N = Q$ and q^2 divides $|Q : M| = q$, a contradiction.

So, we conclude that G has no nonabelian chief factors and consequently G is soluble.

Now we are in the hypothesis of Theorem A for all primes p. Consequently G is p-supersoluble for all primes p. That is to say that G is supersoluble.

Notice that Theorem 1 of [6] is a special case of Theorem C.

To obtain an analogue of Theorem B for supersolubility we notice that the condition $O_{p'}(G) = 1$ in Lemma 4 is used basically to obtain $C_G(F(G)) \leqslant F(G)$. If we restrict ourselves to the soluble universe this condition is satisfied and we can obtain the following.

THEOREM D. *Let G be a group and H a normal subgroup of G such that H is soluble and G/H is supersoluble. Suppose that all maximal subgroups of the Sylow subgroups of $F(H)$ are CAP-subgroups of G. Then G is supersoluble.*

PROOF. We prove the theorem by induction on the order of G.

Suppose $\Phi(G) \neq 1$ and consider a prime p such that p divides $|\Phi(G)|$. Denote $P = O_p(\Phi(G)) \neq 1$. By Satz III.3.5 of [3], $F(HP/P) = F(HP)/P$ and by Lemma 3, $F(HP) = F(H)P$; therefore, $F(HP/P) = F(H)P/P$; it is easy to check the hypothesis and by induction G/P is supersoluble and then so is G.

Hence, we can assume $\Phi(G) = 1$.

The remainder of the proof repeats the arguments of Lemma 4. First we prove that all minimal normal subgroups of G are cyclic. After that, since G is soluble, $C_G(F(G)) \leqslant F(G)$ and by Satz III.4.5 of [3], $F(G) = \mathrm{Soc}(G) = N_1 \times \ldots \times N_r$, where each N_i is a minimal normal subgroup of G. Therefore $\bigcap_{i=1}^{r} C_G(N_i) = C_G(F(G)) = F(G)$ and again

$G/F(G)$ is supersoluble. But all chief factors of G below $F(G)$ are cyclic and hence the whole G is supersoluble.

It is clear again that Theorems C and D give indeed characterizations of supersolubility. As a corollary we easily obtain

COROLLARY 2. *Given a group G the following are equivalent:*

i) *G is supersoluble;*

ii) *all subgroups of G are CAP-subgroups of G;*

iii) *all maximal subgroups of the Sylow subgroups of G are CAP-subgroups of G;*

iv) *there exists a normal subgroup H of G such that G/H is supersoluble and all maximal subgroups of any Sylow subgroup of H are CAP-subgroups of G;*

v) *there exists a normal soluble subgroup H of G such that G/H is supersoluble and all maximal subgroups of $F(H)$ are CAP-subgroups of G.*

Removing the hypothesis of the solubility of H in v), the characterization does not hold.

EXAMPLE 4. Take $G = H = SL(2,5)$; the trivial subgroup is the maximal subgroup of $F(G)$ and G is not supersoluble.

Weakening the hypothesis of Theorem C we obtain a more general result.

THEOREM E. *Let G be a group and let p denote the largest prime dividing $|G|$. Assume that for all prime $q \neq p$, every maximal subgroup of the Sylow q-subgroups of G is a CAP-subgroup of G. Then,*

i) *G possesses a Sylow tower,*

ii) *$G/O_p(G)$ is supersoluble.*

PROOF. i) Consider a minimal counterexample G to the theorem. Repeating some of the arguments of the above proofs, it is not difficult to prove that if N is nontrivial normal subgroup of G then the hypothesis hold in G/N and minimality of G implies that G/N possesses a Sylow tower. Therefore G is a monolithic primitive group such that $G/\mathrm{Soc}(G)$ possesses a Sylow tower (and then is soluble). Denote $S = \mathrm{Soc}(G)$.

Suppose that S is not soluble. If q is the smallest prime dividing $|S|$ then $q \neq p$ and q^2 divides $|S|$ by Satz IV.2.8 of [3]. Take $Q \in \mathrm{Syl}_q(S)$

and $P \in \mathrm{Syl}_q(G)$ with $Q \leqslant P$. Assume that $Q = P$; for any maximal subgroup M of P, M is a *CAP*-subgroup of G and therefore M avoids S; however this means that $M = 1$ and hence $|P| = q$, a contradiction. So, Q is a proper subgroup of P and we can consider a maximal subgroup M of P with $Q \leqslant M$; again M must avoid S and then $Q = 1$, a contradiction. Thus, S is soluble and so is G.

Let $|S| = q^n$, q prime. Of course $q \neq p$. If $n = 1$ then G would be supersoluble and would possess a Sylow tower; so, $n > 1$. If q does not divide $|G/S|$ then $S \in \mathrm{Syl}_q(G)$ and any maximal subgroup M of S must avoid S, i.e. S is cyclic, a contradiction. Consequently q divides $|G/S|$. Now if $Q \in \mathrm{Syl}_q(G)$ and M is maximal subgroup of Q avoiding S then $|S| = |Q: M| = q$ and S would be cyclic, a contradiction. Therefore every maximal subgroup of Q covers S and $S \leqslant \Phi(Q)$.

If K is complement of S in G, $Q = (Q \cap K) S = (Q \cap K) \Phi(Q) = Q \cap \cap K$ and $S \leqslant Q \leqslant K$. This is the final contradiction.

Hence, the minimal counterexample does not exist and the theorem is true.

ii) Apply the equivalence between i) and iii) in Corollary 2 to the group $G/O_p(G)$.

Notice that Theorem 3.6 of [4] is an special case of Theorem E.

5. Final remark.

In [6] a π-quasinormal subgroup of a group G is defined to be a subgroup which permutes with any Sylow subgroup of G. A number of results involving π-quasinormal subgroups are proved in [1] and [6]. The statements of the Theorem 3.2, 4.1 and 4.2 of [1] are analogous to the theorems presented here replacing the cover and avoidance property by π-quasinormality.

However it is easy to find soluble groups with *CAP*-subgroups which are not π-quasinormal. Conversely, there are also soluble groups with π-quasinormal subgroups which are not *CAP*-subgroups.

EXAMPLE 5. Consider $C = \langle a \rangle \cong C_3$ and $A = \mathrm{Alt}(4)$ and construct the wreath product $G = C \,\mathrm{wr}\, A$ with the natural action. Denote $C^{\#}$ the base group of G; $C^{\#}$ is an elementary abelian 3-group of order 3^4 generated by $\{a_1, a_2, a_3, a_4\}$ where the indices are the obvious ones according to the natural action of A. Consider the subgroup $K = \langle a_1 a_2, a_3 a_4 \rangle$. If V is Klein 4-group of A then $N_G(K) = C^{\#} V$ and therefore if $P \in \in \mathrm{Syl}_2(G)$, $PK = KP$ and hence K is a π-quasinormal subgroup of G.

But $Z = \langle a_1 a_2 a_3 a_4 \rangle < K < C^{\#}$ and then K neither covers nor avoids the chief factor $C^{\#}/Z$.

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